Homework 4, solution

March 18, 2024

Exercise 1. [3 points] Show that the sequent rule

$$\frac{\mathbf{\Phi} \vdash \psi}{\mathbf{\Phi}, \mathbf{\Phi'} \vdash \psi}$$

is derivable from the rules in table 2 on page 41 of Pitts' *Categorical Logic*.

Exercise 2. [7 points] Show that in Example 5.2.3, the "inclusive" subsets of a Cpo are the closed sets of a topology.

Solution

Exercise 1:

$$\dfrac{\mathbf{\Phi} \vdash \psi \qquad \dfrac{}{\mathbf{\Phi'}, \psi \vdash \psi}\,(\mathrm{Id})}{\mathbf{\Phi}, \mathbf{\Phi'} \vdash \psi}\,\mathrm{Cut}$$

Exercise 2: we need to show that the inclusive sets contain the empty set and are closed under binary union and arbitrary intersection. The empty set is trivially inclusive (there are no ω-chains).

Suppose C and D are inclusive subsets of a cpo; let $(a_i)_{i\in\omega}$ be an ω-chain in $C \cup D$ with join a. Either $\{i \in \omega \mid a_i \in C\}$ is infinite or $\{i \in \omega \mid a_i \in D\}$ is infinite; by symmetry, assume the former. Then we have an ω-chain $(b_j)_{j\in\omega}$ in C specified by: b_0 is a_i for the least i such that $a_i \in C$; and b_{j+1} is a_i for the least i such that $a_i \in C$, $a_i \geq b_j$ and $i > j+1$. Then the join of $(b_j)_{j\in\omega}$ is in C since C is inclusive; by construction this join is a, so $a \in C$.