

Regular Functors and Relative Realizability Categories

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1 Introduction

This paper is our contribution to the abstract theory of realizability. We generalize *relative realizability* in order to reduce the necessary conditions for the construction of relative realizability categories and regular functors from relative realizability categories into other categories. We hope this will clarify the fascinating properties of the known examples of these categories.

1.1 Universal Properties

We define relative realizability categories by a universal property. Such a property determines the category only up to equivalence; that allows us to switch between equivalent categories whenever convenient. We build on the following research to get this done.

To find a common ground between Grothendieck toposes and realizability toposes, Pitts, Johnstone and Hyland developed tripos theory [11], [16].

Scedrov, Freyd and Carboni [7] showed that every realizability topos could be described as an *exact completion* (definition in subsection 3.3) of its regular subcategory of assemblies. The conditions under which an exact completion is a topos were derived by Menni [14], [15]. Together with Celia Magno [5], Carboni earlier described the *free* exact completion of left exact categories. Rosolini and Robinson showed [17] that realizability toposes constructed over the category of sets are free exact completions of the smaller subcategory of partitioned assemblies too. Carboni noted [4] that the category of assemblies is an intermediate step, being the *free regular completion* of the category of partitioned assemblies. The relation between the various completions is explained in [6].

Longley [13] defined *applicative morphisms* between *partial combinatory algebras* and proved an equivalence between these morphisms and regular functors between categories of assemblies. He also showed that the category of assemblies satisfies a universal property relative to another subcategory of realizability toposes: the category of *modest sets*.

When constructing categories of assemblies over toposes where epimorphisms do not split, the category of assemblies for a partial combinatory algebra no longer is the regular completion of the category of partitioned assemblies. Hofstra developed the alternative notion of *relative completion* [8], to deal with the more general case.

1.2 Generalizations of Realizability

Realizability toposes over the topos of sets are two-valued, because the negation of every unrealized sentence is realized. In relative realizability a sentence is only valid if it is realized by a combinator from a pre-selected set. This implies relative realizability toposes are many valued models of higher order intuitionistic logic.

Relative realizability can be developed inside a model of constructive mathematics to increase the possibilities even further. In fact, Kleene and Vesley introduced relative realizability in [12] to interpret constructive analysis against an intuitionist background, using both generalizations at once. In their example, the realizers are functions $\mathbb{N} \rightarrow \mathbb{N}$, but only recursive functions validate the sentences they realize.

Birkedal and Bauer studied the abstract properties of relative realizability in their Ph. D. theses [2], [1]. In [3] van Oosten and Birkedal described relative realizability as realizability over a PCA object in another topos.

The structure of the set of realizers was generalized by van Oosten and Hofstra [9]. They also characterize the applicative morphisms that correspond to geometric morphisms between toposes. In [10], Hofstra uses *basic combinatory objects* to provide a framework for all kinds of realizability.

1.3 Summary

We generalize realizability in a new direction. In section 2, we define categories of assemblies for *ordered partial combinatory algebras with filters* (in the sense of [9] and [10]) in arbitrary Heyting categories by a universal property. We also construct an example that is a variation on the traditional construction of the category of assemblies we adjusted to ordered partial combinatory algebras that have too few global sections.

In section 3 the underlying category is a topos; under that condition exact completions of categories of assemblies are toposes.

Thanks to the universal property we can easily construct regular functors from relative realizability categories into other regular categories. Section 4 is devoted to examples of such functors. We spend some time there to construct right adjoints to some of these functors, showing that the definition of computationally dense applicative morphisms given in [10] still applies.

1.4 Further Thoughts

Moerdijk and van den Berg [18] show how to construct variants of the effective topos over predicative categories with small maps. We may generalize relative realizability to predicative categories using the constructions in this paper.

We are puzzled by the equivalence of geometric morphisms and computationally dense applicative morphisms. A better theory of exact completions might shed some light on this subject.

There is an algorithm for translating the internal language of a realizability topos to the internal language of the underlying category. We want to know what it is and use it to characterize relative realizability toposes.

The definition we give of the category of assemblies is impredicative. Maybe we can find a good predicative alternative that still is a universal property.

2 Relative Realizability Categories

This is the main section, in which we define the relative realizability categories by a universal property and prove the existence of categories with this property. But we start by defining the basic constructions of generalized relative realizability.

2.1 OPCA pairs

A Heyting category is a category that has first order intuitionistic logic as its internal language. Specifically, \mathcal{E} is a Heyting category if for every object X the class $\text{Sub}(X)$ of subobjects of X is a distributive lattice and for every arrow $f : X \rightarrow Y$ the inverse image map $f^{-1} : \text{Sub}(Y) \rightarrow \text{Sub}(X)$ has both adjoints.

In this subsection we will define *ordered partial combinatory algebras* as combinatory complete ordered partial applicative structures.

Definition 1. An *ordered partial applicative structure* or *OPAS* is an object with an ordering and a monotone partial binary operator called *application*, the domain of which is downward closed. If x, y and z are elements of a OPAS, we write $xy \downarrow z$ for: ‘the application of x to y is defined and is equal to z ’. The formula $xy \downarrow$ simply means there is a z such that $xy \downarrow z$. Finally if $xy \downarrow$, then xy denotes the unique z such that $xy \downarrow z$.

We single out certain partial monotone arrows of OPASes.

Definition 2. For each OPAS A , $n \in \mathbb{N}$, $U \subseteq A^n$ and $f : U \rightarrow A$, we say that f is *representable*, if there is some $a \in A$ such that for all $\vec{x} \in \text{dom}f$, there is a $y \leq f(\vec{x})$ such that $((ax_1) \dots) x_n \downarrow y$. We call such arrows *partial representable arrows*, and $a \in A$ a *realizer* for this arrow.

We interpret this definition in the internal language of the Heyting category. So relative to an OPAS A a partial morphism $f : U \subseteq A^n \rightarrow A$ is representable if and only if the following subobject of A is globally supported.

$$\llbracket f \rrbracket = \{ a \in A \mid \forall \vec{x} \in U. \exists y \in A. y \leq f(\vec{x}) \wedge ((ax_1) \dots) x_n \downarrow y \}$$

This *object of realizers* of f may not have any global section.

We are interested in OPASes in which partial arrows that are constructed by repeated use of application are representable.

Definition 3. The set of *partial combinatory arrows* is constructed from projections by pointwise application. So $(x, y) \mapsto x$ and $(x, y, z) \mapsto xz(yz)$ are both examples of partial combinatory arrows. An OPAS is *combinatory complete*, if every partial combinatory arrow is representable. Combinatory complete OPASes are called *ordered partial combinatory algebras* or *OPCAs* [9]. *Partial combinatory algebras* or *PCAs* are OPCAs that have the discrete ordering.

Remark 4. Let k be a realizer $(x, y) \mapsto x$ and let s be a realizer for $(x, y, z) \mapsto xz(yz)$. We can construct a realizer for each partial combinatory arrow from just these two. Therefore an OPAS is combinatory complete when these two partial combinatory arrows are representable.

OPCAs are models for computation. We can view an OPCA A as the set of codes for programs in a functional programming language. The application operator represents the execution of one program on the code of another. For relative realizability we want to apply a limited set of programs to a larger set of codes. This lead to the following generalization.

Definition 5. A OPCA pair (A', A) is a pair of OPASes, where

- A' is a subobject of A , and application of A' is the restriction of application in A
- A' is closed under the application in A . So if $x, y \in A'$ and there is a $z \in A$ such that $xy \downarrow z$, then $z \in A'$.

- All partial combinatory arrows of A are representable in A' . So if $f : U \subseteq A^n \rightarrow A$ is combinatory, then $\llbracket f \rrbracket$ intersects A' .

Note that if (A', A) is an OPCA pair both A' and A are OPCAs themselves. Also, the last condition is equivalent to the condition that the sets of realizers for the partial combinatory arrows $(x, y) \mapsto x$ and $(x, y, z) \mapsto (xz)(yz)$ intersect A' , for reasons outlined in remark 4. Finally, if A is an OPCA, then (A, A) is an OPCA pair. For this reason ‘absolute’ realizability is a special case of relative realizability.

2.2 Regular Models

For each Heyting category \mathcal{E} and each OPCA pair (A', A) in \mathcal{E} , we would like to construct a slightly larger category $\mathcal{E}[\dot{A}]$, where \dot{A} is a subOPCA of A , where the only partial endomorphisms are restrictions of partial endomorphisms that are representable in A' . We approach this problem as follows. A *pseudoinitial object* in a 2-category is an object for which there is an up to isomorphism unique arrow to every other object. We construct a 2-category of suitable functors with some added structure from \mathcal{E} into other categories, such that a pseudoinitial object should be like the category we hinted at before.

Definition 6. Let \mathcal{E} be a Heyting category, (A', A) an OPCA pair in \mathcal{E} , \mathcal{C} a regular category and $F : \mathcal{E} \rightarrow \mathcal{C}$ a regular functor. An F -filter is a subobject $C \leq FA$ that satisfies:

- If $x \in C$ and $x \leq y$, then $y \in C$.
- If $x, y \in C$ and $xy \downarrow z$ for some $z \in FA$, then $z \in C$.
- If $U \subseteq A$ intersects A' , then FU intersects C .

Like OPAS, filters are preserved by regular functors, because their definition involves only commutative diagrams, pullbacks and images. This also means that for each pair of regular functors $F : \mathcal{E} \rightarrow \mathcal{C}$ and $G : \mathcal{C} \rightarrow \mathcal{D}$ and each F -filter C the object GC is a GF -filter.

Definition 7. Let a *regular model* for (A', A) be a regular functor $F : \mathcal{E} \rightarrow \mathcal{C}$ with an F -filter. For each regular $G : \mathcal{E} \rightarrow \mathcal{D}$, each F -filter C and each G -filter D a morphism $(F, C) \rightarrow (G : \mathcal{E} \rightarrow \mathcal{D}, D)$ is a regular functor $H : \mathcal{C} \rightarrow \mathcal{D}$ with an isomorphism $\eta : HF \rightarrow G$, such that $\eta_A : HFA \rightarrow GA$ restricts to an isomorphism between FC and D . A *regular relative realizability category* for the pair (A', A) is a *pseudoinitial* regular model i.e.: there is an *up to isomorphism* unique regular functor from such a model to any other model.

Remark 8. For each regular model (F, C) , $n \in \mathbb{N}$ and $U \subseteq A^n$ the set of partial arrows $FU \cap C^n \rightarrow C$ contains the images of partial A' -representable arrows $U \rightarrow A$. In that sense it is a model of the regular theory of a subset of A that is closed under a set of partial operators.

Theorem 9. *There is a pseudoinitial regular model for every OPCA pair in every Heyting category.*

In the next couple of subsections, we define a category, a functor and a filter, and prove that these form an pseudoinitial regular model.

2.3 Assemblies

Definition 10. An *assembly* is a pair (X, Y) where $X \in \mathcal{E}$ and where Y is a subobject of $A \times X$, such that if $(a, x) \in Y$ and $b \leq a$, then $(b, x) \in Y$ too. Let $D(X, Y) = \{x \in X \mid \exists a \in A. (a, x) \in Y\}$. A *morphism* $(X, Y) \rightarrow (X', Y')$ is an arrow $f : D(X, Y) \rightarrow D(X', Y')$ for which there exists an $n \in \mathbb{N}$, a $V \subseteq A^n$ that intersects $(A')^n$ and a partial combinatory arrow $g : A^{n+1} \rightarrow A$ such that $V \times Y$ is a subobject of the domain of $g \times f$, while Y' contains the image of $g \times f$. So $g \times f$ is a total arrow $V \times Y \rightarrow Y'$. We will call a pair (V, g) that has this relation to f a *tracking* of f or say that (V, g) *tracks* f .

We summarize this by saying the following diagram must commute.

$$\begin{array}{ccc}
 V \times Y & & \\
 \downarrow (v,y) \mapsto y & \searrow g \times f & \\
 Y & & Y' \\
 \downarrow (a,x) \mapsto x & & \downarrow (a,x) \mapsto x \\
 D(X, Y) & \xrightarrow{f} & D(X', Y')
 \end{array}$$

Lemma 11. *Assemblies and morphisms form a category.*

Proof. The composite of any two morphisms $f : (X, Y) \rightarrow (X', Y')$ and $f' : (X', Y') \rightarrow (X'', Y'')$ is tracked. There are trackings (U, g) of f and (U', g') of f' . Let $h(\vec{x}, \vec{y}, \vec{z}) = g'(\vec{x}, g(\vec{y}, \vec{z}))$, then h is a partial combinatory arrow $A^{m+n+1} \rightarrow A$ and $h \times (f' \circ f) : U' \times U \times Y \rightarrow Y''$. So $(U' \times U, h)$ is a tracking of $f' \circ f$ and in general morphisms are closed under composition. The terminal object A^0 intersects $(A')^0$ and $x \mapsto x$ is partial combinatory, so $1_{D(X, Y)} : (X, Y) \rightarrow (X, Y)$ is a morphism. This is the identity morphism of (X, Y) . \square

Definition 12. We denote the *category of assemblies* by $\text{Asm}(A', A)$.

Our definition of morphism of assemblies is complicated, but equivalent to the conventional definition of a morphism of assemblies (see [19]) in the internal language of \mathcal{E} .

Lemma 13. *For assemblies (X, Y) and (X', Y') and any arrow $f : D(X, Y) \rightarrow D(X', Y')$, let*

$$\begin{aligned}
 \llbracket f : (X, Y) \rightarrow (X', Y') \rrbracket = \\
 \{ a \in A \mid \forall (b, x) \in Y. \exists (c, y) \in Y'. ab \downarrow c, x \in \text{dom} f \wedge f(x) = y \}
 \end{aligned}$$

f is a morphism $(X, Y) \rightarrow (X', Y')$ if and only if $\llbracket f : (X, Y) \rightarrow (X', Y') \rrbracket$ intersects A' .

Proof. We define a useful family of combinatory functions by recursion:

$$\alpha_0(x) = x \quad \alpha_{n+1}(x_0, \dots, x_{n+1}) = \alpha_n(x_0, \dots, x_n)x_{n+1}$$

If $\llbracket f : (X, Y) \rightarrow (X', Y') \rrbracket$ intersects A' , then $(\llbracket f : (X, Y) \rightarrow (X', Y') \rrbracket, \alpha_1)$ tracks f , so f is a morphism.

For each $g : (X, Y) \rightarrow (X', Y')$ there is a tracking $(V \subseteq A^n, h)$. We build a new tracking for g that has a more suitable form.

$$\llbracket h \rrbracket = \{ a \in A \mid \forall \vec{x} \in \text{dom} h. \exists y \leq h(\vec{x}). \alpha_{n+1}(a, \vec{x}) \downarrow y \}$$

$(\alpha_{n+1}, \llbracket h \rrbracket \times V)$ is the new tracking. For all $(a, \vec{x}) \in \llbracket h \rrbracket \times V$, $b \in A$ and $(c, y) \in Y$, if $b = \alpha_n(a, \vec{x})$, then $bc \downarrow$ and $bc = \alpha_{n+1}(a, \vec{x}, c) \leq h(\vec{x}, c)$; therefore $(bc, g(x)) \in Y'$.

This means that $\alpha_n : \llbracket h \rrbracket \times V \rightarrow \llbracket g : (X, Y) \rightarrow (X', Y') \rrbracket$. Because $\llbracket h \rrbracket \times V$ intersects $(A')^{n+1}$ and A' is closed under application, the subobject $\llbracket g : (X, Y) \rightarrow (X', Y') \rrbracket$ intersects A' . \square

Remark 14. $D : \text{Asm}(A', A) \rightarrow \mathcal{E}$ is a faithful functor. For an OPCA pair (A', A) in the category of sets this functor is *not* isomorphic to the global sections functor unless $A' = A$. For that reason we use the D of *domain* rather than the Γ of global section to symbolize this functor.

This category has quite a bit more structure than just any regular category.

Lemma 15. *The category of assemblies is a Heyting category.*

Proof. We start with finite limits. If \top is terminal, then for each assembly (X, Y) the unique map $! : X \rightarrow \top$ is a morphism $(X, Y) \rightarrow (\top, A \times \top)$, as $(A^0, 1_A)$ is a tracking. To help construct pullbacks, let

$$\begin{aligned} \mathsf{T} &= \{ t \in A \mid \forall x, y \in A. \exists z \leq x. (tx)y \downarrow z \} \\ \mathsf{F} &= \{ f \in A \mid \forall x, y \in A. \exists z \leq y. (fx)y \downarrow z \} \\ \mathsf{P} &= \{ p \in A \mid \forall x, y, z \in A. (zx)y \downarrow \rightarrow \exists w \leq (zx)y. ((px)y)z \downarrow w \} \end{aligned}$$

Given $f : (X, F) \rightarrow (Z, H)$ and $g : (Y, G) \rightarrow (Z, H)$ let $p : W \rightarrow D(X, F)$ and $q : W \rightarrow D(Y, G)$ be a pullback cone for f and g in \mathcal{E} . Then let

$$K = \left\{ (a, w) \in A \times W \mid \begin{array}{l} \forall t \in \mathsf{T}. at \downarrow, (at, pw) \in F, \\ \forall f \in \mathsf{F}. af \downarrow, (af, qw) \in G \end{array} \right\}$$

If $h(x, y) = yx$, then (T, h) tracks $p : (W, K) \rightarrow (X, F)$ and (F, h) tracks $q : (W, K) \rightarrow (Y, G)$. Therefore p and q form a commutative square with f and g in the category of assemblies. If (L, l) tracks $r : \xi \rightarrow (X, F)$ and (M, m) tracks $s : \xi \rightarrow (Y, G)$ for any other assembly ξ , let $n(p, x, y, z) = (pl(x, z))m(y, z)$. There exists a unique factorisation $(r, s) : D\xi \rightarrow W$ through p and q and $(\mathsf{P} \times L \times M, n)$ tracks (r, s) . We see both that $\text{Asm}(A', A)$ has all finite limits and that D preserves them.

Next: images. Given $f : (X, Y) \rightarrow (X', Y')$ let $\exists_f(X, Y) = (X', \exists_{1 \times f}(Y))$. By computation $D(X', \exists_{1 \times f}(Y)) = \exists_f(D(X, Y))$, so $f : D(X, Y) \rightarrow D(X', \exists_{1 \times f}(Y))$, and $(A^0, 1_A)$ tracks $f : (X, Y) \rightarrow \exists_f(X, Y)$. If $p, q : \xi \rightarrow (X, Y)$ is a kernel pair for f in $\text{Asm}(A', A)$ then it is a kernel pair for f in \mathcal{E} , because D preserves finite limits. If (V, h) tracks some $g : (X, Y) \rightarrow \psi$ that satisfies $g \circ p = g \circ q$, then it also tracks the factorisation of g through the image of f . Hence $\exists_F(X, Y)$ is a coequalizer for the kernel pair.

We need to show that regular epimorphisms are stable. An epimorphism $e : (X, Y) \rightarrow (X', Y')$ is regular, if $\exists_e(X, Y) \simeq (X', Y')$. Therefore, we can assume that $(X', Y') = (X', \exists_{1 \times e}(Y))$ without loss of generality. Since $D((X', \exists_{1 \times e}(Y))) = \exists_e(D(X, Y))$, the functor D preserves regular epimorphisms and is itself regular. For any $f : (Z, H) \rightarrow (X', \exists_{1 \times e}(Y))$, let $p : (W, K) \rightarrow X$, $q : (W, K) \rightarrow Z$ be a pullback cone for e and f , like the one we constructed above. The arrow q is a regular epimorphism in \mathcal{E} , because D preserves pullbacks and \mathcal{E} is a regular category. Furthermore we find $(A^0, 1_A)$ tracks e , (T, h) tracks p and (F, h) tracks q . If (V, g) tracks f , then $(\mathsf{P} \times V, (p, \vec{v}, x) \mapsto pxg(\vec{v}, x))$ tracks $1_{D(Z, H)} : (Z, H) \rightarrow \exists_q(W, K)$, while (F, h) tracks $1_{D(Z, H)} : \exists_q(W, K) \rightarrow (Z, H)$. So $\exists_q(W, K) \simeq (Z, H)$ and q is a regular epimorphism in $\text{Asm}(A', A)$. So pullbacks of regular epimorphisms are regular epimorphisms.

We see that $\text{Asm}(A', A)$ is a regular category and that $D : \text{Asm}(A', A) \rightarrow \mathcal{E}$ is a regular functor. We now prove the existence of joins of subobjects to show that subobjects form lattices.

A morphisms $m : X \rightarrow Y$ is monic if and only if $\exists_m(X) \simeq X$. Therefore, given a mono $m : (X, Y) \rightarrow (X', Y')$ in $\mathbf{Asm}(A', A)$ we have $(X, Y) \simeq (X', \exists_{1 \times m}(Y))$. Therefore every subobject of an assembly (X, Y) can be represented by a subobject $F \leq A \times X$ in \mathcal{E} , such that $1_{D(X, F)} : (X, F) \rightarrow (X, Y)$ has a tracking. For all $X \in \mathcal{E}$ and all $F, G \leq A \times X$, say that (U, g) tracks $F \leq G$ if it tracks $1_{D(X, F)} : (X, F) \rightarrow (X, G)$.

On to joins. For any pair $F, G \leq A \times X$, let

$$F \vee G = \left\{ (a, x) \in A \times X \left| \begin{array}{c} \exists b \in A, p \in \mathbf{P}, t \in \mathbf{T}. (b, x) \in F, a \leq ptb \\ \vee \\ \exists b \in A, p \in \mathbf{P}, f \in \mathbf{F}. (b, x) \in G, a \leq pfb \end{array} \right. \right\}$$

If $f(x, y, z) = (xy)z$, then $(\mathbf{P} \times \mathbf{T}, f)$ tracks $Y \leq Y \vee Y'$ and $(\mathbf{P} \times \mathbf{F}, f)$ tracks $Y \leq Y \vee Y'$ so $Y \vee Y'$ is an upper bound of $\{Y, Y'\}$. If (U, g) tracks $Y \leq Z$ and (U', g') tracks $Y' \leq Z$, let

$$h(t, f, u, u', a) = ((at)[g(u, af)])[g'(u', af)]$$

$(\mathbf{T} \times \mathbf{F} \times U \times U', h)$ tracks $Y \vee Y' \leq Z$. Therefore $Y \vee Y'$ is the least upper bound.

The initial object \perp of \mathcal{E} allow only one assembly (\perp, \perp) , which is embedded in every other assembly. This is the bottom element of the classes of subobjects, which we may now call lattices of subobjects.

We proceed by constructing right adjoints to the inverse image maps. For each $f : (X, Y) \rightarrow (X', Y')$ and $F \leq Y$ let:

$$\forall_f(F) = \{ (a, y) \in A \times X' \mid \forall (b, x) \in Y. f(x) = y \rightarrow ab \downarrow \wedge (ab, x) \in F \}$$

The inverse image map is induced by pullbacks. Therefore, if $G \leq A \times X'$ represents a subobject of (X', Y') , then its inverse image can be represented by

$$f^{-1}(G) = \left\{ (a, x) \in A \times X \left| \begin{array}{c} \forall t \in \mathbf{T}. at \downarrow, (at, x) \in Y, \\ \forall f' \in \mathbf{F}. af' \downarrow, (af', f(x)) \in G \end{array} \right. \right\}$$

If (U, g) tracks $G \leq \forall_f(F)$, let $h(t, f', \vec{u}, x) = g(\vec{u}, xt)(xf')$. The pair $(\mathbf{T} \times \mathbf{F} \times U, h)$ tracks $f^{-1}(G) \leq F$. If $(V \subseteq A^n, h)$ tracks $f^{-1}(G) \leq F$, let

$$H = \left\{ w \in A \left| \begin{array}{c} \forall p, x, y \in A, u \in A^n. h(v, pxy) \downarrow \rightarrow \\ \exists z \leq h(v, pxy). (((wp)v_1) \cdots v_n)x y \downarrow z \end{array} \right. \right\}$$

and let $k(w, p, \vec{v}, x) = ((wp)\vec{v})x$. Then $(H \times \mathbf{P} \times V, k)$ tracks $G \leq \forall_f(F)$. We see that \forall_f is right adjoint to f^{-1} .

We now have shown that $\mathbf{Asm}(A', A)$ is regular, that subobjects form a lattice and that inverse image maps have both left and right adjoints. The existence of the right adjoint implies that the lattice of subobjects is distributive. Therefore $\mathbf{Asm}(A', A)$ is a Heyting category. \square

On to the functor.

Definition 16. For each object X in A let $\nabla X = (X, A \times X)$. For each arrow $f : X \rightarrow Y$, let $\nabla f = f$.

The arrow ∇f is a morphism $\nabla X \rightarrow \nabla Y$ because $D\nabla X = 1_{\mathcal{E}}$ and $1_A \times f : A \times X \rightarrow A \times Y$. The functor D is a faithful $\mathbf{Asm}(A', A) \rightarrow \mathcal{E}$ and ∇ is a right inverse. In fact ∇ is right adjoint to D , because the inclusion $1_{D(X, Y)} : (X, Y) \rightarrow \nabla D(X, Y)$ is tracked by the identity for every assembly (X, Y) .

Lemma 17. ∇ is regular.

Proof. In regular categories $e : X \rightarrow Y$ is a regular epimorphism if and only if $\exists_e(X) \simeq e$. So let $e \in \mathcal{E}$ be a regular epimorphism. Images lift to $\text{Asm}(A', A)$, and

$$\exists_{\nabla e}(\nabla X) = (Y, \exists_{1_A \times e}(A \times X)) \simeq \nabla Y$$

Therefore ∇ preserves regular epimorphisms. Since ∇ is right adjoint to D , it also preserves all limits. That makes it a regular functor. \square

Remark 18. By the way, a regular functor between Heyting categories preserves OPCAs, because the functor inflates the object of realizers of each partial representable arrow. Let (A', A) be an OPCA pair in \mathcal{E} and let $F : \mathcal{E} \rightarrow \mathcal{F}$ be a regular functor into a Heyting category. For each representable $f : U \subseteq A^n \rightarrow A$, let

$$R(f) = \{ (a, \vec{x}) \in A \times U \mid \exists y \leq f(\vec{x}). \alpha_n(a, \vec{x}) \downarrow y \}$$

If $p : A \times U \rightarrow A$ is the first projection, then by definition $\llbracket f \rrbracket = \forall_p(R(f))$ and that implies $p^{-1}(\llbracket f \rrbracket) \subseteq R(f)$. Just because F preserves finite limits, it preserves OPASes and partial combinatory arrows. For that reason the definition of R makes sense relative to FA . But F also preserves regular epimorphisms and this implies $FR(f) = R(Ff)$. From $p^{-1}(\llbracket f \rrbracket) \subseteq R(f)$ we now deduce $F\llbracket f \rrbracket \subseteq \llbracket Ff \rrbracket$. If f is any partial combinatory arrow, then $\llbracket f \rrbracket$ intersects A' . Therefore $\llbracket Ff \rrbracket$ intersects FA' . So for every partial combinatory arrow $f : FA^n \rightarrow FA$, $\llbracket f \rrbracket$ intersects FA' and this makes (FA', FA) an OPCA pair.

We have a category and we have a functor. Now we need a filter, which is some subobject of ∇A .

Lemma 19. *Let $\{\leq\}$ be $\{(x, y) \in A^2 \mid x \leq y\}$ and let $\mathring{A} = (A, \{\leq\})$. The identity map $1_A : \mathring{A} \rightarrow \nabla A$ is a monomorphism that represent a filter on \mathring{A} .*

Proof. That 1_A is a morphism follows from the fact that $1_A \times 1_A : \{\leq\} \rightarrow A^2$ is just the inclusion. If $f, g : (X, Y) \rightarrow \mathring{A}$ satisfy $1_A \circ f = 1_A \circ g$ then $f = g$, so 1_A is a monomorphism, and monomorphisms represent subobjects.

Because ∇ is regular $\nabla \{\leq\}$ is a partial ordering of ∇A . Relative to this ordering \mathring{A} is an upward closed subobject. The order $\{\leq\}$ has two projections $\{\leq\} \rightarrow A$. By pulling \mathring{A} back along the first projection we get the object of pairs of element of ∇A , where the first is some element \mathring{A} and the second is a greater element of ∇A . The second projection of this pullback to \mathring{A} is tracked by identity. This shows \mathring{A} is upward closed under the ordering $\nabla \{\leq\}$.

If U is a subobject of A that intersects A' , then \mathring{A} intersects ∇U . This means the the support of the pullback of the inclusions of \mathring{A} and ∇U is a terminal object. Using the constructions in the proof of lemma 15 we find this support can be represented by $(\top, \downarrow U)$.

Let $k(x, y) = x$. The unique arrow 1_\top is a morphism $(\top, A) \rightarrow (\top, \downarrow U)$, because U intersects A' and $k \times 1(a, b, c) = (a, c) \in U \times \top$ for all $(a, b, c) \in U \times A \times \top$. Therefore $\mathring{A} \wedge \nabla U$ is globally supported and \mathring{A} intersects ∇U if A' intersects U .

Let $D \subseteq A^2$ be the domain of the application operator. We intersect $\mathring{A} \times \mathring{A}$ with ∇D by pulling back along the inclusion $1_D : D \rightarrow A^2$. To get a simpler representation, we project down along the inclusion of $(\mathring{A} \times \mathring{A}) \cap \nabla D$. This way $\mathring{A}^2 \cap \nabla D \simeq (D, E)$, where

$$E = \{ (a, b, c) \in A \times D \mid \forall t \in \top, f \in \text{F}. at = b, af = c \}$$

Let $g(t, f, x) = (xt)(xf)$. The application operator $\alpha_1 : D \rightarrow A$ is a morphism $(D, E) \rightarrow \mathring{A}$ because $g \times \alpha_1(t, f, a, b, c) = ((xt)(xf), bc)$ and $(xt)(xf) \leq bc$ for all $(t, f, a, b, c) \in \top \times \text{F} \times E$. This means in the internal language of $\text{Asm}(A', A)$ that if $x, y \in \mathring{A}$ and $xy \downarrow$, then $xy \in \mathring{A}$.

The assembly \mathring{A} is a filter because it is downward closed, it intersects ∇U when A' intersects U and it is closed under application. \square

We have a category $\mathbf{Asm}(A', A)$, a regular functor $\nabla : \mathcal{E} \rightarrow \mathbf{Asm}(A', A)$ and a ∇ -filter $\mathring{A} \leq \nabla A$, so we have a regular model for (A', A) . If this model is pseudoinitial, every object and morphism is generated by the common structure of all regular models: the base category, images and preimages and the filter. We show this in the next couple of lemmas.

Lemma 20. *For each assembly (X, Y) let $a : Y \rightarrow A$ be the first projection and $x : Y \rightarrow X$ be the second projection.*

$$(X, Y) \simeq \exists_{\nabla x}((\nabla a)^{-1}(\mathring{A}))$$

This can be computed using the constructions for pullbacks and images given in the proof of lemma 15.

A more traditional way to state this lemma is as follows.

Definition 21. An assembly (X, Y) is *partitioned* if there is an arrow $f : X \rightarrow A$ in \mathcal{E} such that

$$(X, Y) \simeq (\nabla f)^{-1}(\mathring{A})$$

Lemma 22. *Every assembly is covered by a partitioned assembly.*

We will sometimes refer to regular epimorphisms from partitioned assemblies to other assemblies as *partitioned covers*.

Remark 23. While partitioned assemblies are projective objects in realizability categories over the category of sets and other categories that split epimorphisms, this does not generalize to all toposes, let alone all Heyting categories.

The class of morphisms is also generated by the structure of regular models. The proof of the following lemma reveals how our definition of morphism works.

Lemma 24. *$f : (X, Y) \rightarrow (X', Y')$ is the unique factorisation of ∇Df composed with $1_{D(X, Y)} : (X, Y) \rightarrow \nabla D(X, Y)$ through $1_{D(X', Y')} : (X', Y') \rightarrow \nabla D(X, Y)$.*

Proof. Let $(U \subseteq A^n, g)$ track $f : (X, Y) \rightarrow (X', Y')$. According to the definition of morphisms the following diagram commutes and the vertical arrows are regular epimorphisms.

$$\begin{array}{ccc} U \times Y & & \\ \downarrow (u, y) \mapsto y & \searrow g \times f & \\ Y & & Y' \\ \downarrow (a, x) \mapsto x & & \downarrow (a, x) \mapsto x \\ D(X, Y) & \xrightarrow{f} & D(X', Y') \end{array}$$

We will use the internal language here to define some pullbacks. Let

$$\begin{aligned} P &= \left\{ (u, (a, x)) \in \nabla(U \times Y) \mid (u, a) \in \mathring{A}^{n+1} \right\} \\ P' &= \left\{ (a, x) \in Y' \mid a \in \mathring{A} \right\} \end{aligned}$$

The partitioned assembly P covers (X, Y) . The assemblies $\exists_{\nabla((u, (a, x)) \mapsto x)}(P)$ and $\exists_{(a, x) \mapsto x}(Y)$ are the same subobject of ∇X , because ∇U intersects \mathring{A}^n . The restriction of $g \times f$ to P lands in P' , because g is combinatory and \mathring{A} is closed under application. And $(X', Y') = \exists_{\nabla((a, x) \mapsto x)}$ according to lemma 20.

Consider the following diagram.

$$\begin{array}{ccc}
P & \xrightarrow{g \times f} & P' \\
\downarrow (u, (a, x)) \mapsto x & & \downarrow (a, x) \mapsto x \\
(X, Y) & \xrightarrow{f} & (X', Y') \\
\downarrow 1_{D(X, Y)} & & \downarrow 1_{D(X', Y')} \\
\nabla D(X, Y) & \xrightarrow{\nabla D f} & \nabla D(X, Y)
\end{array}$$

We just proved that the outer square commutes and the lower square commutes by the definition of morphism. The upper square commutes because $1_{D(X', Y')}$ is monic.

Conclusion: each morphism of assemblies $f : (X, Y) \rightarrow (X', Y')$ equals the unique factorisation of $\nabla D f \circ 1_{D(X, Y)}$ over $1_{D(X', Y')}$. \square

With these lemmas in hand, we can prove that $\text{Asm}(A', A)$ is a pseudoinitial regular model.

2.4 Existence Theorem

We this section we will show that (∇, \mathring{A}) is a pseudoinitial regular model. Thus we prove theorem 9: *there is a pseudoinitial model for every OPCA pair in every Heyting category.*

Proof. Given a regular model (F, C) for an OPCA pair (A', A) , we choose an object map F_C . For each assembly (X, Y) , let $a : Y \rightarrow A$ and $x : Y \rightarrow X$ be the projections. Let $F_C(X, Y)$ be isomorphic to $\exists_{F_x}(F a^{-1}(C))$. By definition $U(X, Y) = \exists_x(Y)$, therefore $FU(X, Y) = \exists_{F_x}(Y)$ and $F_C(X, Y)$ is a subobject of $FU(X, Y)$.

While the object map requires a strong form of choice or a small category \mathcal{E} , once we have this map, there is a unique way to extend it to a functor, thanks to lemma 24. If $(U \subseteq A^n, g)$ tracks $f : (X, Y) \rightarrow (X', Y')$, then the following square commutes, and the vertical arrows are epic because F is a regular functor.

$$\begin{array}{ccc}
F(U \times Y) & \xrightarrow{F(g \times f)} & F Y' \\
\downarrow (u, (a, x)) \mapsto x & & \downarrow (a, x) \mapsto x \\
F D(X, Y) & \xrightarrow{F f} & F D(X', Y')
\end{array}$$

Because C is a filter, the subobject $\{ (u, (a, x)) \in F(U \times Y) \mid (u, a) \in C^{n+1} \}$ covers $F_C(X, Y)$ and the restriction of $F(g \times f)$ factors through $\{ (a, x) \in F Y' \mid a \in C \}$, the subobject of $F Y'$ that covers $F_C(X, Y)$. Therefore there is a unique factorisation through (X', Y') of $F D f$ restricted to (X, Y) . We define $F_C f$ to be that morphism.

This functor preserves images and preimages by definition and therefore is regular. Also $F_C \nabla X \simeq F X$ and $F_C \mathring{A} \simeq C$, so this regular functor is a morphism of regular models.

Every regular $G : \text{Asm}(A', A) \rightarrow \text{cod} F$ such that $G \mathring{A} \simeq C$ and $G \nabla \simeq F$ is isomorphic to F_C . The isomorphism $F_C(\mathring{A}) \simeq G \mathring{A}$ is preserved by pullbacks, so that the functors have to agree on all partitioned assemblies. The isomorphism $F_C \nabla \simeq G \nabla$, and the relation of each morphism f to $\nabla D f$ now forces the functors to agree on all assemblies.

We conclude that the functor $\nabla : \mathcal{E} \rightarrow \text{Asm}(A', A)$ and the filter $\overset{\circ}{A} \subseteq \nabla A$ together form a pseudoinitial regular model for every OPCA pair (A', A) in every Heyting category \mathcal{E} . \square

We take this result on step further to prove that certain categories of regular functors are equivalent to certain categories of subobjects.

Definition 25. For every Heyting category \mathcal{E} , OPCA pair (A', A) , regular category \mathcal{C} and regular functor $F : \mathcal{E} \rightarrow \mathcal{C}$, a *regular extension* of F is a regular functor $G : \text{Asm}(A', A) \rightarrow \mathcal{C}$ with an isomorphism $\phi : G\nabla \rightarrow F$. A morphism of regular extensions $(G, \phi) \rightarrow (H, \psi)$ is a natural transformation $\eta : G \rightarrow H$ that commutes with the isomorphisms, i.e. $\eta\nabla \circ \phi = \psi$.

Corollary 26. For a fixed regular $F : \mathcal{E} \rightarrow \mathcal{C}$ there is an equivalence of categories between the poset of F -filters ordered by inclusion and the category of regular extensions of F .

Proof. We first show how natural transformations induce inclusions of filters. Let $G, H : \text{Asm}(A', A) \rightarrow \mathcal{C}$ be regular functors, let $\eta : G \rightarrow H$ and let $\eta\Delta : G\Delta \rightarrow H\Delta$ be an isomorphism of functors. Consider the following naturality square.

$$\begin{array}{ccc} G\overset{\circ}{A} & \xrightarrow{\eta_{\overset{\circ}{A}}} & H\overset{\circ}{A} \\ 1_A \downarrow & & \downarrow 1_A \\ G\nabla A & \xrightarrow{\eta_{\nabla A}} & H\nabla A \end{array}$$

Since G and H are both regular, the vertical arrows are monic and the lower arrow is an isomorphism. Therefore $\eta_{\overset{\circ}{A}}$ must be monic too. If there are isomorphisms $\phi : G\nabla \rightarrow F$ and $\psi : H\nabla \rightarrow F$, and if $\eta\nabla$ commutes with these isomorphisms, then $\eta\nabla$ is an isomorphism. Hence $G\overset{\circ}{A} \subseteq H\overset{\circ}{A}$.

Next we construct a natural transformation from an inclusion of filters. Let $C \subseteq C'$ be F -filters. The inclusion $C \subseteq C'$ is preserved by pullback and since partitioned assemblies are pullback, we can define $\eta_P : F_C P \rightarrow F_{C'} P$ to be this pulled back inclusion. Each assembly X has a partitioned cover $e : P \rightarrow X$, which we use to construct this diagram.

$$\begin{array}{ccc} F_C P & \xrightarrow{\eta_P} & F_{C'} P \\ F_C e \downarrow & & \downarrow F_{C'} e \\ F_C(X, Y) & & F_{C'}(X, Y) \\ F_C 1_{D(X, Y)} \downarrow & & \downarrow F_{C'} 1_{D(X, Y)} \\ F_C \nabla D(X, Y) & \xrightarrow{\sim} & F_{C'} \nabla D(X, Y) \end{array}$$

There is a unique arrow $F_C(X, Y) \rightarrow F_{C'}(X, Y)$ that commutes with all the arrows in the diagram, and we define $\eta_{X, Y}$ to equal this arrow. Thus we get a natural transformation η for which $\eta\nabla$ is an isomorphism.

The natural transformation η we constructed in the last paragraph induces the inclusion $C \subseteq C'$. Also, the diagram above shows that any transformation that induces this inclusion must equal η . Therefore, there is an equivalence of categories between the poset of F -filters and the regular extensions of F . \square

2.5 Projective Terminals

The construction of the category of assemblies can be simplified if the terminal object of the underlying category is projective. This is the case in the category of sets for example. Since each globally supported object has a section, each representable arrow of each OPAS and therefore each combinatory function of each OPCA is representable by a global section.

Definition 27. Let A be an OPAS and let $\{\leq\} \subseteq A^2$ be its ordering. For any pair of arrows $f, g : X \rightarrow A$ let $f \leq g$ if the pair $(f, g) : X \rightarrow A^2$ factors through $\{\leq\}$. Let α_n be as in the proof of lemma 13. An arrow $f : A^n \rightarrow A$ is *globally representable* if there is an $x : \top \rightarrow A$ such that $\exists_x(\top) \times U \subseteq \alpha_{n+1}$ and $\alpha_{n+1}(x \times 1_U) \leq f$.

Lemma 28. *In every Heyting category \mathcal{E} every globally representable morphism is representable. If the terminal object is projective, any representable morphism is globally representable.*

Proof. Any arrow $f : U \subseteq A^n \rightarrow A$ is representable if the following object is globally supported.

$$\llbracket f \rrbracket = \{ a \in A \mid \forall \vec{x} \in U. \exists y \in A. y \leq f(\vec{x}) \wedge ((ax_1) \dots) x_n \downarrow y \}$$

If f is globally representable, then $\llbracket f \rrbracket$ has a global section. This makes $\llbracket f \rrbracket$ globally supported and therefore f representable. If g is representable and the terminal object is projective, then $\llbracket f \rrbracket$ has a global section, and this section globally represents f . \square

We can use global representability to construct categories of assemblies for certain pairs of ordered partial applicative structures in categories that have finite limits, but are not necessarily regular or Heyting. In the following lemma we formulate one property global representability that lets us do this.

Lemma 29. *For any finite limit category \mathcal{C} and an ordered partial applicative structure $A \in \mathcal{C}$ let $\Gamma : \mathcal{C} \rightarrow \mathbf{Set}$ be the global sections functor. A partial arrow $f : A^n \rightarrow A$ is globally representable in A if and only if Γf is representable in ΓA .*

Proof. $U \subseteq \Gamma A$ is globally supported precisely when A has a global section, and this makes the definition equivalent. \square

One possible definition of a category of assemblies for a global OPCA pair is now the following.

Definition 30. For any finite limit category \mathcal{C} let $\Gamma : \mathcal{C} \rightarrow \mathbf{Set}$ be the global sections functor. A pair of OPASes $A' \subseteq A$ in \mathcal{C} is a *global OPCA pair*, if $(\Gamma A, \Gamma A')$ is an OPCA pair. The category of assemblies for the global OPCA pair (A', A) is the fibred product of $U : \mathbf{Asm}(\Gamma A', \Gamma A) \rightarrow \mathbf{Set}$ and $\Gamma : \mathcal{C} \rightarrow \mathbf{Set}$.

We return to our own definition of a category of assemblies over arbitrary Heyting categories. Assuming a projective terminal object, we can simplify the definition of a morphism of assemblies.

Lemma 31. *For each morphism $f : (X, Y) \rightarrow (X', Y')$ there is a global combinator $r : \top \rightarrow A$ such that $(ra, f(x)) \in Y'$ for all $(a, x) \in Y$. For every $f' : X \rightarrow X'$ and every pair of assemblies (X, Y) and (X', Y') and each global section $r' : \top \rightarrow A'$, if $r'a \downarrow$ and $(r'a, x) \in Y'$ for all $(a, x) \in Y$, then f' is a morphism.*

Proof. For any tracking (U, g) of f there are global sections $\vec{u} : 1 \rightarrow U \cap (A')^n$ and $y : 1 \rightarrow \llbracket g \rrbracket \cap A'$. Let $r = ((yu_1) \dots)u_n \in A'$ then there is a $z \leq (((yu_1) \dots)u_n)a$ such that $ra \downarrow z$. Since $((yu_1) \dots)u_n a \leq g(\vec{u}, a)$, if $(a, x) \in Y$ then $(g(\vec{u}, a), f(x)) \in Y'$ and therefore $(ra, f(x)) \in Y'$. So for each morphisms $f : (X, Y) \rightarrow (X', Y')$ there is an $r : A \rightarrow A$ that satisfies the requirements.

Let $U = \{r'\}$ and let $g(x, y) = xy$. The pair (U, g) tracks $f' : (X, Y) \rightarrow (X', Y')$, so f' is a morphism. \square

This is the traditional definition of morphism of assemblies (see for example [19]). So known categories of assemblies are special cases of our construction. We now have shown that the category of assemblies is a pseudointitial regular model of an OPCA pair. In the next section we will show a similar definition of realizability toposes.

3 Relative Realizability Toposes

In this section we assume that the underlying category \mathcal{E} is a topos. Under that condition, we can construct a topos out of the category of assemblies.

Definition 32. For every topos \mathcal{E} and every OPCA pair (A', A) in \mathcal{E} an *exact model* is a regular functor F from \mathcal{E} to an exact category \mathcal{C} , together with an F -filter. A *relative realizability topos* $\text{RT}(A', A)$ is a pseudointitial exact model.

Theorem 33. *Relative realizability toposes exist for every OPCA pair in every topos.*

Proof. We start with a constructing that turns regular categories into exact ones. The 2-category of exact categories is a reflective subcategory of the 2-category of regular categories [4]. This means that for every regular category \mathcal{C} there is an exact category $\mathcal{C}_{ex/reg}$ and a regular functor $I : \mathcal{C} \rightarrow \mathcal{C}_{ex/reg}$ such that every regular functor from \mathcal{C} to an exact category \mathcal{D} factors up to isomorphism through I . Categories with this property of $\mathcal{C}_{ex/reg}$ are called *exact completions* of \mathcal{C} .

Let \mathcal{E} be a topos, \mathcal{D} an exact category and $F : \mathcal{E} \rightarrow \mathcal{D}$ a regular functor. If (F, C) is an exact model, then there is an up to isomorphism unique regular functor $F_C : \text{Asm}(A', A) \rightarrow \mathcal{D}$ such that $F_C \nabla \simeq F$ and $F_C \dot{A} \simeq C$, because exact models are regular models. F_C factors up to isomorphism through exact completions of $\text{Asm}(A', A)$ because its codomain is exact. The regular functor $I : \text{Asm}(A', A) \rightarrow \text{Asm}(A', A)_{ex/reg}$ creates an exact model $(I\nabla, I\dot{A})$, and we see now that it is pseudointitial. \square

We give a construction for an exact completion of $\text{Asm}(A', A)$ in section 3.3. Before that we want to prove that relative realizability toposes are indeed toposes. We will use a result from Matthias Menni's thesis [14] for this: if a regular category is locally Cartesian closed and has a generic mono, then its exact completion is a topos.

3.1 Local Cartesian Closure

Local Cartesian closure means Cartesian closure of all slice categories. We prove that the category of assemblies is locally Cartesian closed in two steps. Firstly we prove that if a Heyting category has a Cartesian closed reflective subcategory, then it is Cartesian closed under some conditions on the reflector. Secondly we prove that for each assembly (X, Y) , the slice category $\mathcal{E}/D(X, Y)$ is a reflective subcategory of $\text{Asm}(A', A)/(X, Y)$. For each $Z \in \mathcal{E}$ the slice \mathcal{E}/Z is Cartesian closed because \mathcal{E} is a topos, therefore $\text{Asm}(A', A)$ is locally Cartesian closed.

Lemma 34. *Let \mathcal{E} be a Heyting category, let \mathcal{D} be a Cartesian closed full subcategory and let $L : \mathcal{E} \rightarrow \mathcal{D}$ be a finite limit preserving left adjoint to the inclusion of \mathcal{D} into \mathcal{E} , such that the unit $\eta : L \rightarrow 1$ is a natural monomorphism. Then \mathcal{E} is Cartesian closed.*

Proof. For simplicity, we will use the validity of first order logic and simply typed λ -calculus in the internal languages of respectively Heyting and Cartesian closed categories.

We define for all $Y, Z \in \mathcal{E}$

$$Z^Y = \{ f \in LZ^{LY} \mid \forall y \in Y. \exists z \in Z. f(\eta_Y y) = \eta_Z z \}$$

For all $f : X \rightarrow Y^Z$, $x \in X$ and $y \in Y$, there exists a $z \in Z$ such that $f(x)(\eta_Y y) = \eta_Z z$ and because η_Z is a monomorphism, this z is unique. So let $f^t(x, y) = z$ if $f(x)(\eta_Y y) = \eta_Z z$ for all $x \in X$, $y \in Y$ and $z \in Z$.

For all $g : X \times Y \rightarrow Z$, $x \in X$ and $y \in Y$ we have $\eta_Z \circ g(x, y) = Lg(\eta_X x, \eta_Y y)$. Note that we use $L(X \times Y) \simeq LX \times LY$ by the way. Because the subcategory is Cartesian closed, we can let $g^t(x) = \lambda y. Lg(\eta_X x, \eta_Y y)$.

For each $f : X \rightarrow Y^Z$, $x \in X$ and $y \in Y$ we have $(f^t)^t(x)(\eta_Y y) = \eta_Z z$ if and only if $f(x)(\eta_Y y) = \eta_Z z$. Therefore $(f^t)^t = f$. For each $g : X \times Y \rightarrow Z$, $x \in X$ and $y \in Y$ we have $(g^t)^t(x, y) = z$ if and only if $g^t(x)(\eta_Y y) = \eta_Z z$ while $g^t(x)(\eta_Y y) = \eta_Z \circ g(x, y)$. Since η_Z is mono we have $(g^t)^t = g$. This means that $Z \mapsto Z^Y$ is right adjoint to $X \mapsto X \times Y$ and that \mathcal{E} is Cartesian closed. \square

Lemma 35. *For each $(X, Y) \in \text{Asm}(A', A)$, there is a full and faithful functor $\mathcal{E}/\text{D}(X, Y) \rightarrow \text{Asm}(A', A)/(X, Y)$ with finite limit preserving left adjoint.*

Proof. ∇ is right adjoint to D and the unit of this adjunction $(X, Y) \rightarrow \nabla \text{D}(X, Y)$ is a monomorphism. For each $(X, Y) \in \text{Asm}(A', A)$, we let $\nabla_{(X, Y)} : \mathcal{E}/\text{D}(X, Y) \rightarrow \text{Asm}(A', A)/(X, Y)$ be the functor that maps $f : Z \rightarrow \text{D}(X, Y)$ to $(\nabla f)^{-1}(Y)$: the subobject of $\nabla \text{D}(X, Y)$ represented by Y . This functor is faithful and D acts as reflector $\text{Asm}(A', A)/(X, Y) \rightarrow \mathcal{E}/\text{D}(X, Y)$ that preserves finite limits, and the unit is still a monomorphism. \square

Theorem 36. *For each locally Cartesian closed Heyting category \mathcal{E} and a OPCA pair (A', A) in \mathcal{E} , the category of assemblies is a locally Cartesian closed Heyting category.*

Proof. Lemma 15 tells us $\text{Asm}(A', A)$ is Heyting. For each assembly (X, Y) lemma 35 embeds the Cartesian closed Heyting category $\mathcal{E}/\text{D}(X, Y)$ into the Heyting category $\text{Asm}(A', A)/(X, Y)$ in such way that the inclusion has a finite limit preserving left adjoint. Therefore every slice of $\text{Asm}(A', A)$ is Cartesian closed according to lemma 34, and that means $\text{Asm}(A', A)$ is a locally Cartesian closed Heyting category. \square

3.2 Generic Monomorphisms

We construct a generic monomorphism for the category of assemblies.

Lemma 37. *Let \mathcal{E} be a topos and (A', A) an OPCA pair in \mathcal{E} . Let $DA \leq \Omega^A$ be the object of downward closed subobjects of A , and let $\{\in\}$ be the element-of relation $\{(a, U) \in A \times DA \mid a \in U\}$. The inclusion $1_{\text{D}(DA, \{\in\})} : (DA, \{\in\}) \rightarrow \nabla DA$ is a generic monomorphism.*

Proof. If $m : X \rightarrow Y$ is monic, then $\exists_m(X) \simeq X$. Therefore we can focus on monomorphisms of the form $1_{\text{D}(X, Y)} : (X, Y) \rightarrow (X, Y')$.

To $Y \leq A \times X$ belongs a characteristic map $y : X \rightarrow DA \leq \Omega^A$: $y(x) = \{ a \in A \mid (a, x) \in Y \}$, which by the definition of assemblies is a downward closed set. If we pull back $(DA, \{\in\})$ along y using the constructions from lemma 15, we get the assembly $(X, Y \wedge Y')$, where

$$Y \wedge Y' = \{ (a, x) \in A \times X \mid \forall t \in \mathbb{T}. at \downarrow \wedge (at, x) \in Y, \forall f \in \mathbb{F}. af \downarrow \wedge (af, x) \in Y' \}$$

Since $Y \leq Y'$ we have $Y \wedge Y' \simeq Y$. \square

Theorem 38. *Let \mathcal{E} be a topos and (A', A) an OPCA pair in \mathcal{E} . The relative realizability topos $\text{RT}(A', A) = \text{Asm}(A', A)_{ex/reg}$ is a topos.*

Proof. The category of assemblies is locally Cartesian closed and has a generic monomorphism. This implies that its exact completion is a topos, according to Matias Menni [14]. \square

Remark 39. Given any assembly (X, Y) let $a : Y \rightarrow A$ and $x : Y \rightarrow X$ be the projections. Let $\{\in\} = \{ (a, \xi) \in A \times DA \mid a \in \xi \}$, and let $b : \{\in\} \rightarrow A$ and $d : \{\in\} \rightarrow DA$ be the projections. There is a $y : X \rightarrow DA$ such that the square in the following commutative diagram is a pullback:

$$\begin{array}{ccc} Y & \xrightarrow{\quad a \quad} & \{\in\} & \xrightarrow{\quad b \quad} & A \\ \downarrow x & \xrightarrow{(a,y)} & \downarrow d & & \\ X & \xrightarrow{\quad y \quad} & DA & & \end{array}$$

Because ∇ and $\text{Asm}(A', A)$ are regular and because of lemma 20, this means:

$$(X, Y) \simeq \nabla y^{-1} \exists_{\nabla d} \nabla b^{-1} (\mathring{A})$$

So the generic monomorphism is the inclusion of $\exists_{\nabla d} \nabla b^{-1} (\mathring{A})$ into ∇DA . Note the regular epimorphism $d : \nabla b^{-1} (\mathring{A}) \rightarrow \exists_{\nabla d} \nabla b^{-1} (\mathring{A})$. It is a generic partitioned cover. If \mathcal{C} is regular, $F : \text{Asm}(A', A) \rightarrow \mathcal{C}$ preserves finite limits, $F\nabla$ is regular and Fd is a regular epimorphism, then F is a regular functor.

3.3 Exact Completions

In this section we recall the construction of the exact completion of a regular category and derive a concrete description of the relative realizability topos from it.

Definition 40. Given a regular category \mathcal{C} let a *subquotient* be a pair (X, E) where $X \in \mathcal{C}$, $E \subseteq X^2$ and E satisfies:

$$(x, y) \in E \rightarrow (y, x) \in E \quad (x, y), (y, z) \in E \rightarrow (x, z) \in E$$

Given any two subquotients (X, E) and (X', E') and two subobjects $F, G \subseteq X \times X'$ let $F \simeq_{E \rightarrow E'} G$ if both

$$\begin{aligned} (x, y) \in E &\rightarrow \exists z \in X'. (z, z) \in E' \wedge (x, z) \in F \wedge (y, z) \in G, \\ (x, x) \in E \wedge (x, y) \in F \wedge (x, z) \in G &\rightarrow (y, z) \in E' \end{aligned}$$

If $F \subseteq X \times X'$ satisfies $F \simeq_{E \rightarrow E'}$, then it is called a *functional relation*. A *morphism of subquotients* $(X, E) \rightarrow (X', E')$ is an equivalence class for $\simeq_{E \rightarrow E'}$.

We explain how this definition works. For every subquotient (X, E) , the relation E is symmetric and transitive in the internal language of \mathcal{C} . It defines an equivalence relation on $\{x \in X \mid (x, x) \in E\}$. We use this pair to represent that quotient. The relations $\simeq_{E \rightarrow E'}$ are symmetric and transitive relation on the poset of subobjects of $X \times X'$. This defines an equivalence relation of a subset too, but this relation is external to \mathcal{C} . If $F \subseteq X \times X'$ and $F \simeq_{E, E'} F$, then F induces a function from equivalence classes of E to equivalence classes for E' . Therefore F represents a morphism between quotients. If $G \subseteq X \times X'$, $G \simeq_{E, E'} G$ and $G \simeq_{E, E'} F$, then G induces the same function as F . That is why we define morphism $(X, E) \rightarrow (X', E')$ to be equivalence classes of $\sim_{E \rightarrow E'}$.

Lemma 41. *Subquotients and morphisms together form a category that is an exact completion.*

Proof. We compose relations $F \subseteq X \times Y$ and $G \subseteq Y \times Z$ by letting $G \circ F = \{(x, z) \in X \times Z \mid \exists y \in Y. (y, z) \in G, (x, y) \in F\}$. If $F \simeq F'$ and $G \simeq G'$ relative to some subquotients, then $F \circ G \simeq F' \circ G'$. For every subquotient (X, E) we have $E \simeq_{E, E} E$ and its equivalence class is an identity morphism.

\mathcal{C} is embedded by mapping each object X to the pair (X, Δ_X) , where Δ_X is the diagonal, and each arrow $f : X \rightarrow Y$ to the equivalence class of its graph.

Finally, if $F : \mathcal{C} \rightarrow \mathcal{D}$ is a regular functor to an exact category, and (X, E) is a subquotient, then FE is an equivalence relation on a subobject of X . The subquotient FX/FE exists here because of exactness, and that is where we map (X, E) too. If $G \simeq G'$ between (X, E) and (X', E') , then composition with FG induces a map $FX/FE \rightarrow FX'/FE'$. If $G \simeq G'$ then FG and FG' induce the same map. Thus F can be factored through the category of subquotients in an up to isomorphism unique way. \square

The inclusion $1_{\mathbb{D}(X, Y)} : (X, Y) \rightarrow \nabla \mathbb{D}(X, Y)$ is a monomorphism in $\mathbf{Asm}(A', A)$. That means every assembly is a subobject of an object in the image of ∇ . In turn every subquotient is a subquotient of an object in the image of ∇ . If $m : (Y, E) \rightarrow \nabla X$ is a monomorphism that represents such a relation, then so does the isomorphic assembly $\exists_m(Y, E) \simeq (X^2, \exists_{1_A \times m}(E))$. Therefore every object of the relative realizability topos can be represented by just one assembly (X^2, E) that defines a subquotient of ∇X . We use these facts to get a simpler construction of relative realizability toposes.

Definition 42. Let \mathcal{E} be a topos and let (A', A) be an OPCA pair. The standard relative realizability topos is defined as follows. The objects are pairs $(X, E \subseteq A \times X^2)$ such the the assembly (X^2, E) is a symmetric and transitive relation on ∇X . A morphism $(X, E) \rightarrow (X', E)$ is an isomorphism class of assemblies $(X \times X', Y)$, where Y is a functional relation.

For each OPCA pair, the category of assemblies is the pseudoinitial regular model and the relative realizability topos is the pseudoinitial exact model. That is the main point of our paper. In the next section we explain some consequences of our definitions.

4 Functors

In this section, we use initial models to find examples of regular functors from relative realizability categories into other categories. We no longer demand the the underlying category is a topos. However, when the underlying category is a topos many of the functors we construct have right adjoints and therefore are inverse images parts of geometric morphisms. For completeness we will also proof the existence of these right adjoints.

4.1 Points

For every Heyting category \mathcal{E} the identity functor $1_{\mathcal{E}} : \mathcal{E} \rightarrow \mathcal{E}$ is regular. If (A', A) is an OPCA pair in \mathcal{E} , we can construct regular functors with filters of A . A $1_{\mathcal{E}}$ -filter is a subobject C of A that satisfies:

- For all $x \in C$ and $y \in A$ if $y \geq x$ then $y \in C$.
- For all $x, y \in C$ and $z \in A$ if $xy \downarrow z$ then $z \in C$.
- For all $U \subseteq A$ if U intersects A' then U intersects C .

Remark 43. This last condition must be interpreted externally, not in the internal language of \mathcal{E} . An internal interpretation is possible if \mathcal{E} is a topos, but then the condition implies $A' \subseteq C$.

For each filter $C \subseteq A$, a regular functors induced by C can be constructed as follows. For each assembly (X, Y) we let

$$D_C(X, Y) = \{ x \in X \mid \exists c \in C. (c, x) \in Y \}$$

The functor then maps $f : (X, Y) \rightarrow (X', Y')$ to Df restricted to $D_C(X, Y)$ factored through $D_X(X', Y')$.

The reason to refer to $1_{\mathcal{E}}$ -filters as *points* of $\text{Asm}(A', A)$ is explained by the following lemma.

Theorem 44. *Let \mathcal{E} be a topos, let (A', A) be an OPCA of \mathcal{E} and let C be a $1_{\mathcal{E}}$ -filter. Then the induced regular functor $D_C : \text{RT}(A', A) \rightarrow \mathcal{E}$ has a right adjoint.*

Proof. We use the construction of the relative realizability topos from subsection 3.3 to get a clear picture of $D_C : \text{RT}(A', A) \rightarrow \mathcal{E}$. As D_C must preserve subquotients, we can construct the functor as follows. For each subquotient (X, E) , each $F : (X, E) \rightarrow (X', E')$ and each $\xi \in C(X, E)$ we let

$$\begin{aligned} D_C(X, E) &= \{ \xi \in \Omega^X \mid \exists x \in X. \xi = \{ y \in X \mid \exists a \in C. (a, x, y) \in E \} \} \\ D_C F(\xi) &= \{ y \in X' \mid \exists a \in C. (a, x, y) \in F \} \end{aligned}$$

We now construct a functor $\nabla_C : \mathcal{E} \rightarrow \text{RT}(A', A)$. For each $X \in \mathcal{E}$, let

$$E_X = \{ (a, f, g) \in A \times (\Omega^X)^2 \mid a \in C \rightarrow \exists x \in X. f = g = \{ x \} \}$$

The assembly (X^2, E_X) is a partial equivalence relation on ∇X . For any arrow $f : X \rightarrow Y$ the morphism ∇f commutes with the partial equivalence relation of either side. Therefore we get a functor ∇_C by mapping each X to (X, E_X) and each $f : X \rightarrow Y$ to the morphism of subquotients it induces.

By computation we find that $D_C \nabla_C X$ is isomorphic to X for all $X \in \mathcal{E}$.

$$CR_C X = \{ \xi \in \Omega^{\Omega^X} \mid \exists x \in X. \xi = \{ \{ x \} \} \}$$

Let $e_X : D_C \nabla_C X \rightarrow X$ be the inverse of $x \mapsto \{ \{ x \} \}$.

For each $(X, E) \in \text{RT}(A', A)$ define $f_{(X, E)} : X \rightarrow \Omega^{D_C(X, E)}$ by

$$f(x) = \{ \{ y \in X \mid \exists a \in C. (a, x, y) \in E \} \}$$

If $(a, x, y) \in E$ and $a \in C$, then there is an $z \in D_C(X, E)$ such that $f(x) = f(y) = \{ z \}$, namely $z = \{ y \in X \mid \exists a \in C. (a, x, y) \in E \}$. So $(a, f(x), f(y)) \in E_{D_C(X, E)}$, and therefore f is a morphism of the partial equivalence relations. Hence $f_{(X, E)} : (X, E) \rightarrow \nabla_C D_C(X, E)$.

For $\xi \in D_C(X, E)$ we have $D_C f_{X, E}(\xi) = \{ f(x) \mid x \in \xi \} = \{ \{ \xi \} \}$. Therefore $e_{D_C(X, E)} \circ D_C f_{(X, E)} = 1_C$. For $g \in \Omega^X$ we have $f_{\nabla_C X}(g) = \{ \{ g \} \}$. Therefore $\nabla_C e \circ f_{\nabla_C X} = 1_{\nabla_C}$. Hence we have an adjunction $D_C \dashv \nabla_C$. \square

Points ordinarily are geometric morphisms from \mathbf{Set} to other toposes. Here we are dealing with geometric morphisms from \mathcal{E} to $\mathbf{RT}(A', A)$. So $1_{\mathcal{E}}$ -filters represent \mathcal{E} -points rather than genuine points of relative realizability toposes.

Apparently relative realizability toposes can have many points, just like Grothendieck toposes. We explore the connection between these two kinds of toposes in the following subsection.

4.2 Characters

If P is a preordered set a topos \mathcal{E} , we can form a topos of internal presheaves $\mathcal{E}^{P^{op}}$ over it. Each internal presheaf is an arrow $p : X \rightarrow P$ in \mathcal{E} , plus a restriction operator $r : \{ (x, u) \in P \times X \mid p \leq bx \} \rightarrow X$ that satisfies $p \circ r(x, u) = u$. Each morphism $f : (p, r) \rightarrow (p', r')$ is just an arrow $f : X \rightarrow X'$ such that $p' \circ f = p$ and $r' \circ f = f \circ r$.

The constant sheaf functor $\Delta : \mathcal{E} \rightarrow \mathcal{E}^{P^{op}}$ has both adjoints and is therefore a regular functor. Let DP be the object of downsets of P . The Δ -filters of (A', A) correspond to arrows $A \rightarrow DP$.

Definition 45. Let P be a preordered and (A', A) and OPCA pair in \mathcal{E} . A character γ is an arrow $A \rightarrow DP$ that satisfies:

- If $x \leq y$ then $\gamma(x) \leq \gamma(y)$.
- If $xy \downarrow z$ then $\gamma(x) \cap \gamma(y) \leq \gamma(z)$.
- If $a \in A'$ then $\gamma(a) = P$.

We derive the next corollary from theorem 9.

Corollary 46. *Characters correspond to regular functors $\mathbf{RT}(A', A) \rightarrow \mathcal{E}^{P^{op}}$.*

Proof. There is a bijection between $\mathcal{E}(A, DP)$ and the subobjects of ΔA :

$$\mathcal{E}(A, DP) \simeq \mathcal{E}(A, \Gamma\Omega) \simeq \mathcal{E}^{DP^{op}}(\Delta A, \Omega) \simeq \mathbf{Sub}(\Delta A)$$

This bijection turns characters into Δ -filters. □

Because the functor $\gamma^* : \mathbf{RT}(A', A) \rightarrow \mathcal{E}^{(P^{op})}$ is regular, we can give an explicit definition. Let (X, E) be any object and let for all $x \in X$

$$\llbracket x \rrbracket_u = \{ y \in X \mid \exists a \in A. u \in \gamma(a), (a, x, y) \in E \}$$

Let $\gamma^*(X, E) = (X', p, r)$ with

$$X' = \{ (u, \xi) \in P \times \Omega^X \mid \exists x \in X. \xi = \llbracket x \rrbracket_u \}$$

$p : \gamma^*(X, E)$ is just the projection to the first coordinate. We let $r(\xi, u) = \bigcup_{x \in \xi} \llbracket x \rrbracket_u$; r now satisfies $r(\llbracket x \rrbracket_u, v) = \llbracket x \rrbracket_v$ for $v \leq u$. Let $f : (X, E) \rightarrow (X', E')$ be any functional relation. Let for all $(u, \xi) \in \gamma^*(X, E)$

$$\gamma_* f(u, \xi) = (u, \{ y \in X' \mid \exists a \in A, x \in \xi. u \in \gamma(a) \wedge (a, x, y) \in f \})$$

In the case that the underlying category is a topos, the functors that characters induce are not just regular, however.

Theorem 47. *Let \mathcal{E} be a topos, P a preordered set and (A', A) an OPCA pair. Characters $A \rightarrow DP$ induce geometric morphisms $\mathcal{E}^{P^{op}} \rightarrow \mathbf{RT}(A', A)$.*

Proof. For each object $(X, p, r) \in \mathcal{E}^{P^{op}}$, let

$$\begin{aligned} x, y \in X \quad e(x, y) &= \{ u \in P \mid u \leq px, u \leq py, r(x, u) = r(y, u) \} \\ \gamma_*(X, p, r) &= (X, \{ (a, x, y) \in A \times X \times Y \mid \gamma(a) \subseteq e(x, y) \}) \end{aligned}$$

For each morphism $f : (X, p, r) \rightarrow (X', p', r')$ we let:

$$\gamma_* f = \{ (a, x, y) \in A \times X \times X' \mid \gamma(a) \subseteq e(fx, y) \}$$

By writing out the definitions we find that if $\gamma^* \gamma_*(X, p, r) = (X', p', r')$ then

$$(u, \xi) \in X' \iff \exists x \in X. \xi = \{ y \in X \mid u \leq p(y), r(y, u) = r(x, u) \}$$

This new sheaf is isomorphic to (X, p, r) with the isomorphism determined by:

$$\begin{aligned} g(x) &= (px, \{ y \in X \mid u \leq p(y), r(y, u) = x \}) \\ \forall x \in \xi. \quad \epsilon_{(X, p, r)}(u, \xi) &= r(x, u) \end{aligned}$$

The second family of morphisms acts as counit.

If $\gamma_* \gamma^*(X, E) = (X', E')$, then

$$X' = \{ (u, \llbracket x \rrbracket_u) \in P \times \Omega^X \mid x \in X \}$$

with $\llbracket x \rrbracket_u$ defined as before. The partial equivalence relation can be simplified.

$$E' = \{ (a, (u, \llbracket x \rrbracket_u), (v, \llbracket y \rrbracket_v)) \mid (\forall w \in \gamma(a). w \leq u, w \leq v) \wedge (a, x, y) \in E \}$$

We define a family of functional relations $(X, E) \rightarrow \gamma_* \gamma^*(X, E)$ by

$$\eta_{(X, E)} = \{ (a, x, (u, \llbracket x \rrbracket_u)) \in A \times X \times X' \mid u \in \gamma(a) \}$$

These are all tracked by identity and together form the unit.

We conclude that $\epsilon_{\gamma^*} \circ \gamma^* \eta = 1_{\gamma^*}$ because

$$\gamma^* \eta_{(X, E)}(u, \xi) = (u, \{ (v, \llbracket x \rrbracket_v) \in X' \mid x \in \xi \}) = g(u, \xi)$$

By writing out definitions we also find that $(a, (u, \xi), y) \in \gamma_* \epsilon_{(X, p, r)}$ if for all $v \in \gamma(a)$ and $x \in \xi$, $v \leq u$ and $r(x, v) = r(y, v)$, while $(b, x, (u, \llbracket x \rrbracket_u)) \in \eta_{\gamma_*(X, E)}$ if $u \in \gamma(b)$. We have $\gamma_* \epsilon \circ \eta_{\gamma^*} = 1_{\gamma^*}$, because for any $p \in P$ we have $\gamma(pab) = \gamma(a) \cap \gamma(b)$.

So γ_* is right adjoint to γ^* . Since γ^* is regular their combination is a geometric morphism. \square

Remark 48. An internal Grothendieck topology J on a preordered object P allows us to define a topos of sheaves $\mathbf{Sh}(P, J)$. This topos is embedded in $\mathcal{E}^{P^{op}}$ by a geometric morphism. Therefore, we can relate geometric morphisms $\mathbf{Sh}(P, J) \rightarrow \mathbf{RT}(A', A)$ to characters $\gamma : A \rightarrow DP$ of which the values are J -closed sets.

Remark 49. For a trivial poset that consists of a terminal object \top we have $\mathcal{E}^{\top^{op}} \cong \mathcal{E}$ and characters are points.

Toposes of sheaves are better understood than relative realizability toposes. By inducing geometric morphisms between these two kinds of toposes, characters may clarify the theory of relative realizability.

4.3 Applicative Morphisms

In this subsection we consider regular functors between realizability categories for different OPCA pairs. The filters that induce these functors are applicative morphisms as defined by Longley [13] and Hofstra [10].

Definition 50. Let (A', A) and (B', B) be two OPCA pairs in an Heyting category \mathcal{E} . An *applicative morphism* $\gamma : (A', A) \rightarrow (B', B)$ is a B -assembly (A, C) over A , such that the following subobjects of B intersect B' .

$$\begin{aligned} & \{ u \in B \mid \forall(x, y) \in C, y' \in A.y \leq y' \rightarrow (ux \downarrow \wedge (ux, y') \in C) \} \\ & \{ r \in B \mid \forall(x', x), (y', y) \in C.xy \downarrow \rightarrow ((rx')y' \downarrow \wedge ((rx')y', xy) \in C) \} \\ \forall a \in A' & \quad \{ b \in B \mid (b, a) \in C \} \end{aligned}$$

Theorem 51. For each applicative morphism $\gamma : (A', A) \rightarrow (B', B)$ there is an up to isomorphism unique regular functor $F : \mathbf{Asm}(A', A) \rightarrow \mathbf{Asm}(B', B)$ such that $F\overset{\circ}{A} \simeq (A, C)$ and $F\nabla \simeq \nabla$. For each regular functor $F : \mathbf{Asm}(A', A) \rightarrow \mathbf{Asm}(B', B)$ such that $F\nabla \simeq \nabla$, there is an up to isomorphism unique applicative morphism $\gamma : (A', A) \rightarrow (B', B)$.

Proof. γ is a filter for $\nabla : \mathcal{E} \rightarrow \mathbf{Asm}(B', B)$, so (∇, γ) is a regular model for (A', A) . Therefore there is an up to isomorphisms unique regular functor $\mathbf{Asm}(A', A) \rightarrow \mathbf{Asm}(B', B)$ satisfying the conditions.

Any regular functor F such that $F\nabla \simeq \nabla$ will map $1_A : \overset{\circ}{A} \rightarrow \nabla A$ to some monomorphism $F\overset{\circ}{A} \rightarrow F\nabla A$. The image of $F\overset{\circ}{A}$ along the composition of $F1_A$ with the isomorphism $F\nabla A \rightarrow \nabla A$ is an applicative morphism because F preserves filters. \square

Unlike characters, applicative morphisms do not generally induce geometric morphisms if the underlying category is a topos. The ones that do have the following property.

Definition 52. For $\gamma : (A', A) \rightarrow (B', B)$ we define the arrow $\gamma : A \rightarrow DB$ by $\gamma(a) = \{ b \in B \mid b \in \gamma(a) \}$. We define the following relation on DB : $UV \downarrow W$ if and only if

$$\forall x \in U, y \in V. \exists z \in W. xy \downarrow z$$

The term UV stand for the least $W \in DB$ such that $UV \downarrow W$ and remains undefined if no such W exists. The applicative morphism γ is *computationally dense* if there is some $\mu \subseteq B$ intersecting B' such that for each $U \in DB$ that intersects B' the following subobject of A intersects A' .

$$U^\mu = \{ a \in A \mid \forall x \in A. U\gamma(x) \downarrow \rightarrow ax \downarrow \wedge \mu\gamma(ax) \downarrow U\gamma(x) \downarrow \}$$

Theorem 53. *Computationally dense applicative morphisms induce geometric morphisms between relative realizability toposes.*

Proof. We leave to the reader to check that for each relative realizability topos $\mathbf{RT}(A', A)$ over a base topos \mathcal{E} the assignment $X \rightarrow \mathbf{Sub}(\nabla -)$ is a tripos over \mathcal{E} and that an adjoint pair of transformations of triposes induces a geometric morphism [19].

For clarity, let $(\nabla_A, \overset{\circ}{A})$ be an initial exact model for (A', A) and $(\nabla_B, \overset{\circ}{B})$ for (B', B) . Since the regular functor that γ preserves ∇ and subobjects, the functor relates to a transformation of triposes $\mathbf{Sub}(\nabla_A -) \rightarrow \mathbf{Sub}(\nabla_B -)$. So we need to find a right adjoint to that transformation.

Fixing $X \in \mathcal{E}$, we may represent subobjects of $\nabla_A X$ by subobjects of $A \times X$ and subobjects of ∇_B by subobjects of $B \times X$. We can represent the transformation induced by $\gamma = (A, C)$ with the following map.

$$\gamma^* Y = \{ (b, x) \in B \times X \mid \exists a \in A. (b, a) \in C \wedge (a, x) \in Y \}$$

Now we finally start constructing a right adjoint.

$$\gamma_\mu Y = \{ (a, x) \in A \times X \mid \mu\gamma(a) \downarrow \{ b \in B \mid (b, x) \in Y \} \}$$

Automatically $(\mu, (x, y) \mapsto xy)$ tracks the inclusion $1_{D(X, \gamma^* \gamma_\mu Y)} : (X, \gamma^* \gamma_\mu Y) \rightarrow (X, Y)$. To find a tracking for the inclusion $(X, Y) \rightarrow (\gamma_\mu \gamma^* Y)$ let

$$\iota = \{ b \in B \mid \forall x \in B. \exists y \leq x. bx \downarrow y \}$$

Since the identity arrow is combinatory, the subobject ι intersects B' and ι^μ intersects A' . The tracking we need is $(\iota^\mu, (x, y) \mapsto xy)$.

To establish that γ_μ is a well defined mapping $\mathbf{Sub}(\nabla_B X) \rightarrow \mathbf{Sub}(\nabla_A X)$, let (X, Y) and (X', Y') be any pair of assemblies for (B', B) , and let

$$U = \{ b \in B \mid \forall (x, y) \in Y. bx \downarrow, (bx, y) \in Y' \}$$

If $(a, x) \in \gamma_\mu(Y)$ and $u \in U^\mu$, then $ua \downarrow$ and $\mu\gamma(ua) \downarrow U\gamma(a)$. This implies $(U^\mu, (x, y) \mapsto xy)$ tracks the inclusion of $(X, \gamma_\mu(Y))$ into $(X, \gamma_\mu(Y'))$.

Thus we get a right adjoint to γ^* , and a geometric morphism of relative realizability topos. \square

With this theorem we end our paper on regular functors and relative realizability categories and leave the reader to his or her own thoughts.

References

- [1] Andrej Bauer. *The realizability approach to computable analysis and topology*. ProQuest LLC, Ann Arbor, MI, 2000. Thesis (Ph.D.)—Carnegie Mellon University.
- [2] Lars Birkedal. *Developing theories of types and computability via realizability*, volume 34 of *Electronic Notes in Theoretical Computer Science*. Elsevier Science B.V., Amsterdam, 2000.
- [3] Lars Birkedal and Jaap van Oosten. Relative and modified relative realizability. *Ann. Pure Appl. Logic*, 118(1-2):115–132, 2002.
- [4] A. Carboni. Some free constructions in realizability and proof theory. *J. Pure Appl. Algebra*, 103(2):117–148, 1995.
- [5] A. Carboni and R. Celia Magno. The free exact category on a left exact one. *J. Austral. Math. Soc. Ser. A*, 33(3):295–301, 1982.
- [6] A. Carboni and E. M. Vitale. Regular and exact completions. *J. Pure Appl. Algebra*, 125(1-3):79–116, 1998.
- [7] Aurelio Carboni, Peter J. Freyd, and Andre Scedrov. A categorical approach to realizability and polymorphic types. In *Mathematical foundations of programming language semantics (New Orleans, LA, 1987)*, volume 298 of *Lecture Notes in Comput. Sci.*, pages 23–42. Springer, Berlin, 1988.

- [8] P. J. W. Hofstra. Relative completions. *J. Pure Appl. Algebra*, 192(1-3):129–148, 2004.
- [9] Pieter Hofstra and Jaap van Oosten. Ordered partial combinatory algebras. *Math. Proc. Cambridge Philos. Soc.*, 134(3):445–463, 2003.
- [10] Pieter J. W. Hofstra. All realizability is relative. *Math. Proc. Cambridge Philos. Soc.*, 141(2):239–264, 2006.
- [11] J. M. E. Hyland, P. T. Johnstone, and A. M. Pitts. Tripos theory. *Math. Proc. Cambridge Philos. Soc.*, 88(2):205–231, 1980.
- [12] Stephen Cole Kleene and Richard Eugene Vesley. *The foundations of intuitionistic mathematics, especially in relation to recursive functions*. North-Holland Publishing Co., Amsterdam, 1965.
- [13] John R. Longley. *Realizability Toposes and Language Semantics*. PhD thesis, University of Edinburgh, 1994.
- [14] Matias Menni. *Exact Completions and Toposes*. PhD thesis, University of Edinburgh, 2000.
- [15] Matías Menni. More exact completions that are toposes. *Ann. Pure Appl. Logic*, 116(1-3):187–203, 2002.
- [16] A. M. Pitts. *The Theory of Tripases*. PhD thesis, University of Cambridge, 1981.
- [17] Edmund Robinson and Giuseppe Rosolini. Colimit completions and the effective topos. *J. Symbolic Logic*, 55(2):678–699, 1990.
- [18] Benno van den Berg and Ieke Moerdijk. Aspects of predicative algebraic set theory, ii: Realizability. *Theoretical Computer Science*, 412(20):1916 – 1940, 2011. Girard’s Festschrift.
- [19] Jaap van Oosten. *Realizability: an introduction to its categorical side*, volume 152 of *Studies in Logic and the Foundations of Mathematics*. Elsevier B. V., Amsterdam, 2008.