Homework Assignment 3

February 24, 2024

Throughout these exercises we fix some arbitrary theory T and consider the category $\mathcal{R}(T)$.

Exercise 1. (4 points) Let $\{\gamma\} : \{\bar{x} \mid p\} \to \{\bar{y} \mid q\}$ and $\{\chi\} : \{\bar{y} \mid q\} \to \{\bar{z} \mid r\}$ be morphisms of $\mathcal{R}(T)$. Recall that the composition $\chi\gamma$ is defined by $\chi\gamma(\bar{x}, \bar{z}) = \exists \bar{y}.\ \gamma(\bar{x}, \bar{y}) \wedge \chi(\bar{y}, \bar{z})$. Prove that this relation is indeed functional. In other words, prove that

$$T \vdash_{\bar{x}, \bar{z}_1, \bar{z}_2} \chi\gamma(\bar{x}, \bar{z}_1) \wedge \chi\gamma(\bar{x}, \bar{z}_2) \Rightarrow \bar{z}_1 = \bar{z}_2.$$

Exercise 2. (7 points) Recall that the identity morphism for an object $\{\bar{x} \mid p(\bar{x})\}$ is defined by the formula $\mathrm{id}(\bar{x}_1, \bar{x}_2) = (p(\bar{x}_1) \wedge \bar{x}_1 = \bar{x}_2)$. Prove that an object $\{\bar{x} \mid p(\bar{x})\}$ of $\mathcal{R}(T)$ is a subobject of the terminal object if and only if there is some sentence q such that $\{\bar{x} \mid p(\bar{x}\}$ is isomorphic to $\{\cdot \mid q\}$.

Exercise 3. (3 + 2 points) Suppose that the language of T has no relation symbols and that T includes no axioms involving $\exists$. Then we can interpret T as a theory both in the sense of Butz and in the sense of Pitts. One may ask themselves if the categories $\mathcal{R}(T)$ and $\mathcal{C}\ell(T)$ are equivalent.

(a) Suppose that our language consists of a single sort σ and a single function symbol $F : \sigma \to \sigma$. Prove that if $\mathcal{C}\ell(T)$ had equalizers then there would be some $n \in \mathbb{N}$ such that

$$F^n(x) = F^{n+1}(x)\ [x : \sigma]$$

is provable (as in Pitts) in the theory T.

(b) Assume that T contains no axioms. Prove that $\mathcal{R}(T)$ and $\mathcal{C}\ell(T)$ are not equivalent.