Comparing Realizability over $\mathcal{P}\omega$ and K_2 (Draft)

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1 Introduction

We compare the realizability toposes over the graph model $\mathcal{P}\omega$ and Kleene's second model K_2. In the topos $\mathbf{RT}(\mathcal{P}\omega)$ many choice principles are valid, whereas $\mathbf{RT}(K_2)$ supports strong continuity principles. Both toposes allow for the embedding of the category of countably based $\mathcal{T}_0$-spaces, but the standard embedding of the space $\mathbb{R}$ in $\mathbf{RT}(\mathcal{P}\omega)$ does not coincide with the real number object, while both objects do coincide in $\mathbf{RT}(K_2)$.

2 Choice versus Continuity

2.1 Choice Principles in $\mathbf{RT}(\mathcal{P}\omega)$

For a topos $\mathcal{E}$ and objects $X, Y \in \mathcal{E}$ we define the axiom of choice AC_{XY} as follows

$$P:\mathcal{P}(X \times Y) \mid \forall x:X.\exists y:Y.P(x,y) \vdash \exists f:X \to Y.\forall x:X.P(x,f(x))$$

Furthermore, we say AC_X is valid in $\mathcal{E}$ if AC_{XY} is valid for all $Y \in \mathcal{E}$.

For an object X in some *realizability* topos $\mathbf{RT}(\mathcal{A})$, AC_X holds if and only if X is projective. An object X is projective in $\mathbf{RT}(\mathcal{A})$ if and only if it is isomorphic to a partitioned assembly, i.e. an assembly where every element has exactly one realizer.

Next, we are going to show that for a surprisingly large number of pairs of objects X, Y in $\mathbf{RT}(\mathcal{P}\omega)$ including all pairs of finite types over the natural number object, the axiom AC_{XY} holds. So in particular, $\mathbf{RT}(\mathcal{P}\omega)$ is a model for extensional $\mathrm{HA}^\omega + \mathrm{AC}$.

As all of the objects we consider in the following are modest sets, we choose to present them as PERs. If R is a PER let $|R|$ denote the carrier of R, i.e. the set $|R| = \{a \in \mathcal{P}\omega \mid aRa\}$.

Definition 2.1. Let us call a PER R *well-behaved* if

1. R has the *join-property*, i.e. every equivalence class of R is closed under binary joins.
2. for all $a, b, c \in |R|$, $a, b \subseteq c$ implies $a\,R\,b$.

Note that this implies that every equivalence class of R is downward closed *within* $|R|$.

As we shall see in section 3.1, well behaved modest sets are often used for the interpretation of datatypes.

Theorem 2.2. *The class of well behaved PERs is closed under finite products, finite sums and it is an exponential ideal in* $\mathbf{PER}(\mathcal{P}\omega)$.[1]

Proof. The empty PER and hence the initial object is well-behaved and for the terminal object we choose the biggest equivalence relation on $\mathcal{P}\omega$ which is also well-behaved. The standard pairing function $\langle \_ , \_ \rangle : \mathcal{P}\omega \times \mathcal{P}\omega \to \mathcal{P}\omega$ is defined by $n \in \langle a, b \rangle \Leftrightarrow \exists p \in a.\ n = 2p \vee \exists q \in b.\ n = 2q + 1$. If R, S are well-behaved and if we build $R \times S$ using the standard pairing function then $R \times S$ is again well-behaved. In fact, this holds for any pairing whose projection functions preserve binary joins. The same is true for the binary sum. In order to show that the exponential is well-behaved one makes use of the fact that $(f \cup g).a = f.a \cup g.a$ holds in $\mathcal{P}\omega$. □

Theorem 2.3. *If R, S are well-behaved PERs then* AC_{RS} *holds.*

Proof. It is sufficient to show that any realizer sending $|R|$ to $|S|$ is extensional. That means, if $e \in \mathcal{P}\omega$ is such that $r \in |R|$ implies $e.r \in |S|$ then for all $r, r' \in |R|$ $r\,R\,r'$ implies $e.r\,S\,e.r'$. Assume $r\,R\,r'$. Then $r \cup r' \in |R|$. Therefore $e.(r \cup r') \in |S|$. By monotonicity of the application $e.r$ and $e.r'$ are below $e.(r \cup r')$ and as S is well-behaved $e.r\,S\,e.r'$ follows. □

Remark 2.4. The theorems apply to $\mathcal{P}\omega_{\mathrm{r.e.}}$ without modification of the proofs. In fact everything works fine for any combinatory algebra which is a sup-semilattice such that application is monotone and there is a pairing such that the projection functions preserve the sup and application commutes with sup on the left.

Note also that in theorem 2.3 one only uses that each equivalence class of R is directed and that S satisfies condition 2 of the definition of well-behavedness.

An analogous version of the result, where AC_{XY} is to be replaced by any of its first order instances and the symbols $\subseteq, \cup$ have to be replaced by $\leq, \vee$, also holds for the logos **Equ** which is introduced in [BBS98].

2.2 Continuity Principles in $\mathbf{RT}(K_2)$

The topos $\mathbf{RT}(K_2)$ also validates some choice principles like AC_0 and AC_1 but not $\mathrm{AC}_{2,0}$ for example. As a characteristic feature, $\mathbf{RT}(K_2)$ validates the axiom

$$F : (N \to N) \to N \vdash \exists \gamma. \forall \alpha. F(\alpha) = \gamma(\alpha) \tag{1}$$

where $\gamma(\alpha) = n$ is short for $\exists m. \gamma(\bar{\alpha} m) = n + 1 \wedge \forall i < m.\ \gamma(\bar{\alpha} i) = 0$, the letters α, γ range over type 1 and n, m, i range over type 0. Axiom (1) formally expresses that every type 2 functional has an associate, i.e. is realizable in K_2, so it doesn't come as a surprise that axiom (1) is valid in $\mathbf{RT}(K_2)$. In conjunction with AC_1 axiom (1) entails the so called axiom of *strong continuity*.

2.3 Incompatibility of Choice and Continuity

In particular, axiom (1) implies the axiom

$$F : (N \to N) \to N \vdash \forall \alpha. \exists m. \forall \beta.\ \bar{\beta} m = \bar{\alpha} m \supset F(\beta) = F(\alpha) \tag{2}$$

which formally expresses that every type 2 functional is continuous. In conjunction with $\mathrm{AC}_{2,0}$ axiom (2) entails the existence of a modulus of continuity functional of type 3, which is known to be inconsistent with extensional HA^ω (see e.g. [TvD88]).

This implies that axiom (2) does not hold in $\mathbf{RT}(\mathcal{P}\omega)$ even though, from the external point of view, all type 2 functionals *are* continuous. On the other hand,

[1] more precisely one should probably say that *there is* always a well-behaved product, sum and exponential, as the class of well-behaved PERs is clearly not closed under isomorphy.

$\mathrm{AC}_{2,0}$ cannot hold in $\mathbf{RT}(K_2)$. In particular $\mathcal{P}\omega$ and K_2 are inequivalent as PCAs. All one can show is that K_2 is a retract of $\mathcal{P}\omega$ in the 2-category of PCAs and applicative morphisms in the sense of [Lon94].

3 Embedding spaces in realizability toposes

3.1 Embedding $\mathcal{T}_{0,\omega}$ in $\mathbf{RT}(\mathcal{P}\omega)$

Let us denote the category of countably-based $\mathcal{T}_0$-spaces by $\mathcal{T}_{0,\omega}$ and its well known embedding into $\mathbf{RT}(\mathcal{P}\omega)$ by $\mathcal{I} : \mathcal{T}_{0,\omega} \to \mathbf{RT}(\mathcal{P}\omega)$. Then for all $\mathcal{T}_{0,\omega}$-spaces X the object $\mathcal{I}X$ is modest and projective. If, in addition, X is a $\mathcal{T}_1$-space then $\mathcal{I}X$ is also well-behaved. That means, when reasoning about $\mathcal{T}_{1,\omega}$-spaces and all finite types over them inside of $\mathbf{RT}(\mathcal{P}\omega)$, the axiom of choice always holds. Figuratively speaking, there is no place for intensional information.

Note also that the NNO in $\mathbf{RT}(\mathcal{P}\omega)$ coincides with the embedding of the discrete space $\mathbb{N}$. All realizability toposes validate AC_N and therefore the Cauchy and the Dedekind construction of the real numbers coincide. Let us denote the real number object by $\mathcal{R}$. An obvious question to ask is whether $\mathcal{I}\mathbb{R}$ coincides with $\mathcal{R}$.

Theorem 3.1. *The real number object $\mathcal{R}$ in $\mathbf{RT}(\mathcal{P}\omega)$ is not isomorphic to the object $\mathcal{I}\mathbb{R}$.*

Proof. The sentence $\forall r\!:\!\mathcal{R}.\exists n\!:\!N.\ r < n$ holds in any topos. If $\mathcal{R}$ were isomorphic to $\mathcal{I}\mathbb{R}$ then by $\mathrm{AC}_{\mathcal{I}\mathbb{R},N}$ (and because supports split in realizability toposes) there would actually be a Skolem function $f : \mathcal{I}\mathbb{R} \to N$ for the sentence which, by the fullness of the embedding, would have to be continuous, hence constant, which is absurd since $\neg\exists n\!:\!N.\forall r\!:\!\mathcal{R}.\ r < n$ holds in any topos, too. $\square$

Remark 3.2. By analysis of the object $\mathcal{R}$ it can be seen that, externally, all morphisms from $\mathcal{R}$ to N are constant, too, which allows us to strengthen theorem 3.1 in the following way: $\mathcal{R}$ is not projective, hence it is not isomorphic to *any* object of the form $\mathcal{I}X$.

3.2 Embedding $\mathcal{T}_{0,\omega}$ in $\mathbf{RT}(K_2)$

For a space X in $\mathcal{T}_{0,\omega}$ with some enumeration $(O_n)_{n\in\mathbb{N}}$ of a set of basic opens let $\mathcal{I}'X$ be the following modest set in $\mathbf{RT}(K_2)$: The underlying set is defined to be $|X|$ and $\alpha \Vdash x$ iff $(O_{\alpha_n})_{n\in\mathbb{N}}$ is a basis for the neighborhood filter of x. Note that $\mathcal{I}'X$ is always modest but rarely projective. Like the functor $\mathcal{I}$, $\mathcal{I}'$ is full and faithful and $\mathcal{I}'\mathbb{N} \cong N$, but this time also $\mathcal{I}'\mathbb{R} \cong \mathcal{R}$.

4 Remarks

A special case of the theorems 2.2 and 2.3 was shown by I. Bethke and can be found on page 488 in [TvD88]. I have learned about the notion of well-behavedness, which allows a clearer proof of a more general fact, from the talk given by Martin Hyland at the workshop of realizability '99 in Trento.

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