Weak systems of Gandy, Jensen and Devlin.

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Abstract. In Part I, we formulate and examine some systems that have arisen in the study of the constructible hierarchy; we find numerous transitive models for them, among which are supertransitive models containing all ordinals that show that Devlin's system BS lies strictly between Gandy's systems PZ and BST'; and we use our models to show that BS fails to handle even the simplest rudimentary functions, and is thus inadequate for the use intended for it in Devlin's treatise. In Part II we propose and study an enhancement of the underlying logic of these systems, build further models to show where the previous hierarchy of systems is preserved by our enhancement; and consider three systems that might serve Devlin's purposes: one the enhancement of a version of BS, one a formulation of Gandy-Jensen set theory, and the third a subsystem common to those two. In Part III we give new proofs of results of Boffa by constructing three models in which, respectively, TCo, AxPair and AxSing fail; we give some sufficient conditions for a set not to belong to the rudimentary closure of another set, and thus answer a question of McAloon; and we comment on Gandy's numerals and correct and sharpen other of his observations.

0: Introduction

During the 1960's, as knowledge of the constructible hierarchy advanced, pre-eminently through the work of Jensen [J1] [J2], there was a drive to study various weak systems of set theory, all weaker than that of Kripke–Platek. Those systems included Δ_0 separation but weakened Δ_0 collection in various ways, and their purpose was to give a finer account of the growth of the constructible hierarchy. As is well-known, this move has been extraordinarily fruitful.

Gandy [G] proposed four systems which he called PZ (for "predicative Zermelo"), BST', BRT and PZF. and which he proved to be strictly ascending in strength. Devlin in his treatise [Dev] proposed a further system, which he called BS. We shall, starting in Section 1, introduce new names for those five systems and others which have suggested themselves, but shall use the old in this introduction.

So, roughly, PZ is a weak base theory plus Δ_0 separation. BS adds cartesian product to that. BST' is the result of adding an axiom of infinity to Gandy's theory BST, of which the transitive models are precisely the rudimentarily closed sets. BRT has what Gandy calls the bounded replacement axiom; and PZF has Δ_0 replacement, making it weaker than but close to and equiconsistent with the system of Kripke–Platek with an axiom of infinity. We shall also look briefly at what Gandy would have called the bounded collection axiom, and at our preferred formulation of the system of Kripke and Platek.

When, in the next section and later, we give precise formulations of systems, we shall put names of systems and axioms in $\mathsf{nine\text{-}point\ sans\text{-}serif}$ type to indicate that it is our particular formulations that are being discussed, as defined either in this paper or in [M2]. In our formulations we shall change some of Gandy's terminology and notation, since Gandy uses the term "basic" for the functions that Jensen called "rudimentary"; and further Gandy studies two versions of the axiom of replacement, calling the one "basic" and the other "bounded", an unfortunate combination of adjectives as both begin with 'b'. Therefore we shall follow Jensen's usage, often shortening "rudimentary" to "rud", and shall use "RR" to name what Gandy called the basic replacement axiom. We shall reserve the word "basic" for a proper subclass $\mathcal{B}$ of the class $\mathcal{R}$ of rudimentary functions, namely those generated by composition from Gödel's functions $\mathcal{F}_1, \dots, \mathcal{F}_8$, and we shall use "flat" where Gandy used "bounded" in naming axioms.

In discussing these systems it will, as in *The Strength of Mac Lane Set Theory* [M2], at times be necessary to maintain a careful distinction between three levels of language, which we call the *metalanguage*, which is English, the *language of discourse*, which is a language of set theory formulated with atomic predicates $\in$ and $=$, and various *object languages*, again set-theoretical in nature, with atomic predicates symbolised by ϵ and $\mathsf{=}$. We use Fraktur lower case letters $\mathfrak{k}, \mathfrak{l}, \mathfrak{m}, \dots$ for *concrete* integers, which are quantified only in the metalanguage, and the corresponding terms for them in the language of discourse. This visual aid may be used to mark the distinction between a system T being able, for each $\mathfrak{k}$, to prove some statement $\Phi(\mathfrak{k})$ and being able to prove $\forall k \Phi(k)$.

Three areas of uncertainty in the choice of axioms

The above authors differ in their treatment of the scheme of foundation: Gandy makes no mention of foundation in his formulations, whereas Devlin calls for the full scheme of foundation in his. Without foundation, his system is intermediate between PZ and BST'. The question of the amount of foundation possessed by a system is not idle: in our paper [M2] we showed that in terms of consistency strength Π_2

foundation is in some cases strictly stronger than Π_1 foundation—see Metacorollary 9.21 and Metatheorem 9.34 of [M2]—and there is evidence that Π_1 foundation is the "right" amount to have in formulating the system of Kripke–Platek; see Corollary 1.22(ii) and Proposition 3.14 (ii') of [M2]. The investigations of the present paper suggest that Π_1 foundation is also the "right" amount to have in these weaker systems.

A second area of uncertainty is the axiom of transitive containment, TCo, which asserts that every set is a member of a transitive set. It was shown by Boffa [B1],[B2] that TCo is not provable in Zermelo set theory: we give a new proof of that result in Section 12. TCo is, however, provable in our formulation of Kripke-Platek.

Finally it is of interest to see to what extent the axiom of infinity can be avoided.

So our policy will be, at least initially, to exclude the "special" axioms of infinity and transitive containment from the general axioms of our systems, and explicitly to note each use of those special axioms as it occurs. As for foundation, we shall include the scheme of Π_1 foundation in our systems, and draw attention to areas where foundation can be avoided, and where the full scheme of foundation is required.

In many sections of the paper, our focus will be chiefly not on the consistency strength of the various theories but on constructing transitive models for them; and in such models, the full scheme of foundation will be inherited from our ambient set theory. Further TCo and AxInf will be true in most of our models. We remark that we are not in this paper concerned to find the minimal ambient set theory in which our examples can be built. ZF is certainly too strong; $\mathsf{Z} + \mathsf{KP}$ is usually enough, apart from the occasional appeal to the existence of $V_{\omega+\omega}$ and similar sets. The axiom of choice is used only in a very few peripheral remarks.

Some differences

In the calibration of these systems, certain sets function as litmus paper:

0·0 Definition We write $\mathcal{S}(x)$ for the set of finite subsets of x; for each $\mathfrak{k} > 0$, $[\omega]^{\mathfrak{k}}$ for the class of subsets of ω of size $\mathfrak{k}$; **HF** for the class of hereditarily finite sets, which in appropriate set theories will coincide with the classes notated V_ω, L_ω and J_1; Even for the class of even numbers, Ack for the Ackermann relation on ω, defined as $\{(m,n)_2 \mid 2^m$ is one of the summands in the expression of n as a sum of powers of $2\}$; and $\mathcal{G}_+$ for the graph of integer addition, defined as the class $\{(p,m,n)_3 \mid m+n=p\}$.

We shall see that PZ cannot prove the existence even of $[\omega]^1$; BS can prove the existence of $[\omega]^1$ and $[\omega]^2$ but not of $[\omega]^3$; BST' can prove the existence of each $[\omega]^{\mathfrak{k}}$; BST' with Π_1 foundation can prove that $\forall k\, [\omega]^k \in V$ but cannot prove the existence of $\mathcal{S}(\omega)$; BRT can prove the existence of $\mathcal{S}(\omega)$ but not of **HF**; and PZF proves the existence of **HF**. Further, we shall see that BRT proves that $\mathcal{G}_+$ is a set but that BST' fails to do so.

The contents of the paper

In the first of the three parts of the paper, we shall formulate, in Section 1, eight systems, with variants, and note in Section 2 various results provable in them. In Section 3, we review some simple techniques for building transitive models of weak systems. In the next four sections, we work through the systems in order of increasing strength, summarising Gandy's model-theoretic constructions and giving new ones of our own; our models will demonstrate the unprovability of various results.

The second part begins with the heavily syntactic Section 8, in which we examine the result of strengthening our previous systems by uniformly adding an axiom of infinity and the principle that the class of all finite subsets of any given set is a set; and study the effect of enhancing those strengthened systems by adding limited quantifiers of the form "for some finite subset of a" and "for all finite subsets of a". In Section 9, we give further models illustrating the limitations of our strengthened systems. Then in Section 10 we turn to an examination of Devlin's book *Constructibility*, of which certain passages have been known since its publication to be problematical; we use our models to shed light on those passages, and draw attention to three of our systems that might serve Devlin's purposes better than his system BS.

We begin the final part of the paper by showing in Section 11 that Gandy's remarks concerning certain variants of his systems are not correct. In Section 12, we return to model-building and give a new proof of the result of Boffa that TCo is not provable in Zermelo set theory; Section 13 looks briefly at the axiom of pairing; in Section 14 we find an answer to a question raised by McAloon in the 1970's by giving criteria for one set not to lie in the rudimentary closure of another; finally in Section 15 we apply the technique of Section 14 to show that the set of Gandy numerals is not in the rudimentary closure of ω.

PART I

1: Formulations of the various systems

We outline the syntactical development of our systems: various aspects will be discussed in the projected sequel [M4] in greater detail.

We start with enough syntax to introduce the axioms of our first, very weak system, and to define for each $\mathfrak{n}$ the ordered $\mathfrak{n}$-tuple; then we shall enlarge the syntax to include some convenient extensions of the class-forming operator, and shall then be able to enunciate in the language of discourse the axioms of the systems we intend to study.

We begin therefore with two undefined binary relations $\in$, $=$; propositional connectives $\neg$, $\vee$, $\&$, $\Longrightarrow$, $\Longleftrightarrow$; unrestricted quantifiers $\forall x$, $\exists x$; and restricted quantifiers $\forall x_{\in y}$, $\exists x_{\in y}$, where x and y are not permitted to be the same letter, and the quantifier binds x but not y, in harmony with the axioms that express their intended meaning:

$$\exists x_{\in y}\,\mathfrak{A} \Longleftrightarrow \exists x[x \in y \;\&\; \mathfrak{A}]; \qquad \forall x_{\in y}\,\mathfrak{A} \Longleftrightarrow \forall x[x \in y \Longrightarrow \mathfrak{A}].$$

The rules of formation are the usual ones of classical logic.

We then define a Δ_0 formula or a Δ_0 class to be one containing no unrestricted quantifiers; a Π_1 formula is one of the form $\forall x\mathfrak{A}$ where $\mathfrak{A}$ is Δ_0; a Σ_2 formula is one of the form $\exists y\mathfrak{B}$ where $\mathfrak{B}$ is Π_1; a Σ_1 formula is one of the form $\exists x\mathfrak{A}$ where $\mathfrak{A}$ is Δ_0, and so on.

We have the usual axioms of classical propositional and predicate logic; we should (but do not) define the result of substituting one variable for another, indicated informally by such usages as $\mathfrak{A}(x)$ and $\mathfrak{A}(y)$.

It is convenient to permit the use of the class-forming operator $\{\cdot \mid \ldots\}$, but we emphasize that our language is unramified and that our logic includes the Church conversion schema

$$x \in \{y \mid \mathfrak{A}(y)\} \Longleftrightarrow \mathfrak{A}(x)$$

so that all occurrences of the class-forming operator are in principle eliminable. We adopt appropriate axioms interpreting the result of substituting a class for a variable in a formula, which we summarise in these three equivalences, in which A is a class and t a class or a variable:

$$\exists x_{\in A}\,\mathfrak{A} \Longleftrightarrow \exists x[x \in A \;\&\; \mathfrak{A}]; \qquad t = A \Longleftrightarrow \forall y_{\in t}\, y \in A \;\&\; \forall y_{\in A}\, y \in t; \qquad A \in t \Longleftrightarrow \exists z_{\in t}\, z = A.$$

With this syntax, we may give axioms for our first, very weak, system:

S_0 The axiom of extensionality, $[\forall x_{\in a}\, x \in b \;\&\; \forall x_{\in b}\, x \in a] \Longrightarrow a = b$, and axioms of empty set, pair set, difference and sumset (or union):

$$\varnothing \in V, \quad \{x,\, y\} \in V, \quad x \setminus y \in V, \quad \bigcup x \in V.$$

In this system we introduce, successively, ordered $\mathfrak{k}$-tuples, in the Wiener-Kuratowski manner:

$$\begin{aligned}
(y_1)_1 &=_{\mathrm{df}} y_1\\
(y_1, y_2)_2 &=_{\mathrm{df}} \{\{y_1\},\, \{y_1,\, y_2\}\};\\
(y_1, y_2, y_3)_3 &=_{\mathrm{df}} (y_1, (y_2, y_3)_2)_2\\
(y_1, y_2, y_3, y_4)_4 &=_{\mathrm{df}} (y_1, (y_2, y_3, y_4)_3)_2\\
(y_1, y_2, y_3, y_4, y_5)_5 &=_{\mathrm{df}} (y_1, (y_2, y_3, y_4, y_5)_4)_2\\
&\ldots
\end{aligned}$$

1·0 REMARK Thus all WK-tuples are generated from the single binary function $\{x,\, y\}$.

We may now develop the usual theory of relations, $\mathfrak{k}$-ary functions and so on: we treat functions as a subclass of their image $\times$ their domain. We shall see that these weak systems are sensitive to the choice of implementation of function, and so it is necessary to distinguish notationally between concepts that "the working mathematician" would often conflate. Thus we adopt a policy of writing ${}^3X$ for the set of 3-sequences of members of X, reserving X^3 for the set of WK 3-tuples of members of X; thus $\omega^3 = \omega \times (\omega \times \omega)$.

1·1 It is convenient further to enlarge the syntax to permit certain classes with quantified terms, namely those where the terms are WK-tuples: where there might otherwise be ambiguity, we indicate the variables to be quantified in a list placed subscript to the vertical bar, for example:

$$\{(x,y)_2 \mid_{x,y} \mathfrak{A}(x,y)\}; \quad \{(x,a)_2 \mid_x \mathfrak{A}(x,a)\}.$$

The first of those will equal $\{z \mid \exists x \exists y[z = (x,y)_2 \ \& \ \mathfrak{A}(x,y)]\}$; the second, $\{z \mid \exists x[z = (x,a)_2 \ \& \ \mathfrak{A}(x,a)]\}$ for the given a: such equalities are accomplished by adding the following scheme to our system:

$$x \in \big\{(y_1, y_2, \ldots, y_{\mathfrak{k}})_{\mathfrak{k}} \;\big|_{y_1,y_2,\ldots,y_{\mathfrak{k}}} \mathfrak{A}\big\} \iff \exists y_1 \exists y_2 \ldots \exists y_{\mathfrak{k}}[x = (y_1, y_2, \ldots, y_{\mathfrak{k}})_{\mathfrak{k}} \ \& \ \mathfrak{A}].$$

1·2 We informally permit classes with other quantified terms, for example $\{\bigcup x \mid_x x \in a\}$.

1·3 Definition $\mathsf{Foundation}$, the axiom of (set) foundation, is $x \neq \varnothing \implies \exists y_{\in x}\, x \cap y = \varnothing$.

S_0' S_0 + $\mathsf{Foundation}$

A calculus of Δ_0 terms

1·4 Definition We call a term A, possibly with free variables, T-*semi-suitable*, where T is some system of set theory, if whenever Φ is Δ_0, and the variable w is not free in A, then $\forall w_{\in A}\, \Phi$ is Δ_0^{T}, meaning "equivalent, provably in the system T, to a Δ_0 formula". If in addition, T proves that A is a set, we call A T-*suitable*.

1·5 Remark S_0 is adequate for the development of a surprisingly large number of suitable terms. In particular, $\bigcup x$ is S_0-suitable, as is each $\bigcup^{\mathfrak{l}} x$, where we define inductively $\bigcup^{\mathfrak{k}+1} x =_{\mathrm{df}} \bigcup(\bigcup^{\mathfrak{k}} x)$. S_0 easily proves that if $x = (y,z)_2$, then $y \in \bigcup^2 x$ and $z \in \bigcup^2 x$; hence if $\mathfrak{A}$ is Δ_0 then the $\mathfrak{k}$-class $\big\{(y_1, y_2, \ldots, y_{\mathfrak{k}})_{\mathfrak{k}} \;\big|_{y_1,y_2,\ldots,y_{\mathfrak{k}}} \mathfrak{A}\big\}$ is equal, provably in S_0, to a Δ_0 1-class.

With $\mathsf{Foundation}$ added, the formulation of "ordinal" becomes Δ_0 and much of the elementary theory of ordinals can then be developed.

1·6 Remark Gandy in [G] proves that the term ω is S_0'-semi-suitable in that if Φ is Δ_0 then the formula $\exists y_{\in\omega}\, \Phi$ is equivalent in S_0' to a Δ_0 formula. His proof will work for appropriate terms for each ordinal strictly less than ω^ω, an interesting ordinal shown by Delhommé [Del] to be the first non-automatic ordinal, but, by [DoMT, page 44, Theorem 38], no further.

1·7 Remark Gandy [G] and Dodd [Do] have a concept of "substitutable" which is similar to our "suitable" but formulated semantically rather than syntactically. Jensen [J2] and Devlin [Dev] have the same concept but call it "simple". In the present author's opinion, that concept has the danger of blurring the levels of language. If one considers a rudimentary function to be defined by a class of the language of discourse, then implicitly there is a quantification taking place in the meta-language whenever one uses such phrases as "rud closed" or "the class of rud functions". That is scarcely satisfactory, though the situation is saved by defining a rud closed set to be one closed under, say, the explicit list of nine functions given in 2·61. What would be better would be to resort to some mild recursion theory, and to list terms of an object language defining certain (set-theoretical) computations, and then when one speaks of closure the quantification will indeed be going on in the language of discourse.

Thus it would seem that the axiom TCo, not adopted by Mac Lane, expresses a characteristic of set theory, namely that it is often concerned with computations going on in small portions of the universe, perhaps the transitive sets, or else the transitive sets closed under pairing functions. Not adopting TCo is a sop to the structuralists; but adopting it is what set theorists should do if they are to be true to their underlying intuitions. The point is linked to the meaning of Δ_0 and will recur in Remark 10·1.

Names of systems

Our policy will be this: if we have a system X, X_0 will mean the variant of that system with no axiom of foundation, no TCo, and no axiom of infinity. Without that subscript, Π_1 foundation will be customary. We use "restricted" to mean Δ_0. We use "flat", where Gandy used "bounded", to mean that a certain quantifier limits its variable to subsets of a named set.

Four of our names will reflect the fact that a significant part of the system is the scheme of restricted separation, flat restricted replacement, flat restricted collection or restricted replacement: ReS, fReR, fReC, ReR.

We shall add the letter I to indicate the adjunction of an axiom of infinity, usually in the form $\omega \in V$. In Section 8 we shall add the letter S, either in upper or lower case, to existing names to indicate the adjunction of both the axiom of infinity and the axiom $\mathcal{S}(x) \in V$. TCo will be listed by name when needed.

Gandy's first system

Gandy called his weakest system PZ, for "predicative Zermelo", and his strongest PZF, for "predicative Zermelo–Fraenkel". They are both something of a misnomer as he overlooked the power-set axiom; and without that axiom, as shown by Zarach [Z], the difference between replacement and separation-with-collection becomes significant. We use ReS, for "restricted separation".

ReS_0 S_0 plus the Δ_0 separation axiom: $x \cap A \in V$ for A a Δ_0 class.

ReS ReS_0 plus the scheme of Π_1 foundation: $A \neq \varnothing \Longrightarrow \exists x_{\in A}\, x \cap A = \varnothing$ for A a Π_1 class.

We shall call functions of the form $x \mapsto x \cap A$ *separators*.

Devlin's system and variant

The next system, which we call DB for "Devlin Basic", adds the existence of cartesian product to ReS_0, but as it thereby becomes finitely axiomatisable, by a result of which many variants are found in the literature, and presumably going back to Bernays, we give it officially as that finite axiomatisation.

DB_0 The system of which the set-theoretic axioms are Extensionality and the following nine set-existence axioms:

$$\begin{array}{lll} \varnothing \in V & \bigcup x \in V & a \cap \{(x,y)_2 \mid x \in y\} \in V \\ \{x,\, y\} \in V & \mathrm{Dom}\,(x) \in V & \{(y,x,z)_3 \mid (x,y,z)_3 \in b\} \in V \\ x \smallsetminus y \in V & x \times y \in V & \{(y,z,x)_3 \mid (x,y,z)_3 \in c\} \in V \end{array}$$

DB DB_0 plus Π_1 foundation.

1·8 Remark All those nine are theorems of ReS_0 + cartesian product, without foundation.

1·9 Definition Although in one model that we consider, we must use a different formulation, we shall usually take the axiom of infinity in the form $\omega \in V$, ω being defined as the class of all von Neumann ordinals such that they and all their predecessors are either 0 or successor ordinals.

1·10 Remark If we add the axiom of infinity plus the scheme of foundation for all classes to DB we obtain the system BS as formulated on page 36 of Devlin's book *Constructibility*:

BS ReS_0 + Cartesian product + full foundation + $\omega \in V$.

The Gandy–Jensen system

The next system, called BST by Gandy, represents a considerable step forward, in that it involves the class of rudimentary functions. Foundation apart, it is finitely axiomatisable, and indeed needs only one axiom beyond those of DB_0. We give first the scheme of Gandy, and in Remark 1·12 shall indicate why all instances of it are derivable in the finitely-axiomatisable version.

GJ_0 S_0 + the rudimentary replacement axiom:

$$(RR) \qquad \forall x \exists w \forall \vec{v}_{\in x}\, \exists t_{\in w} \forall u (u \in t \Longleftrightarrow .u \in x \;\&\; \phi[u, \vec{v}]).$$

for ϕ any Δ_0 formula.

1·11 Remark At first glance, it might seem more appropriate to call that a collection axiom, since it says that a certain family of sets is included in a set, rather than being a set. But if φ is Δ_0, x a set and $\vec{v}$ parameters, not necessarily in x, then a term x_1 and a Δ_0 formula φ_1 are readily found so that S_0 proves that x_1 is a set containing each parameter in the list $\vec{v}$, that $x \cap \{u \mid \varphi\} = x_1 \cap \{u \mid \varphi_1\}$ and that the latter is a set. So GJ_0 indeed proves Δ_0 separation.

1·12 REMARK GJ_0 is the result of adding a single axiom, which I call R_8, to DB_0:

$$(R_8) \qquad \{x``\{w\} \mid w \in y\} \in V$$

To see that, use the fact remarked above and reformulated again as Proposition 2·65 that DB_0 generates all Δ_0 separators; each instance of (RR) then follows by taking the F of the Gandy-Jensen Lemma 2·72 to be an appropriate such separator.

GJ GJ_0 + the scheme of Π_1 foundation.

Flat restricted replacement

The next system has what Gandy called the bounded replacement axiom, but we shall prefer to use the adjective "flat".

fReR_0 S_0 plus the flat Δ_0 replacement axiom: namely, for any ϕ in Δ_0,

$$(\textit{Flat } \Delta_0 \textit{ Replacement}) \qquad \forall x_{\in u}\, \exists! y(\phi(x,y) \;\&\; y \subseteq z) \Longrightarrow \exists v \forall y[y \in v \Longleftrightarrow \exists x_{\in u}\,(\phi(x,y) \;\&\; y \subseteq z)].$$

In words, the image of a set by a function whose values are all included in a set is itself a set.

fReR fReR_0 + the scheme of Π_1 foundation.

Flat restricted collection

fReC_0 S_0 plus Δ_0 separation plus the following scheme, for ϕ any Δ_0 formula:

$$(\textit{Flat } \Delta_0 \textit{ Collection}) \qquad \forall x_{\in u}\, \exists y(\phi(x,y) \;\&\; y \subseteq z) \Longrightarrow \exists v \forall x_{\in u}\, \exists y_{\in v}\, (\phi(x,y) \;\&\; y \subseteq z)].$$

fReC fReC_0 + the scheme of Π_1 foundation.

1·13 REMARK Π_1 Foundation aside, the axioms of the above systems are all provable in the system M_0 studied in [M2], which is the system ReS_0 + the power set axiom, $\mathcal{P}(x) \in V$ and is a subsystem of Mac Lane's system ZBQC, which in turn, shorn of the axiom of choice, is a subsystem of Zermelo's system Z.

Restricted replacement

We depart now from a linearly ordered set of systems: we shall see that ReR is not a subsystem of fReC, and I suspect that methods of Zarach will show that fReC is not a subsystem of ReR.

ReR_0 S_0 + the following scheme, for ϕ any Δ_0 formula:

$$(\Delta_0 \textit{ Replacement}) \qquad \forall x_{\in u}\, \exists! y\; \phi(x,y) \Longrightarrow \exists v \forall y[y \in v \Longleftrightarrow \exists x_{\in u}\; \phi(x,y)].$$

ReR ReR_0 + the scheme of Π_1 foundation.

Kripke–Platek

Finally we arrive at Kripke–Platek set theory, KP which we formulate with Π_1 foundation.

KP Δ_0 separation, Π_1 foundation, and Δ_0 collection, in the formulation of which u and v are to be variables having no occurrence in the Δ_0 formula ϕ:

$$(\Delta_0 \textit{ Collection}) \qquad \forall x \exists y \phi \Longrightarrow \forall u \exists v \forall x_{\in u}\, \exists y_{\in v}\, \phi(x,y).$$

We shall indicate the addition of the axiom of infinity to one of the above systems by adding the letter I: thus $\mathsf{DB}_0\mathsf{I}$, KPI.

1·14 REMARK By a result of Boffa, TCo, the statement that every set is a member of a transitive set, is not provable in Z, and therefore not in its subsystems. It is, however, provable in KP when that system is formulated, as here, to include Π_1 foundation, and in ReRI: see Proposition 2·108 and Problem 2·107.

On ReS and finite sets:

We shall work with two definitions of *finite*: we get an easy Σ_1 definition of **HF** by taking “finite” to mean “in bijection with a member of ω”; we shall get an easy proof that the union of two finite sets is finite by taking “finite” to mean “possesses a double well-ordering”; and we need Π_1 foundation to prove the equivalence of the two definitions (or to develop the arithmetic necessary were we to work only with the “member of ω” definition).

2·0 Definition x is *finite* if x carries a double well-ordering, that is, a linear ordering such that every non-empty subset has both a least and a greatest element.

The natural ordering of any member of ω is a double well-ordering.

2·1 Proposition (ReS) *If a set is finite then it is in bijection with some member of ω.*

Proof: Let X be a set with a double well-ordering $\leqslant_X$. We say that f is *an attempt at x in X* if $\mathrm{Dom}\,(f) = \{y \mid y \leqslant_X x\}$ and for all y in $\mathrm{Dom}\,(f)$, $f(y) = \{f(z) \mid z <_X y\}$. The class

$$\{x \mid x \in X \;\&\; \neg\exists f \; f \text{ is an attempt at } x\}$$

is Π_1 and if non-empty, has a $\leqslant_X$-least element $\bar{x}$. $\bar{x}$ is not the first member of X, as an attempt at that point is easily built; nor can $\bar{x}$ be a successor, as an attempt at its predecessor is easily extended. So that class is in fact empty. Let ϖ be an attempt at the largest element of X: then a further induction shows that ϖ maps $(X, <_X)$ bijectively to a member of ω. ⊣ (2·1)

As we are working without assuming that cartesian products exist in general, the converse, which is true, requires some preparation.

2·2 Lemma (ReS) *For all m and k in ω, $\{m\} \times k$ is a set.*

Proof: Fix m. Use the fact that

$$\{(m,n)_2 \mid_n n < k+1\} = \{(m,n)_2 \mid_n n < k\} \cup \{(m,k)_2\}. \qquad ⊣ (2·2)$$

2·3 Lemma (ReS) *For all m and k in ω, $k \times \{m\}$ is a set.*

Proof: Fix m. Use the fact that

$$\{(n,m)_2 \mid_n n < k+1\} = \{(n,m)_2 \mid_n n < k\} \cup \{(k,m)_2\}. \qquad ⊣ (2·3)$$

2·4 Proposition (ReS) *For all m and n in ω, $m \times n$ is a set.*

Proof: Use the fact that

$$(m+1) \times (m+1) = (m \times m) \cup (\{m\} \times m) \cup (m \times \{m\}) \cup (\{m\} \times \{m\}). \qquad ⊣ (2·4)$$

2·5 Remark Note that that cannot lead to a proof that $\omega \times \omega$ is a set. We cannot form the collection of attempts.

2·6 Corollary (ReS) *The cartesian product of two sets, each in bijection with a member of ω, is a set.*

Proof: First, reason thus: if $g : m \longleftrightarrow a$ and $h : n \longleftrightarrow b$, define the function f with domain $m \times n$ by

$$f((i,j)_2) = (g(i), h(j))_2.$$

Then the image of that function is $a \times b$.

But that reasoning, though sound in GJ, is not available in BS or ReS_0. Hence we must do an induction structured as above: first for $m = 1$ prove, by induction on n, that for any n, and g and h as above, the cartesian product exists. Then do an induction on m. ⊣ (2·6)

2·7 Remark In systems without the Axiom of Cartesian Products, it cannot be assumed that the inverse of an injective function will always exist: see the variant of Model 4 mentioned in 4·8.

2·8 PROPOSITION (ReS) *If X is in bijection with some member of ω, then it is finite.*

Proof : From the above we know that $X \times X$ and each $n \times X$ exist. Now given $f : n \longleftrightarrow X$, we may form its inverse g thus:

$$g := (n \times X) \cap \{(a, b)_2 \mid_{a,b} (b, a)_2 \in f\}$$

and we may then form the set $X \times X \cap \{(x, y)_2 \mid_{x,y} g(x) \leqslant g(y)\}$, which will be a double well-ordering. ⊣ (2·8)

2·9 PROPOSITION *Every subset of a finite set is finite.*

Proof : a restriction of a double well-ordering is ditto. ⊣ (2·9)

2·10 PROPOSITION *If x and y are finite, so is $x \cup y$.*

Proof : a double well-ordering of $x \cup y$ can easily be constructed given ones of x and of $y \setminus x$. ⊣ (2·10)

2·11 LEMMA *Let z be a finite set, and $a \notin \bigcup z$. Then $\{y \cup \{a\} \mid_y y \in z\}$ is a set and is finite.*

Proof : let $f : n \longleftrightarrow z$. Define

$$g(0) = \{f(0) \cup \{a\}\}$$
$$g(k + 1) = g(k) \cup \{f(k) \cup \{a\}\}$$

Then $g(n)$ will be defined—appeal to Π_1 foundation if not !—and will be the desired set, which is evidently in bijection with z and therefore finite. ⊣ (2·11)

2·12 LEMMA (S_0) *Let z be a set, and $a \notin z$. Then $\mathcal{P}(z \cup \{a\}) = \mathcal{P}(z) \cup \{y \cup \{a\} \mid_y y \in \mathcal{P}(z)\}$.*

2·13 PROPOSITION *Let w be finite. Then $\mathcal{P}(w)$ is a set and is finite.*

Proof : write $F(a, z)$ for $\{y \cup \{a\} \mid_y y \in z\}$. Let $f : n \longleftrightarrow w$. Define

$$g(0) = \{\varnothing\}$$
$$g(k + 1) = g(k) \cup F(f(k), g(k))$$

As before, we consider the least m for which there is no attempt at m for this recursion; and obtain a contradiction. So $g(n)$ will be the desired set $\mathcal{P}(w)$.

To see that $\mathcal{P}(w)$ is finite, argue, again by induction on $k \leq n$, and using 2·10, 2·11 and 2·12, that each $g(k)$ is finite, (the class of failures being again Π_1, the argument succeeds); so $g(n)$ is finite. ⊣ (2·13)

2·14 PROPOSITION *The cartesian product of two finite sets is finite.*

Proof : by a similar argument, starting from the observation that $x \times (z \cup \{a\}) = (x \times z) \cup (x \times \{a\})$. ⊣ (2·14)

2·15 PROPOSITION *A surjective image of a finite set is finite.*

[trivial if the surjection is a set; if it is defined by some formula, we may need full foundation.]

2·16 DEFINITION $\mathcal{S}(x) =_{\mathrm{df}} \{y \mid y \subseteq x \;\&\; y \text{ is finite}\}$.
[It is not assumed that $\mathcal{S}(x)$ is a set.]

2·17 DEFINITION Let $\Psi_{\mathcal{S}}(q, y)$ be the Δ_0 formula $\varnothing \in q \;\&\; \forall w_{\in q} \forall x_{\in y}\, w \cup \{x\} \in q$.

2·18 LEMMA (ReS) $\Psi_{\mathcal{S}}(q, y) \Longrightarrow q \supseteq \mathcal{S}(y)$.

2·19 LEMMA (ReS) $x \in \mathcal{S}(y) \Longleftrightarrow \exists f(x \subseteq y \;\&\; \exists n_{\in \omega} \mathrm{Fn}(f) \;\&\; f : n \longleftrightarrow x)$.

Hence, using the semi-suitablility of the constant ω recorded in Remark 1·6:

2·20 COROLLARY *"$x \in \mathcal{S}(y)$" is Σ_1^{ReS}.*

2·21 LEMMA (ReS) $\mathcal{S}(y) \in V \Longrightarrow \forall x[x \in \mathcal{S}(y) \Longleftrightarrow \forall q(\Psi_{\mathcal{S}}(q, y) \Longrightarrow x \in q)]$.

2·22 LEMMA (ReS) $\mathcal{S}(y) \in V \Longrightarrow [z \subseteq \mathcal{S}(y) \Longleftrightarrow \forall q(\Psi_{\mathcal{S}}(q, y) \Longrightarrow q \supseteq z)]$.

2·23 LEMMA (ReS) $\mathcal{S}(y) \in V \Longrightarrow [z = \mathcal{S}(y) \Longleftrightarrow z \subseteq \mathcal{S}(y) \;\&\; \Psi_{\mathcal{S}}(z, y)]$

Next, a principle of collection for finite sets.

2·24 METATHEOREM *Let $\mathfrak{A}$ be a $\Pi_{\mathfrak{k}}$ wff; then it is provable in ReS_0 with $\Pi_{\mathfrak{k}+1}$ foundation that for v finite, $\forall x_{\in v} \exists y \mathfrak{A} \Longrightarrow \exists w \forall x_{\in v} \exists y_{\in w} \mathfrak{A}$.*

Proof : let $f : n \longleftrightarrow v$. Let $P(k)$ say that there is a function g with domain k such that $\forall i < k\, \mathfrak{A}(f(i), g(i))$. Find the least $k \leqslant n$ such that $P(k)$ fails. By taking cases on k, we see that it cannot exist. So $P(n)$ holds. Take the image of a corresponding g for w. ⊣ (2·24)

2·25 Remark The above result is self-strengthening to the case that $\mathfrak{A}$ is $\Sigma_{\mathfrak{k}+1}$.

Proof that HF models ZF minus infinity

2·26 Definition We define **TF** to be the class of all finite transitive sets, and **HF** to be its union.

2·27 Remark In a set theory without an axiom of foundation, **HF** might be strictly greater than V_ω; for example, any *Quine atom*, that is, a set x which equals its own singleton $\{x\}$, would be in **HF** as we have defined it. To exclude such ill-founded sets we should define **HF** as the union of transitive finite sets u which are well-founded in the sense that $\forall x_{\subseteq u}(x \neq \varnothing \implies \exists y_{\in x}\, y \cap x = \varnothing)$; and would then have to add occasional remarks to the discussion below. But as our chief focus is on contexts where the axiom of foundation is true, we may leave our definition of **HF** as it is.

2·28 Metatheorem *Let $\mathfrak{A}$ be any axiom of* ZF *other than that of infinity. Then $(\mathfrak{A})^{\mathbf{HF}}$ is a theorem of* ReS_0 + *full foundation.*

We begin a sequence of verifications. We frequently use the fact that for Δ_0 concepts it suffices to prove that the object in question is in **HF** as its definition will relativise without difficulty.

2·29 Lemma **HF** *is transitive.*

2·30 Lemma $(\mathsf{Extensionality})^{\mathbf{HF}}$.

Proof : assured by the transitivity of **HF**. ⊣ (2·30)

2·31 Lemma $\mathbf{TF} \subseteq \mathbf{HF}$.

Proof : since u transitive and finite implies $u \cup \{u\}$ is too; and hence u is in **HF**. ⊣ (2·31)

2·32 Corollary $(\mathsf{TCo})^{\mathbf{HF}}$.

2·33 Lemma $(\mathsf{Emptyset})^{\mathbf{HF}}$

Proof : $\{\varnothing\}$ is transitive and finite. ⊣ (2·33)

2·34 Lemma $(\mathsf{Pairing})^{\mathbf{HF}}$

Proof : by Proposition 2·10 and the fact that the union of two transitive sets is transitive. ⊣ (2·34)

2·35 Lemma $(\mathsf{Sumset})^{\mathbf{HF}}$

Proof : if $x \in u \in \mathbf{TF}$, then $\bigcup x \subseteq u$ and $\bigcup x \in u \cup \{\bigcup x\} \in \mathbf{TF}$. ⊣ (2·35)

2·36 Lemma $(\Delta_0\ \mathsf{Separation})^{\mathbf{HF}}$

Proof : Δ_0 separation will relativise to any transitive set. ⊣ (2·36)

2·37 Remark Indeed an "external" version of Δ_0 separation holds, in that $x \cap A \in \mathbf{HF}$ whenever $x \in \mathbf{HF}$ and A is a Δ_0 class, possibly with parameters that are not in **HF**.

2·38 Lemma $(\mathsf{Powerset})^{\mathbf{HF}}$

Proof : By Proposition 2·13 and the fact that if u is transitive and $\forall x_{\in a}\, x \subseteq u$ then $u \cup a$ is transitive. ⊣ (2·38)

2·39 Lemma (set foundation) $(\mathsf{Foundation})^{\mathbf{HF}}$

2·40 Remark Foundation is definitely needed here: the result would be false if **HF** contained Quine atoms. ⊣ (2·40)

At this point we have proved that all of M_1 is true in **HF**.

2·41 Definition $u^\star =_{\mathrm{df}} u \cup [u]^1 \cup [u]^2 \cup (u \times u)$.

2·42 Lemma *If u is finite and transitive then so is $u^\star$.*

Proof : $[u]^1 \cup [u]^2$ is a Δ_0 subclass of $\mathcal{P}(u)$, $u \times u$ is finite by what we have seen, and the transitivity is easily verified. ⊣ (2·42)

2·43 PROPOSITION *"all sets are finite" is true in* $\mathbf{HF}$.

Proof : if $x \in u \in \mathbf{TF}$ and $f : n \longleftrightarrow u$, then $f \subseteq u \times n$; $(u \cup n)^\star$ is in $\mathbf{TF}$, and so is $(u \cup n)^\star \cup u \times n \cup \{u \times n\}$. ⊣ (2·43)

2·44 LEMMA *"$x \in \mathbf{HF}$" is* Σ_1^{ReS}.

Proof : $x \in \mathbf{HF} \iff \exists u \exists f \exists n[n \in \omega \;\&\; \bigcup u \subseteq u \;\&\; f : n \longleftrightarrow u]$. ⊣ (2·44)

2·45 REMARK Here we benefit from the "simplified" definition of $\mathbf{HF}$: if we had to say that u is well-founded, that would introduce a Π_1 clause.

2·46 LEMMA (ReS) $((\Pi_1 \text{ foundation}))^{\mathbf{HF}}$.

Proof : Let Φ be Δ_0 and $B = (\{x \mid \forall b\, \Phi\})^{\mathbf{HF}}$.

Let $C = \{x \mid \forall b[b \in \mathbf{HF} \Longrightarrow \Phi]\}$. Then C is Π_1 and $B \subseteq C$; indeed $B = C \cap \mathbf{HF}$. Suppose that B is non-empty and that x is a member. Then there is $u \in \mathbf{TF}$ with $x \in u$. Then $C \cap u$ is Π_1 and non-empty; let $\bar{x}$ be a minimal element. Then $\bar{x}$ is a minimal element of B. ⊣ (2·46)

2·47 COROLLARY (ReS) $(\Delta_0 \text{ collection})^{\mathbf{HF}}$.

Thus ReS proves the relative consistency of the system MOST (as defined in [M2]) less infinity.

2·48 REMARK The above sheds some light on relative consistency strengths: reasoning in ReS we have shown the relative consistency of adding the power set axiom.

With Full Foundation

By results of [M3] we could now conclude that all axioms of ZF save that of infinity are true in $\mathbf{HF}$ provided we established the truth of the principle called Repcoll in [M3] and shown there to imply all the axioms of ZF in the system M_1, which is $\mathsf{M}_0 + \mathsf{TCo} +$ set Foundation. M_0 is ReS_0 plus $\mathcal{P}(x) \in V$.

2·49 LEMMA (ReS + full Foundation) $(\text{Repcoll})^{\mathbf{HF}}$

We shall not give the proof, because we shall derive the truth of $\mathsf{ZF}{-}\infty$ in $\mathbf{HF}$ by another route.

2·50 LEMMA *Let A be any class: then* ReS + full Foundation *proves* $A \cap \mathbf{HF} \neq \varnothing \Longrightarrow \exists x_{\in A \cap \mathbf{HF}}\, x \cap A = \varnothing$.

2·51 REMARK Here we definitely need the "simplied" version of $\mathbf{HF}$ that does not mention well-foundedness.

If we use full foundation we can establish an "external" form of full separation, as in the following scheme:

2·52 LEMMA (ReS + full Foundation) $x \in \mathbf{HF} \Longrightarrow x \cap A \in \mathbf{HF}$ *for A any class.*

Proof : let $f : n \longleftrightarrow x$. Consider the class

$$B := \{k \leqslant n \mid_k \neg\exists y[y \subseteq x \;\&\; \forall m :\leqslant k(f(m) \in y \iff f(m) \in A\}.$$

By full Foundation, that, if non-empty has a minimal element, $\bar{k}$, say. The case $\bar{k} = 0$ is easily dismissed; if $\bar{k} = k+1$, we know that $z \;=_{\mathrm{df}}\; \{f(i) \mid i \leqslant k\} \cap A$ is a set, and $\{f(i) \mid i \leqslant \bar{k}\} \cap A$ will be either z or $z \cup \{f(\bar{k})\}$; as both are sets, we have a contradiction; so the class B is empty and the theorem is proved. ⊣ (2·52)

2·53 THEOREM (ReS + full Foundation) $(\text{full Collection})^{\mathbf{HF}}$.

Proof : from the above, since we know from Lemma 2·50 that $\mathbf{HF}$ models full foundation and from Proposition 2·43 that $\mathbf{HF}$ thinks that all sets are finite.

With $\mathbf{HF} \in V$

2·54 LEMMA $(\mathsf{ReS}_0 + \mathbf{HF} \in V)$ $(\text{full Separation})^{\mathbf{HF}}$

Proof : by re-writing the formula relativising all quantifiers to the set $\mathbf{HF}$, and then applying Δ_0 Separation.

2·55 LEMMA $(\mathsf{ReS}_0 + \mathbf{HF} \in V +$ set Foundation$)$ $(\text{full Foundation})^{\mathbf{HF}}$

Proof : by Lemma 2·54 and Corollary 2·32. ⊣ (2·55)

Another example of the amount of foundation needed for a proof being reduced by the assumption that $\mathbf{HF} \in V$ is furnished by the next sub-section.

Do graphs of recursive functions exist ?

2·56 Consider the following argument, intended to prove that addition on ω is total:

Let $\phi(m,n)$ say that there is no function with domain $(m+1)\times(n+1)$ which satisfies the definition of addition for $m'+n'$ for $m'\leqslant m$ and $n'\leqslant n$. [We call such functions *attempts at integer addition.* "f is an attempt at integer addition" is Δ_0, and therefore rudimentary.]

Consider the class of $m\in\omega$ such that there is some $n\in\omega$ for which $\phi(m,n)$ is true. If non-empty, use Π_1 foundation to find its least member, $\bar{m}$, which cannot be 0, as the function $f(0,n)\equiv n$ would work: a subset of $(n+1)\times(\{0\}\times(n+1))$, and so is some $m+1$. Now minimise n. Again it cannot be 0. So it is some $n+1$. But we have a function h defined up to $m+1,n$, and can extend it to g by setting $g(m+1,n+1)=h(m+1,n)\dot{+}1$, a contradiction. We have proved the following:

2·57 Proposition (ReS) *Every pair (m,n) of integers is in the domain of some attempt at integer addition.*

2·58 Now comes the great task of putting all the attempts together: what does it take to prove that the graph of integer addition is a set ? The axiom of infinity is certainly necessary, but not sufficient: we shall see in Proposition 2·95 that fReRI would do this very well, and in Remarks 5·21 and 6·0, that neither BS nor GJI can do it, though see also Remark 5·24 for a fine point. Happily, our system DS does prove it. $\mathbf{HF}\in V$ would also do it.

2·59 Remark Proposition 2·1, taken with Propositions 2·10 and 2·14, suggests the possibility of using ideas from cardinal rather than ordinal arithmetic to define addition and multiplication within ReS.

On $\mathsf{DB}_0\mathsf{I}$:

2·60 Proposition ($\mathsf{DB}_0\mathsf{I}$) *$[\omega]^1$ and $[\omega]^2$ exist.*

Proof : $\omega\in V$ is an axiom of $\mathsf{DB}_0\mathsf{I}$. By the definition of ordered pair, $[\omega]^1\cup[\omega]^2\subseteq\bigcup(\omega\times\omega)$, and the result follows by Δ_0 separation. ⊣ (2·60)

On GJ and the class of rudimentary functions

The companion papers *Rudimentary recursion* and *Rudimentary forcing* will contain more detailed material on rudimentary functions and related topics. Here we merely give a summary, drawing on but in places differing from the material in Jensen [J2], Gandy [G], Devlin [Dev] and Dodd [Do].

2·61 Corresponding to the systems of DB_0 and GJ_0, we introduce the rudimentary functions $R_0,\ldots R_8$ and certain auxiliary functions $A_0\ldots A_{15}$ generated by them: this is not the shortest possible list, but one that conveniently extends the list that generates the Δ_0 separators. Of the auxiliaries, we list only the most important, A_{14}.

$R_0(x,y)=\{x,y\}$
$R_1(x,y)=x\setminus y$
$R_2(x)=\bigcup x$
$R_3(x)=\mathrm{Dom}\,(x)$
$R_4(x,y)=x\times y$
$R_5(x)=x\cap\{(a,b)_2\mid a\in b\}$
$R_6(x)=\{(b,a,c)_3\mid (a,b,c)_3\in x\}$
$R_7(x)=\{(b,c,a)_3\mid (a,b,c)_3\in x\}$
$\quad A_{14}(x,y)=x``\{y\}\,[=\mathrm{Dom}\,((x\cap([\bigcup\bigcup x]\times\{y\}))^{-1})]$
$R_8(x,y)=\{x``\{w\}\mid w\in y\}$

2·62 Proposition *Each of $R_0\ldots R_7$ and $A_0,\ldots A_{14}$ is DB_0-suitable; R_8 is GJ_0-suitable.*

2·63 Definition Let $\mathcal{B}$ be the closure of $R_0\ldots R_7$ under composition.

2·64 Proposition *Each function in $\mathcal{B}$ is DB_0-suitable.*

2·65 Proposition *For each Δ_0 class A the map $x\mapsto x\cap A$ is in $\mathcal{B}$.*

2·66 Remark That corresponds to the derivability of Δ_0 separation in DB_0.

2·67 DEFINITION Let $\mathcal{R}$ be the closure of $R_0 \dots R_8$ under composition.

2·68 PROPOSITION *Each function in* $\mathcal{R}$ *is* GJ_0*-suitable.*

The collection of functions in $\mathcal{R}$ is also closed under formation of images: by which is meant that if F is in $\mathcal{R}$ so is $x \mapsto F``x$. To prove this we introduce the notion of a *companion*. We will actually have two such notions.

Let T be some system of set theory extending DB, and let G and F be Δ_0 classes such that T proves that both G and F are total functions.

2·69 DEFINITION G is a *1-companion* of F in T if G is T-suitable and

$$\vdash_{\mathsf{T}} \vec{x} \in \vec{u} \Longrightarrow F(\vec{x}) \downarrow\in G(\vec{u})$$

2·70 DEFINITION H is a *2-companion* of F in T if H is T-suitable and

$$\vdash_{\mathsf{T}} \vec{x} \in \vec{u} \Longrightarrow F(\vec{x}) \downarrow\subseteq H(\vec{u})$$

where $\vec{x} \in \vec{u}$ abbreviates $x_1 \in u_1 \;\&\; \dots x_n \in u_n$ for an appropriate n.

The collection of functions with a 1-companion is easily seen to be closed under composition; but usually it is much easier to spot a 2-companion of a function. The following is easily verified by inspection.

2·71 PROPOSITION *Each of the functions* $R_0, \dots, R_7$ *and* A_{14} *has a 2-companion in* DB_0.

Generation of 1-companions from 2-companions and separators.

The Gandy–Jensen Lemma is the core of the proof that $\mathcal{R}$ is closed under formation of images. Versions of it are to be found in the papers of Gandy [G] and Jensen [J2]. We discuss it only for 1-ary functions.

2·72 THE GANDY–JENSEN LEMMA *Suppose that* H *is a 2-companion of* F*, and that '*$a \in F(b)$*' is* Δ_0*. Then* F *is generated by composition from* H *and members of* $\mathcal{B}$*; further* $F``x \in V$ *and* $F``$ *(as a function) is generated by* H *and members of* $\mathcal{R}$ *and (as a term) is* $\mathcal{S}$*-suitable and is a 1-companion of* F *in* $\mathcal{S}$*.*

Proof : We have

$$x \in u \Longrightarrow F(x) \subseteq H(u).$$

Form

$$h(u) \;=_{\mathrm{df}}\; \big(H(u) \times u\big) \cap \{(a,b)_2 | b \in u \;\&\; a \in F(b)\}.$$

Actually, we could just take

$$h(u) \;=_{\mathrm{df}}\; \big(H(u) \times u\big) \cap \{(a,b)_2 | a \in F(b)\}.$$

Since $a \in F(b)$ is Δ_0 and for each Δ_0 A, the separator $x \mapsto x \cap A$ is in $\mathcal{F}$ and is DB-suitable, we have that h is generated by H and functions in $\mathcal{F}$.

Now note that for $b \in u$, $F(b) = h(u)``\{b\} = A_{13}(h(u), b)$, so $F$ is built from $H$ and functions in $\mathcal{F}$; if $R_8$ is available, we may argue further that $F``u = R_8(h(u), u)$ so $F``$ is built from $H$ and rudimentary functions; hence $F``u \in V$, and this function $F``$ now forms a 1-companion of F. $\dashv$ (2·72)

Proofs that $\mathcal{R}$ is closed under the rudimentary schemata may be found in the cited works on fine structure.

A single generating function for rud(u)

Following Jensen, we define rud(u) to be the rud closure of $u \cup \{u\}$. Various functions with properties similar to those of the following may be found in the literature.

2·73 DEFINITION

$$
\begin{aligned}
\mathbb{T}(u) = u \cup \{u\}\\
&\cup [u]^1 \cup [u]^2\\
&\cup \{x \smallsetminus y \mid_{x,y} x, y \in u\}\\
&\cup \left\{\bigcup x \;\Big|_x\; x \in u\right\}\\
&\cup \left\{\mathrm{Dom}\,(x) \;\Big|_x\; x \in u\right\}\\
&\cup \{u \cap (x \times y) \mid_{x,y} x, y \in u\}\\
&\cup \left\{x \cap \{(a,\, b)_2 \mid_{a,b} a \in b\} \;\Big|_x\; x \in u\right\}\\
&\cup \left\{u \cap \{(b, a, c)_3 \mid_{a,b,c} (a, b, c)_3 \in x\} \;\Big|_x\; x \in u\right\}\\
&\cup \left\{u \cap \{(b, c, a)_3 \mid_{a,b,c} (a, b, c)_3 \in x\} \;\Big|_x\; x \in u\right\}\\
&\cup \left\{x``\{w\} \;\Big|_{x,w}\; x \in u, w \in u\right\}\\
&\cup \left\{u \cap \left\{x``\{w\} \;\Big|_w\; w \in y\right\} \;\Bigg|_{x,y}\; x, y \in u\right\}.
\end{aligned}
$$

2·74 REMARK The successive lines of the definition of $\mathbb{T}$, after the first, may be written more prosaically as $R_0``(u \times u)$, $R_1``(u \times u)$, $R_2``u$, $R_3``u$, $\{u \cap R_4(x, y) \mid_{x,y} x, y \in u\}$, $R_5``u$, $\{u \cap R_6(x) \mid_x x \in u\}$, $\{u \cap R_7(x) \mid_x x \in u\}$, $A_{14}``(u \times u)$ and $\{u \cap R_8(x, y) \mid_{x,y} x, y \in u\}$. It will be notationally convenient to treat all these functions as having three variables, so let us define $S_i(u; x, y) := R_i(x, y)$ for $i = 0, 1$; $S_i(u; x, y) := R_i(x)$ for $i = 2, 3, 5$; $S_i(u; x, y) := u \cap R_i(x, y)$ for $i = 4, 8$; $S_i(u; x, y) := u \cap R_i(x)$ for $i = 6, 7$; and $S_9(u; x, y) := A_{14}(x, y)$.

Then each of those lines is of the form $S_i``(\{u\} \times (u \times u))$ for some i. If we further define $S_{10}(u; x, y) := u$ and $S_{11}(u; x, y) := x$, then we are still within the class of rudimentary functions, as $\varnothing = R_1(x, x)$, $S_{10}(u; x, y) = R_1(u, \varnothing)$ and $S_{11}(u; x, y) = R_1(x, \varnothing)$, and, easily, $S_{11}``(\{u\} \times (u \times u)) = u$ and for non-empty $u$, $S_{10}``(\{u\} \times (u \times u)) = \{u\}$, so that $\mathbb{T}(\varnothing) = \{\varnothing\}$ and for u non-empty, $\mathbb{T}(u) = \bigcup_{i<12} S_i``(\{u\} \times (u \times u))$.

We have proved the first clause of the following, and the others are easy.

2·75 PROPOSITION *$\mathbb{T}$ is rudimentary, $u \subseteq \mathbb{T}(u)$ and $u \in \mathbb{T}(u)$. Further, if u is transitive, then $\mathbb{T}(u)$ is a set of subsets of u, and hence $\mathbb{T}(u)$ is transitive.*

2·76 REMARK It will not in general be true that $u \subseteq v \Longrightarrow \mathbb{T}(u) \subseteq \mathbb{T}(v)$, the problem being that $u \in \mathbb{T}(u)$, but if v is countably infinite, so is $\mathbb{T}(v)$ which therefore cannot contain all the subsets of v. Fortunately, $u \subseteq \mathbb{T}(u) \subseteq \mathbb{T}^2(u) \ldots$

2·77 LEMMA *For x, y in u, $R_4(x, y) = x \times y \subseteq u \times u \subseteq \mathbb{T}^2(u)$.*

2·78 COROLLARY *For x, y in u, $R_4(x, y) \in \mathbb{T}^3(u)$.*

2·79 LEMMA *For a, b c in u, $(a,\, c)_2 \in \mathbb{T}^2(u)$ and $(b, a, c)_3 \in \mathbb{T}^4(u)$.*

2·80 COROLLARY *For $x \in u$, $R_6(x)$ and $R_7(x)$ are in $\mathbb{T}^5(u)$.*

2·81 LEMMA *For x, $y \in u$, $R_8(x, y) \in \mathbb{T}^2(u)$.*

Proof : For x, w in u, $x``w \in \mathbb{T}(u)$, so $R_8(x, y) = \mathbb{T}(u) \cap \{x``w \mid_w w \in y\}$; $x, y \in \mathbb{T}(u)$, so $R_8(x, y) \in \mathbb{T}^2(u)$. $\dashv$ (2·81)

2·82 PROPOSITION *For any transitive u, $\bigcup_{n\in\omega} \mathbb{T}^n(u)$ is the rudimentary closure of $u \cup \{u\}$, and in it,* TCo *holds.*

2·83 PROBLEM I do not see how to form a single rud function which will in similar fashion give the rud closure of u. Perhaps this has something to do with the question of MacAloon and Stanley discussed inSection 14.

Other remarks on GJ

2·84 REMARK RR produces a collection of subsets of x.

2·85 PROPOSITION (Gandy; Jensen) *A transitive set is rud closed (= basically closed) iff it models* GJ_0.

2·86 REMARK GJ_0 proves that the cartesian product of two sets is a set.

2·87 REMARK Δ_0 separation is a theorem scheme of GJ_0.

2·88 PROPOSITION *RR is self-strengthening to*

$$(RR^+) \qquad \forall x_1 \forall x_2 \exists w \forall \vec{v}_{\in x_1} \exists t_{\in w} \forall u (u \in t \iff .u \in x_2 \ \& \ \phi[u, \vec{v}]).$$

for ϕ any Δ_0 formula.

2·89 PROBLEM Does GJ prove the existence of a bijection between ω and $\omega \times \omega$?

I suspect that BS does, as everything necessary is in **HF**.

The next result is a scheme of theorems:

2·90 PROPOSITION (GJ_0) *Each $[\omega]^{\mathfrak{k}}$ exists; indeed, each $[a]^{\mathfrak{k}}$ exists for any set a.*

Proof : $[a]^0 = \{\varnothing\} \in V$. $[a]^1 = A_0 \text{“} a \in V$. $[a]^{\mathfrak{k}+1} = \big\{ s \cup \{x\} \bigm| (s,x)_2 \in ([a]^{\mathfrak{k}} \times a) \cap \{(s,x)_2 \mid x \notin s\} \big\}$, which is in V, being of the form $h\text{“}b$ for some set b and rudimentary function h. ⊣ (2·90)

2·91 THEOREM (GJ) $\forall a \forall k_{\in \omega} \ [a]^k \in V$.

We omit the proof, it being similar to that of Theorem 2·93.

2·92 PROBLEM Is the quantified form provable without Π_1 foundation ?

2·93 THEOREM (GJ) $\forall a \forall m_{\in \omega} \, {}^m a \in V$.

Proof : Fix a, and consider the Π_1 class

$$\omega \cap \{m \mid \neg \exists x [\forall y_{\in x} (y : m \longrightarrow a \ \& \ \forall k_{\in m} \forall t_{\in a} \exists z_{\in x} (z \restriction k = y \restriction k \ \& \ z(k) = t))]\}.$$

The theorem states that that class is empty: if it is not, let m be its minimal element. But then m is either 0 or a successor; if 0, nothing to prove; if $m = k+1$, then ${}^k a$ exists and we can then form ${}^m a$ as the image of a rudimentary function applied to ${}^k a \times a$, since

$${}^{k+1} a = \big\{ f \cup \{(t,k)_2\} \bigm|_{f,t} f \in {}^k a \ \& \ t \in a \big\}. \qquad ⊣ (2·93)$$

2·94 PROBLEM Is ${}^m a$ suitable in any sense ? What seems to be true is that each ${}^{\mathfrak{k}} a$ is rud, and each $[a]^{\mathfrak{k}}$ but that $[b]^n$ is not a rud function of two variables, as, if it were, $\mathcal{S}(b,x) =_{\text{df}} \bigcup_{n \in x} [b]^n$ would be a rud function; but by Gandy the rud closure of $\omega + 1$ omits $\mathcal{S}(\omega) = \mathcal{S}(\omega, \omega)$.

On fReR

That GJ is a subsystem of fReR would follow from the theory of companions.

2·95 PROPOSITION (fReRI) *The graph of addition, and indeed of every primitive recursive function is a set.*

Proof : we prove first that $\forall n \exists f \ f \subseteq \omega \times (\omega \times \omega)$ with $\mathrm{Dom}\,(f) = n \times n$ and

$$\forall m :< n \forall k :< n [f(m,0) = m \ \& \ f(m, k \dot{+} 1) = f(m,k) \dot{+} 1].$$

The collection of all such f's is a set, of which the union will be the graph of addition. ⊣ (2·95)

2·96 COROLLARY *The Ackermann relation may be proved to exist in* fReR.

2·97 COROLLARY (fReRI) $\mathcal{S}(\omega) \in V$.

For another proof, one may reflect that every finite set of natural numbers is of the form

$$\{i \mid \mathsf{p}_i \text{ divides } n\}$$

for some n, where p_i is the i^{th} rational prime.

2·98 COROLLARY (fReRI) EVEN *is a set.*

2·99 PROPOSITION (fReRI) *If x is countable then $\mathcal{S}(x)$ exists.*

2·100 PROBLEM Does fReR prove that each $\mathcal{S}(x)$ is a set ? or at least that each $\mathcal{S}(\zeta)$ exists ?

It may be that in a model with amorphous sets in the sense of Truss, there will be difficulties.

2·101 PROPOSITION fReR *is self-strengthening to allowing ϕ in (BdR) to have further free variables.*

Proof : Note that if $\mathrm{Rel}(s)$ and $\mathrm{Dom}\, s \neq \varnothing$ and $s \subseteq z \times \{w\}$, then $s = y \times \{w\}$ for some $y \subseteq z$; further, $\mathrm{Dom}\, s = \{w\}$, $\bigcup \mathrm{Dom}\, s = w$ and $\mathrm{Im}\, s = y$.

Let $\psi(x, s) \iff_{\mathrm{df}} \mathrm{Rel}(s)\ \&\ \mathrm{Dom}\, s \neq \varnothing\ \&\ \phi(x, \mathrm{Im}\, s, \bigcup \mathrm{Dom}\, s)$. Then ψ is Δ_0. Let $z_1 = z \times \{w\}$, and suppose that $\forall x_{\in u}\, \exists! y[\phi(x, y, w)\ \&\ y \subseteq z.]$ That tells us that

$$\forall x_{\in u}\, \exists! s[\psi(x, s)\ \&\ s \subseteq z_1],$$

so applying (BdR), we deduce that the class $\{y \times \{w\} \mid \exists x_{\in u}\, \phi(x, y, w)\ \&\ y \subseteq z\}$ is a set, v, say. Then applying an appropriate rudimentary function, we see that the class $\{\mathrm{Im}\, t \mid t \in v\}$ is a set; but that class is $\{y \mid \exists x_{\in u}\, \phi(x, y, w)\ \&\ y \subseteq z\}$, as desired. $\dashv$ (2·101)

On ReRI

2·102 PROPOSITION (ReRI) $\omega + \omega \in V$.

Proof : $\forall n_{\in\omega}\, \exists f[\mathrm{Fn}(f)\ \&\ \mathrm{Dom}\,(f) = n + 1\ \&\ (f(0) = \omega)\ \&\ \forall m :< n f(m \dot{+} 1) = f(m) \dot{+} 1]$, by an easy application of Π_1 foundation, and for each n there cannot be two distinct such f's. Hence by Δ_0 replacement, the set F of those f's exists, and $\omega + \omega$ will be $\mathrm{Im}\,(\bigcup F)$. $\dashv$ (2·102)

2·103 PROPOSITION (ReRI) $\mathcal{S}(x) \in V$

Proof : Fix x. Let G be the rudimentary function given by $G(y, z) = \{a \cup b \mid a \in y\ \&\ b \in [z]^1\}$. We seek to define a function $f : \omega \longrightarrow V$ by the following recursion:

$$f(0) = [x]^1; f(n + 1) = G(f(n), x).$$

We call f a *G-attempt at n* if

$$Fn(f)\ \&\ \mathrm{Dom}\,(f) = n + 1\ \&\ f(0) = [x]^1\ \&\ \forall k_{\in n}\, f(k + 1) = G(f(k), x).$$

Using set foundation it is easily seen that any two G-attempts agree on their common domain, so that there is at most one attempt at n; and, using Π_1 foundation to obtain a minimal element of the class of those n at which there is no attempt, we see that that class in fact must be empty, and hence that there is a unique attempt at each n.

Since being an attempt is Δ_0 in our present system, ReRI proves that there is a set containing (exactly) the attempts for each n. The union of that set is therefore a set and a function, and the union of its image is $\mathcal{S}(x)$. $\dashv$ (2·103)

2·104 REMARK A similar argument will show in ReRI that the transitive closure of any set exists.

2·105 PROPOSITION (ReRI) $\mathbf{HF} \in V$

We omit the proof as the Proposition is a special case of Proposition 8·28.

2·106 REMARK I would guess that ReRI suffices to define the relation $u \models \varphi$, and the constructible hierarchy; and that the L of a model of ReRI is a model of KPI, so that indeed the two theories are equiconsistent.

2·107 PROBLEM Does ReR prove TCo?

On KP

2·108 PROPOSITION (KP) TCo

Proof : Let $A = \{x \mid \forall u \bigcup u \subseteq u \Longrightarrow x \notin u\}$. By Π_1 foundation, A, if non-empty, has an $\in$-minimal element $\bar{x}$. So $\forall x_{\in \bar{x}}\, \exists u \bigcup u \subseteq u\ \&\ x \in u$. By Δ_0 Collection there is a v such that $\forall x_{\in \bar{x}}\, \exists u_{\in v} \bigcup u \subseteq u\ \&\ x \in u$. Let $w = v \cap \{u \mid \bigcup u \subseteq u\}$. w is a set by Δ_0 separation; let $\bar{u} = \bigcup w$. The $\bar{u}$ is transitive and $\bar{x} \subseteq \bar{u}$. Hence $\bar{x}$ is a member of the transitive set $\bar{u} \cup \{\bar{x}\}$, and is therefore in A, a contradiction. $\dashv$ (2·108)

3: Remarks on transitive models

Many of our models are of the following simple kind. We define a class $\mathbf{A}$ of transitive sets, and take $\mathbf{M} = \bigcup \mathbf{A}$.

3·0 PROPOSITION *i) Such an* $\mathbf{M}$ *will always be transitive, and will model the Axiom of Extensionality and the full scheme of Foundation for all classes, and be absolute for all* Δ_0 *formulæ.*

ii) If $\mathbf{A}$ *is non-empty, the axiom* $\varnothing \in V$ *will be true in* $\mathbf{M}$*; if* $\omega + 1 \in \mathbf{A}$ *then* $\mathbf{M}$ *will model* $\omega \in V$.

iii) If $u \in \mathbf{A}$ and $y \subseteq u$ *implies* $u \cup \{y\} \in \mathbf{A}$*, then* $\mathbf{M}$ *will model the sumset axiom; further* $\mathbf{M}$ *will be supertransitive and will therefore model the full separation scheme; and* $\mathbf{A}$ *will be a subclass of* $\mathbf{M}$*, which will therefore model* TCo*, and indeed the transitive closure of any member of* $\mathbf{M}$ *will also be a member of* $\mathbf{M}$.

iv) If the hypothesis of (iii) holds and, additionally, $u \in \mathbf{A}$ *and* $v \in \mathbf{A}$ *implies* $u \cup v \in \mathbf{A}$*, then* $\mathbf{M}$ *will model* AxPair.

The proof is straightforward. Models of that kind, therefore, are always models of Gandy's system ReS_0 with TCo, and with full foundation and full separation.

3·1 REMARK Just to clarify that last remark: to prove full foundation in the model, we require (if the model be a proper class) full foundation in the background theory; and similarly for full separation.

Slim models of weak systems

Many such classes $\mathbf{A}$ can be found by modifying a definition to be found in *Slim Models of Zermelo Set Theory* [M1]:

3·2 DEFINITION $\mathcal{T}$ is *weakly fruitful* if

(i) every x in $\mathcal{T}$ is transitive;

(iii) $x \in \mathcal{T}$ & $y \in \mathcal{T} \Longrightarrow x \cup y \in \mathcal{T}$;

(iv') $x \in \mathcal{T}$ & $a \subseteq x \Longrightarrow x \cup \{a\} \in \mathcal{T}$.

The missing condition (ii) lists three possible conditions on the ordinals in the class $\mathcal{T}$:

(ii) $1 \in \mathcal{T}$; $\omega + 1 \in \mathcal{T}$; $ON \subseteq \mathcal{T}$, respectively;

So our theorem above gives the following:

3·3 PROPOSITION *If* $\mathcal{T}$ *is weakly fruitful, then* $\bigcup \mathcal{T}$ *will be a supertransitive model of* ReS_0 *with* TCo*, full separation and full foundation, and if* $1 \in \mathcal{T}$*, of Empty Set; if* $\omega + 1 \in \mathcal{T}$*, the axiom of infinity will hold in* $\bigcup \mathcal{T}$ *in the form* $\omega \in V$*, and in the third case, the model* $\bigcup \mathcal{T}$ *will contain all ordinals.*

There is a simple further requirement on $\mathbf{A}$ that ensures that $\bigcup \mathbf{A}$ is closed under cartesian products. Recall our definition from section 2:

DEFINITION $u^\star =_{\mathrm{df}} u \cup [u]^1 \cup [u]^2 \cup (u \times u)$.

3·4 LEMMA $u^\star$ *is* BS *suitable; if* u *is transitive, so is* $u^\star$*, and* $u \times u \subseteq u^\star$.

3·5 PROPOSITION *If* $\mathbf{A}$ *is a collection of transitive sets closed under* **, union of two elements, adding a subset to an element, and containing the set* $\omega + 1$*, then* $\bigcup \mathbf{A}$ *will model* BS *with* TCo *and* $\mathsf{full\ Separation}$.

As in *Slim Models*, we may obtain some interesting examples of such models by estimating the rate of growth of various transitive sets. Given a function $Q : \omega \longrightarrow V$, set $f^Q_x(n) = \overline{\overline{x \cap Q(n)}}$. For $\mathcal{G}$ a class of functions, form $\mathcal{T}^{Q,\mathcal{G}} =_{\mathrm{df}} \{x \mid \bigcup x \subseteq x \ \&\ f^Q_x \in \mathcal{G}\}$.

3·6 PROPOSITION *If* $\mathcal{G}$ *has these properties then* $\mathcal{T}^{Q,\mathcal{G}}$ *will be weakly fruitful:*

$f \leqslant g \in \mathcal{G} \Longrightarrow f \in \mathcal{G}$;

$f, g \in \mathcal{G} \Longrightarrow f + g \in \mathcal{G}$;

$\bigcup x \subseteq x \ \&\ f^Q_x \in \mathcal{G} \Longrightarrow f^Q_x + 1 \in \mathcal{G}$.

The three conditions on ordinals considered correspond to the three requirements

$f^Q_1 \in \mathcal{G}$; $f^Q_{\omega+1} \in \mathcal{G}$; $\forall \zeta f^Q_\zeta \in \mathcal{G}$.

3·7 PROPOSITION *A sufficient further condition on $\mathcal{G}$ for cartesian products to exist in $\bigcup \mathcal{T}^{Q,\mathcal{G}}$, when $Q(n) = V_n$, is this:*

$$(f \in \mathcal{G} \;\&\; g \in \mathcal{G} \;\&\; C \in \omega) \Longrightarrow C.f.g \in \mathcal{G}$$

Proof : We must show that in these circumstances, $u \in \mathcal{T} \Longrightarrow u^\star \in \mathcal{T}$. Note that for $n \geqslant 2$,

$$\overline{\overline{[u]^1 \cap V_n}} = \overline{\overline{u \cap V_{n-1}}}; \qquad \overline{\overline{[u]^2 \cap V_n}} \leqslant (\overline{\overline{u \cap V_{n-1}}})^2; \qquad \overline{\overline{(u \times u) \cap V_n}} = (\overline{\overline{u \cap V_{n-2}}})^2.$$

Hence $f^Q_{u^\star}(n) = \overline{\overline{u^\star \cap V_n}} \leqslant f^Q_u(n) + f^Q_u(n-1) + (f^Q_u(n-1))^2 + (f^Q_u(n-2))^2$. Since $V_n \subseteq V_{n+1}$ each f^Q_u is monotonic; the proposition now follows by elementary analysis. ⊣ (3·7)

Of our collection, Models 3, 5 and 8 are obtained by the above rate-of-growth method, of which the last two model the Axiom of Cartesian Products. Models 1, 2, 4, 6, 7, 9, and 10 are obtained by a different method, which we now describe.

3·8 PROPOSITION *Let X be a class. Put $\mathbf{A}^X$ = the class of those transitive u whose intersection with X is finite. Then $\mathbf{M}^X =_{\text{df}} \bigcup \mathbf{A}^X$ will be supertransitive and will model extensionality; foundation; full separation, difference and $\bigcup$; pairing; and* TCo, *since $\mathbf{A}^X \subseteq \mathbf{M}^X$; as long as X contains only finitely many ordinals, $\mathbf{M}^X$ will model infinity; if u in $\mathbf{A}^X$ implies $u^\star$ is in $\mathbf{A}^X$ then $\mathbf{M}^X$ will be closed under cartesian products.*

Models 11–15 are obtained by yet other methods. TCo holds in all these models; all are supertransitive save for Model 14 and some variants of Model 11.

4: Models of ReS

Gandy: A set which models PZ but not BST.

We take $\mathbf{G}_1$ to be the class of all x such that everything in $tcl(\{x\})$ is either finite or differs from ω by a finite set. Gandy remarks that (a) $\mathbf{G}_1$ is transitive; (b) if x is in $\mathbf{G}_1$ $\bigcup x$ is a subset of $\mathbf{G}_1$; (c) $\omega \in \mathbf{G}_1$; (d) $\mathbf{G}_1$ contains every finite subset of itself, and every x in $\mathbf{G}_1$ is a substitutable constant in his sense. (e) $\mathbf{G}_1$ satisfies Δ_0 separation, the proof of which uses the fact that every Δ_0 subset of ω is finite or cofinite, by his quantifier elimination lemma. (f) $\omega \times \omega$ is not in A.

It follows from those remarks that $\mathbf{G}_1$ is not supertransitive and that $\mathbf{G}_1 \cap ON = \omega + \omega$. We verify the following in detail:

4·0 PROPOSITION *If $x \in \mathbf{G}_1$ then so are $\bigcup x$ and* $\mathrm{tcl}(x)$.

Now $\mathrm{tcl}(\{\bigcup x\}) = \{\bigcup x\} \cup \mathrm{tcl}(\bigcup x)$ and $\mathrm{tcl}(\bigcup x) \subseteq \mathrm{tcl}(x) \subseteq \mathrm{tcl}(\{x\})$, so it is enough to prove that if x is in $\mathbf{G}_1$, $\bigcup x$ is either finite or almost ω.

First note that if x is finite and in $\mathbf{G}_1$, then $x = y \cup z$, where y is the set of finite members of x and z is the set of members of x which are infinite and therefore almost equal to ω. If z is empty, then $\bigcup x = \bigcup y$, and is thus finite. If z is non-empty, then $\bigcup x = \bigcup y \cup \bigcup z$; y and z are both finite, and so $\bigcup y$ will be finite, and $\bigcup z$ will be almost equal to ω. Hence $\bigcup x$ is almost equal to ω.

Thus we have verified that if x is a finite member of $\mathbf{G}_1$ then $\bigcup x \in \mathbf{G}_1$.

If on the other hand, x almost equals ω, then we can write $x = y \cup z$ where z is a cofinite subset of ω, and y is a finite set disjoint from $\omega = \varnothing$. As $\mathbf{G}_1$ is transitive, y is a a finite subset of it, and therefore a member of it, and therefore $\bigcup y \in \mathbf{G}_1$, by the previous paragraph. So $\bigcup x = \bigcup y \cup \omega$; $\bigcup y$ is either finite or almost ω; either way, $\bigcup x$ is almost ω.

To show that $x \in \mathbf{G}_1 \Longrightarrow \mathrm{tcl}(x) \in \mathbf{G}_1$, suppose that x is a counterexample of minimal rank. It is enough to show that $\mathrm{tcl}(x)$ is either finite or almost ω.

$$\mathrm{tcl}(x) = x \cup \bigcup_{t \in x} \mathrm{tcl}(t),$$

where by the minimality of x each $\mathrm{tcl}(t)$ is in $\mathbf{G}_1$.

4·1 REMARK The displayed formula implies easily that $\mathrm{tcl}(a \cup b) = \mathrm{tcl}(a) \cup \mathrm{tcl}(b)$.

So if x is finite, $\mathrm{tcl}(x)$ is the union of a finite set and finitely many sets each either finite or almost ω, so that $\mathrm{tcl}(x)$ itself must be either finite or almost ω, and therefore in $\mathbf{G}_1$. Thus the minimal counterexample must be almost ω.

But now we may write x as the union of a finite set y disjoint from ω and a cofinite subset z of ω. We know that $\mathrm{tcl}(y) \in \mathbf{G}_1$ by the argument of the previous paragraph, the rank of y not exceeding that of x, and that $\mathrm{tcl}(z) = \omega$, so that again $\mathrm{tcl}(x)$, being the union of a pair of elements of $\mathbf{G}_1$ is itself in $\mathbf{G}_1$.

Model 1: A model of ReS with full separation in which cartesian products are absent

Consider, working in some suitable theory such as ZF, the class $\mathbf{A}_1$ of all transitive sets which contain but finitely many ordered pairs.

Then $\mathbf{M}_1 = \bigcup \mathbf{A}_1$, which is the same as the class of all sets x such that $\mathrm{tcl}(x)$ contains but finitely many ordered pairs,is supertransitive and contains all ordinals, and models Extensionality, AxPair, Sum Set, Infinity and full Separation, full foundation and TCo. $\omega \in \mathbf{M}_1$ but $\omega \times \omega$ is not. Indeed the cartesian product of an infinite set and a non-empty set is never there; but the cartesian product of two finite sets is there, so in this model a set a is finite if and only if $a \times a \in V$.

4·2 REMARK Note also that the graph of addition is not present in this model, since its domain would be $\omega \times \omega$, and the domain can be recovered using the axioms of union and Δ_0 separation.

4·3 REMARK $\mathcal{S}(\omega) \in \mathbf{M}_1$; indeed for each ordinal ζ, $\mathcal{S}(\zeta) \in \mathbf{M}_1$.

4·4 REMARK $\mathbf{M}_1$ contains no bijection between ω and $\mathcal{S}(\omega)$. For a bijection would be an infinite set of ordered pairs. Indeed, $\mathbf{M}_1$ contains no functions with infinite domain !

Model 1a

Write $\mathcal{S}(x)$ for the set of finite subsets of x. Then in $\mathbf{M}_1$, $\mathcal{S}(\omega)$ exists, but $\mathcal{S}(\mathcal{S}(\omega))$ does not. Indeed if a is infinite, $\mathcal{S}(\mathcal{S}(a))$ never exists. So let $\mathbf{M}_{1a}$ be the set of members x of $\mathbf{M}_1$ such that $\mathcal{S}(y)$ exists in $\mathbf{M}_1$ for each member y of $\mathrm{tcl}(\{x\})$. Then the model $\mathbf{M}_{1a}$ contains all ordinals but not $\mathcal{S}(\omega)$, and in it, a is finite iff $\mathcal{S}(a)$ exists iff $\mathcal{P}(a)$ exists. What else is true there ?

Model 2: A model of ReS with full separation in which $[\omega]^1$ and $[\omega]^2$ do not exist

Take $\mathbf{A}_2$ to be the class of those transitive u such that $\{x \in u \mid \overline{\overline{x}} \leqslant 2\}$ is finite, and $\mathbf{M}_2$ to be $\bigcup \mathbf{A}_2$.

4·5 REMARK If we look at $\mathbf{C}$, the class of those x such that $\mathrm{tcl}(x)$ contains only finitely many sets of cardinality 2, we get a model that is nearly the same as the model $\mathbf{M}_1$; the chief difference seems to be that $[\omega]^\omega$ is not a member of $\mathbf{C}$, but is a member of $\mathbf{M}_1$.

4·6 REMARK We shall return to this mode of construction for Model 6.

Model 3: ringing the changes

Consider for any given $\mathfrak{k}$ the set $\mathbf{A}_{3,\mathfrak{k}}$ of those u with f_u $O(n^{\mathfrak{k}})$. This gives a model $\mathbf{M}_{3,\mathfrak{k}}$ of full separation in which Cartesian product will fail. $[\omega]^{\mathfrak{k}}$ will be in the model but not $[\omega]^{\mathfrak{k}+1}$.

The arguments are modifications of those of [M1]: a similar argument is worked in detail below.

Model 4: asymmetry of cartesian product

Let $\mathbf{A}_4 = \{u \mid u \text{ is transitive and } (V \times \{\omega\}) \cap u \text{ is finite }\}$.

Put $\mathbf{M}_4 = \bigcup \mathbf{A}_4$. Then $\omega \times \{\omega\} \notin \mathbf{M}_4$, but both $\{\omega\} \times \omega$ and $\omega \times \{\omega + 1\}$ are in $\mathbf{M}_4$.

4·7 In one of our later systems we would be able to define the *right Wiener–Kuratowski rank* of a set by this rudimentary recursion:

$$\varrho_{rWK}(x) = \begin{cases} 0 & \text{if } x \text{ is not an ordered pair} \\ 1 + \varrho_{rWK}(\mathrm{right}(x)) & \text{otherwise} \end{cases}$$

and prove that for any x, $\varrho_{rWK}(x) < \omega$.

For the moment we content ourselves with a weak form, for which S_0 is adequate, and which will be useful for some of our model-building:

4·8 Definition The *weak right Wiener–Kuratowski rank* is defined by cases:

$$\varrho_{wrWK}(x) = \begin{cases} 0 & \text{if } x \text{ is not an ordered pair} \\ 1 & \text{if } x \text{ is an ordered pair but right}(x) \text{ is not} \\ 2 & \text{if both } x \text{ and right}(x) \text{ are ordered pairs} \end{cases}$$

Now, for a variant of Model 4, take X to be the class of those sets of weak right WK rank 2. Then $\omega \times (\omega \times \omega)$ will not be in M_X, whereas $(\omega \times \omega) \times \omega$ will be.

Hence we have the curiosity that in this model, there will be a bijection one way but not the other.

5: Models of DB

Model 5: A slim model for Devlin

5·0 Proposition *There is a supertransitive model of* DB *containing all ordinals but omitting the set of finite sets of natural numbers.*

Write f_u for the map $n \mapsto \overline{\overline{u \cap V_n}}$. Write g_k for the map $n \mapsto n^k$.

5·1 Definition Let $\mathbf{A}_5$ be the class of transitive sets u such that the map f_u is dominated (i.e. eventually majorised) by some g_k. Let $\mathbf{M}_5 = \bigcup \mathbf{A}_5$.

5·2 Lemma $\mathbf{A}_5 \subseteq \mathbf{M}_5$.

Proof : If $u \in \mathbf{A}_5$, then $u \in u \cup \{u\} \in \mathbf{A}_5$. ⊣ (5·2)

5·3 Lemma $\mathbf{M}_5$ *is transitive, being the union of transitive sets.*

5·4 Lemma $\mathbf{M}_5$ *is supertransitive.*

Proof : If $x \subseteq y \in u \in \mathbf{A}_5$ then $x \subseteq u$; put $v = u \cup \{x\}$. v is transitive and for each n $\overline{\overline{v \cap V_n}} \leqslant \overline{\overline{u \cap V_n}} + 1$, so $v \in \mathbf{A}_5$. ⊣ (5·4)

5·5 Corollary (Z) $\mathbf{M}_5$ *models extensionality, difference, full foundation and full separation.*

5·6 Lemma $\omega \in \mathbf{M}_5$*: indeed,* $\mathbf{A}_5$ *contains all ordinals.*

5·7 Lemma *For each* k*,* $[\omega]^k$ *is in* $\mathbf{M}_5$.

Proof : $u_k =_{\text{df}} \omega \cup [\omega]^k \cup \{[\omega]^k\}$ is transitive. $\overline{\overline{(u_k \cap V_n)}} = \binom{n}{k} < n^k$. ⊣ (5·7)

5·8 Remark Indeed for each $x \in \mathbf{M}_5$ and each $k \in \omega$, $[x]^k \in \mathbf{M}_5$.
Proof : Fix $x \in u \in \mathbf{A}_5$ and $k \in \omega$. It is enough to show that $[u]^k$ is in $\mathbf{M}_5$. Let $v = u \cup [u]^k \cup \{[u]^k\}$: then v is transitive. We shall show that v is in $\mathbf{A}_5$. Note that $a \in [u]^k \cap V_{n+1} \iff a \subseteq V_n \;\&\; a \in [u]^k$, so that $[u]^k \cap V_{n+1} = [V_n \cap u]^k$. So if $V_n \cap u$ is of size $O(n^\ell)$, then $V_{n+1} \cap [u]^k$ is of size $O(n^{k\ell})$. ⊣ (5·8)

5·9 Lemma $[\omega]^{<\omega}$ *is not in* $\mathbf{M}_5$.

Proof : Suppose $[\omega]^\omega \in u$, a transitive set. Then $\overline{\overline{u \cap V_n}} \geqslant 2^n$, and the map $n \mapsto 2^n$ eventually strictly dominates all the $n \mapsto n^k$'s. ⊣ (5·9)

5·10 Corollary $\mathcal{P}(\omega) \notin \mathbf{M}_5$.

5·11 Lemma $\varnothing \in \mathbf{M}_5$.

5·12 Lemma *If* a *and* b *are in* $\mathbf{M}_5$ *so is* $\{a, b\}$.

Proof : Let $a \in u \in \mathbf{A}_5$ and $b \in v \in \mathbf{A}_5$. Put $w = u \cup v$. Then f_w is dominated by $f_u + f_v$, so if f_u is dominated by g_k and f_v by g_ℓ, then f_w is dominated by $g_{\max(k,\ell)+1}$. ⊣ (5·12)

5·13 Lemma *If* a *is in* $\mathbf{M}_5$*, so is* $\bigcup a$.

Proof : Let $a \in u \in \mathbf{A}_5$. Then $a \subseteq u$, so $\bigcup a \subseteq \bigcup u \subseteq u$; as before $\{\bigcup a\} \cup u$ will be in $\mathbf{A}_5$. ⊣ (5·13)

5·14 Lemma TCo *holds in* $\mathbf{M}_5$*; indeed* $x \in \mathbf{M}_5 \Longrightarrow tcl(x) \in \mathbf{M}_5$.

Proof : Let $v = tcl(x)$ where $x \in u \in \mathbf{A}_5$. Then $v \subseteq u$ and is therefore in $\mathbf{M}_5$ by supertransitivity. ⊣ (5·14)

5·15 LEMMA *If a and b are in $\mathbf{M}_5$ so is $a \times b$.*

Proof : it is enough to show that if u is in $\mathbf{A}_5$, then $u^\star \in \mathbf{A}_5$. By the reasoning in the proof of Proposition 3·7, if f_u is dominated by g_k then $f_{u^\star}(n)$ for sufficiently large n is at most $n^k + (n-1)^k + (n-1)^{2k} + (n-2)^{2k}$ which in turn is at most $4g_{2k}(n)$; thus $f_{u^\star}$ is dominated by g_{2k+1} and $u^\star$ is accordingly in $\mathbf{A}_5$. ⊣ (5·15)

5·16 LEMMA *if $x \in u \in \mathbf{A}_5$, then* $\mathrm{Dom}\, x \subseteq u$ *and is thus in* $\mathbf{M}_5$.

The following verifications are related to the finite axiomatisation of DB. We check that for a in $\mathbf{M}_5$,

$$a \cap \{(p,q)_2 \mid p \in q\} \in V$$
$$\{\langle q,p,r\rangle \mid \langle p,q,r\rangle \in a\} \in \mathbf{M}_5$$
$$\{\langle q,r,p\rangle \mid \langle p,q,r\rangle \in a\} \in \mathbf{M}_5$$

The first is immediate by supertransitivity, and for the other two, if $a \in u \in \mathbf{A}_5$, both the given classes are contained in $u \times (u \times u)$, and are thus in $\mathbf{M}_5$ by supertransitivity.

5·17 REMARK The model being supertransitive, the set of even numbers is in it. That is of interest, because that was Gandy's test set, studied in Section 2. His arguments use quantifier elimination; our examples do not.

We show that $\mathbf{M}_5$ is not a model of GJ. Recall the definition of the Ackermann relation $\mathrm{ACK} \subseteq \omega \times \omega$: $m\, \mathrm{ACK}\, n$ if and only if 2^m is one of the summands in the binary expression of n as a sum of powers of 2.

5·18 LEMMA $\mathrm{ACK} \in \mathbf{M}_5$.

Proof : $\omega \times \omega \in \mathbf{M}_5$ and $\mathbf{M}_5$ is supertransitive. ⊣ (5·18)

5·19 PROPOSITION $\mathbf{M}_5$ *is not a model of* GJ.

Proof : $\{\mathrm{ACK}``\{n\} \mid n \in \omega\} = [\omega]^{<\omega}$. By Lemmata 5·9 and 5·18, Axiom R_8 fails in $\mathbf{M}_5$. ⊣ (5·19)

5·20 REMARK The graph of addition is present in this model, as it will be in any supertransitive model of DB_0 containing ω; one may also argue directly that if u is the transitive closure of the singleton of that graph, f_u is dominated by g_3.

5·21 REMARK Gandy's model $\mathbf{G}_2$, given below, is a model of GJI without the graph of addition; the submodel $\bigcup(\mathbf{G}_2 \cap \mathbf{A}_5)$ will be supertransitive relative to $\mathbf{G}_2$, and will be a transitive model of DB, indeed of BS, in which GJ fails and in which the graph of addition is absent.

Model 6

We consider a variant of the construction $\mathbf{M}_2$ of section 2.

Here we wish to study the extent to which DB proves the existence of the sets $[\omega]^{\mathfrak{k}}$

5·22 PROPOSITION *For any $\mathfrak{k} \geqslant 3$,* DB*, if consistent, fails to prove that $[\omega]^{\mathfrak{k}}$ exists.*

Fix $\mathfrak{k} \geqslant 3$. We shall exhibit a supertransitive model $\mathbf{M}_{6,\mathfrak{k}}$ of DB in which $[\omega]^\ell$ exists iff $\ell \neq \mathfrak{k}$.

5·23 REMARK Indeed the existence of $[\omega]^\ell$ for different ℓ is independent. So we can code an arbitrary subset of ω into the theory of such a model.

Guided by Proposition 3·8, we let $X_{6,\mathfrak{k}}$ be the class of all sets of cardinality $\mathfrak{k}$, we take $\mathbf{A}_{6,\mathfrak{k}}$ to be the class of all transitive u such that $u \cap X_{6,\mathfrak{k}}$ is finite, and $\mathbf{M}_{6,\mathfrak{k}}$ to be $\bigcup \mathbf{A}_{6,\mathfrak{k}}$. Then that will model S_0 with full separation and full foundation; for $\mathfrak{k} \geqslant 3$, it will model Cartesian Product, since then for u transitive, $X_{6,\mathfrak{k}} \cap \big([u]^1 \cup [u]^2 \cup (u \times u)\big) = \varnothing$, and so $u \in \mathbf{A}_{6,\mathfrak{k}} \Longrightarrow u^\star \in \mathbf{A}_{6,\mathfrak{k}}$.

If $\mathfrak{l} \neq \mathfrak{k}$, then for each x in $\mathbf{M}_{6,\mathfrak{k}}$, $[x]^{\mathfrak{l}}$ will be in $\mathbf{M}_{6,\mathfrak{k}}$: if $x \in u \in \mathbf{A}_{6,\mathfrak{k}}$, $[x]^{\mathfrak{l}} \subseteq [u]^{\mathfrak{l}}$; $u \cup [u]^{\mathfrak{l}}$ is transitive, and its intersection with $X_{6,\mathfrak{k}}$ equals $u \cap X_{6,\mathfrak{k}}$, and is therefore finite. By the supertransitivity of $\mathbf{M}_{6,\mathfrak{k}}$, $[x]^{\mathfrak{l}} \in \mathbf{M}_{6,\mathfrak{k}}$.

On the other hand for no infinite member x of $\mathbf{M}_{6,\mathfrak{k}}$ will $[x]^{\mathfrak{k}}$ be in $\mathbf{M}_{6,\mathfrak{k}}$, as no member of $\mathbf{M}_{6,\mathfrak{k}}$ can have infinitely many members of cardinality $\mathfrak{k}$.

So it will also be true that ${}^{\mathfrak{k}}\omega$ is not in the model, although $\omega \times (\omega \times (\ldots))$ ($\mathfrak{k}$ times) will be.

5·24 REMARK Consider the case $\mathfrak{k} = 3$: the graph of addition, implemented (as we do) as a subset of $\omega \times (\omega \times \omega)$, is a member of $\mathbf{M}_{6,3}$, but implemented as a set of 3-tuples is not, since in that model, no infinite subset of ${}^3\omega$ exists. Thus these weak theories are extremely sensitive to the implementation of functions, a point that is touched on by Stanley in his review [St] of Devlin's book [De].

5·25 Remark If we ask that for each k u contains only finitely many sets of size k, the resulting model, though containing all the ordinals, will contain none of the sets $[\omega]^k$; if we ask for u to contain only finitely many finite sets, the resulting model will be **HF**, given that we are using the Axiom of Foundation. In a universe with Quine atoms, of course, the situation would be different.

A variant of Model 6

Let $\mathbf{A} = \{u \mid \bigcup u \subseteq u \ \&\ u \cap {}^3[\omega, \omega+\omega) \text{ is finite}\}$, and let $\mathbf{M} = \bigcup \mathbf{A}$. Then $\mathbf{HF} \in \mathbf{M}$ but ${}^3[\omega, \omega+\omega)$ is not. $\mathbf{M}$ contains all ordinals and is a supertransitive model of BS.

Model 7: a failure of $\bigcup$ “

Here we shall exhibit a transitive model of BS in which the following failure of GJ occurs: there is a set B such that $\{\bigcup x \mid x \in B\}$ is not a set.

Following Proposition 3·8, take X to be the class of transitive sets of limit rank, $\mathbf{A}_7$ to be A^X, the class of all transitive sets u such that only finitely many transitive sets of limit rank are members of u, and $\mathbf{M}_7$ to be $\bigcup \mathbf{A}_7$.

Then $\mathbf{M}_7$ is a supertransitive model of ReS_0 + full Foundation + TCo; “$x \times y \in V$” will be true in it since for u transitive, $u^\star \cap X = u \cap X$, as all members of $[u]^1 \cup [u]^2 \cup (u \times u)$ are non-empty finite sets and therefore of successor rank; and it contains all the ordinals below ω^2, and thus models the axiom of infinity. To prove the failure of GJ, we turn to the idea of a *Zermelo tower* from [M1], which is defined thus:

5·26 Definition For a any set, put

$Z_0(a) = \varnothing$; $Z_1(a) = \{a\}$; $Z_{n+1}(a) = \{a\} \cup (\mathcal{P}(Z_n(a)) \smallsetminus \{\varnothing\})$; $Z(a) = \bigcup_{n\in\omega} Z_n(a)$.

If one thinks of **HF** as a collection of words in $\varnothing$, $\{$ and $\}$ then $Z(a)$ is the collection of the corresponding words with a substituted for $\varnothing$ throughout. Thus every member either is a finite non-empty set or equals a.

Now let $\mathcal{X}$ be the set of those subsets a of $\omega+1$ of which ω is a member.

For each such a let $x(a) =_{\mathrm{df}} \{Z_n(a) \mid n \in \omega\}$. The rank of $x(a)$ is $\omega+\omega$.

Let $x^*(a) = x_a \cup \{\omega+1\}$. All the members of $x^*(a)$ are of successor rank, and so $x^*(a)$ is not transitive, but $\bigcup x^*(a) = Z(a) \cup (\omega+1)$ which is transitive, and of rank $\omega+\omega$; its only transitive member of limit rank is ω; thus each $x^*(a)$ is in $\mathbf{M}_7$.

Take B to be $\{x^*(a) \mid a \in \mathcal{X}\}$. Note that

$$\mathrm{tcl}(\{B\}) = \{B\} \cup B \cup \{Z_n(a) \mid n \in \omega \ \&\ a \in \mathcal{X}\} \cup \{\omega+1\} \cup \omega+1,$$

a transitive set of which the sole transitive member of limit rank is ω. Hence $B \in \mathbf{M}_7$; but $\{\bigcup x \mid x \in B\}$ will not be, since it is an infinite set of transitive sets of limit rank.

Model 8: in which $\mathcal{S}(\omega)$ exists but not $\mathcal{S}(\omega \times \omega)$

Note that the cardinality of $\mathcal{S}(\omega\times\omega) \cap V_n$ is about $2^{(n-2)^2}$, an order of magnitude higher than that of $\mathcal{S}(\omega) \cap V_n$; we have to take the transitive closure of course, but that will only make it higher.

So take $\mathbf{A}_8$ to be the class of all transitive u such that the map f_u defined by $f_u(n) = \overline{\overline{u \cap V_n}}$ is eventually dominated, for some k, by $n \mapsto 2^{kn}$, and $\mathbf{M}_8$ to be $\bigcup \mathbf{A}_8$.

By Proposition 3·3 and Proposition 3·7, $\mathbf{M}_8$ models BS.

5·27 Remark By estimating the number of ordered triples in V_n, and considering those transitive u with f_u dominated by $n \mapsto 2^{kn^2}$ for some k, we would obtain a model containing $\mathcal{S}(\omega\times\omega)$ but omitting $\mathcal{S}(\omega\times(\omega\times\omega))$.

Model 9: a failure of Seq

The importance of this example will be explained in our discussion in 10·6: it provides a model of BS containg **HF** that refutes Devlin's claim that BS proves $\forall a \forall n_{\in\omega} \exists u\, \mathrm{Seq}(u, a, n)$.

Let $\mathbf{A}_9$ be $\{u \mid \bigcup u \subseteq u \ \&\ u \cap {}^3A \text{ is finite}\}$, where we have yet to choose A.

5·28 Lemma $\mathbf{HF} \cap {}^3A = {}^3(\mathbf{HF} \cap A)$.

So take A to be $\{\omega\} \times \omega$. The resulting model $\mathbf{M}_9 = \bigcup \mathbf{A}_9$ will have **HF** as a member; ${}^3(\{\omega\}\times\omega)$ will not be there, but ${}^3(\omega \times \{\omega\})$ will be. The model will contain a bijection between the two sets $\omega \times \{\omega\}$ and $\{\omega\} \times \omega$, and therefore will fail to model GJ.

We should check that $\star$-closure holds in Model 9. Recall that $u^\star = u \cup [u]^1 \cup [u]^2 \cup u \times u$.

The members of ${}^3A$ are 3-sequences, which are neither singletons nor doubletons nor ordered pairs. So in this case

$$u^\star \cap {}^3A = u \cap {}^3A,$$

and all is well.

5·29 REMARK In the next section we give Gandy's model of GJI, which thus contains for each a and n a u such that $\mathrm{Seq}(u, a, n)$ but which does not, for $a = \omega$, contain the set of all finite sequences of members of a.

Model 10: from sheer perversity

Let P be an almost disjoint family of infinite subsets of ω; for X in P, consider the class A_X of all transitive sets having finite intersection with ${}^3X$. Take for Q any subset of P, $\mathbf{A}_Q$ to be the intersection of all the A_X for $X \in Q$. Then, for X in P, $\bigcup \mathbf{A}_Q$ will contain ${}^3X$ iff X is not in Q, and will model BS.

6: Models of GJ

Gandy: A set that models GJ but not fReR

Take $\mathbf{G}_2$ to be the rudimentary closure of $\{\omega\}$.

The set of even numbers is not in $\mathbf{G}_2$, not being Δ_0. Π_1, indeed full, foundation is true in $\mathbf{G}_2$; TCo will be true there as ω is transitive, by Proposition 2·82. But as we saw in Section 2, fReR proves the existence of EVEN.

The next two remarks are semantical versions of [G, Theorems 2.2.2(ii) and 3.1.1].

6·0 REMARK It follows that the graph G of addition is not a member of this model, for

$$\mathrm{EVEN} = \omega \cap \{n \mid n = 0 \vee \exists m_{\in n}\, (n, m, m) \in G\}.$$

6·1 REMARK The graph of concatenation is not in this model.

The unprovability of $\mathcal{S}(\omega) \in V$ in GJI

6·2 REMARK If Δ_0 separation is true and $\mathcal{S}(\omega) \in V$, then the set of even numbers can be built as

$$\bigcup\bigl(\mathcal{S}(\omega) \cap \{x \mid x \subseteq \omega \;\&\; 0 \in x \;\&\; \forall n :< \bigcup x\, (n \in x \iff n+1 \notin x)\}\bigr)$$

6·3 COROLLARY *"$\mathcal{S}(\omega) \in V$" is false in the rud closure of $\{\omega\}$.*

Proof : by Gandy, who showed that EVEN is not there.

6·4 COROLLARY *"$\mathcal{S}(\omega) \in V$" is not provable in* GJI.

6·5 COROLLARY *Since the existence of $\mathcal{S}(\omega)$ is derivable in* GJ *from the existence of* ACK*, the existence of* ACK *is not provable in* GJI.

Gandy: A set that models fReR but not ReR

Take $\mathbf{G}_3$ to be $V_{\omega+\omega}$.

Model 11:

Write **HC** for the union of all countable transitive sets. Then, assuming choice for countable families, $\mathbf{M}_{11} =_{\text{df}} V_{\omega+\omega} \cap \mathbf{HC}$, that is, the union of all countable transitive sets of rank less than $\omega + \omega$, will be a model of fReRI but not, by Proposition 2·102, ReR.

Variants of Model 11:

As often in this paper, we can obtain further models by carrying out one construction within another. Let $\mathbf{N}$ be an admissible set of height $\kappa > \omega$. For $0 < \eta = \bigcup \eta < \kappa$, let $\mathbf{N}_{11,\eta}$ be the union of transitive sets in $\mathbf{N}$ of rank less than η. Then that will be a model of fReC, and of AxInf if $\eta > \omega$. For a second example, assume that AC holds in $\mathbf{N}$ and consider the union $\mathbf{P}$ of all transitive sets which are members of $\mathbf{N}$ and countable there. Then $\mathbf{P}$ will be a model of fReC. Further $\mathbf{P}$ will be a model of $\mathcal{S}(x) \in V$.

Model 12: of fReR omitting HF

Since fReR_0 is a subtheory of Z, it is enough to find a transitive model of Z in which **HF** is not a set. The construction of one such model is sketched in Remark 14·24; for others, see [M1] and the further references there.

7·0 PROBLEM For which λ and α are L_λ and J_α a model of fReR or fReC? Material in Section 9 suggests that a necessary condition will be that $\alpha = \omega\alpha$. Is that also sufficient ?

Zarach: a set that models ReR but not KPI

See [Z], Theorem 6.4.

Model 13: a model of Z + TCo in which rank is not everywhere defined

Let λ be a limit ordinal. Define

$$\mathbf{A}_{13,\lambda} =_{\text{df}} \{u \mid \bigcup u \subseteq u \ \&\ u \cap \lambda < \lambda\}; \qquad \mathbf{M}_{13,\lambda} = \bigcup \mathbf{A}_{13,\lambda};$$

Note that if u and v are members of $\mathbf{A}_{13,\lambda}$ then $u \cup v \in \mathbf{A}_{13,\lambda}$, and $u \cup \mathcal{P}(u) \cup \{\mathcal{P}(u)\} \in \mathbf{A}_{13,\lambda}$; so $\mathbf{M}_{13,\lambda}$ will be a supertransitive model of all of Z except (in the case $\lambda = \omega$) the axiom of infinity. As $\mathbf{A}_{13,\lambda} \subseteq \mathbf{M}_{13,\lambda}$, $\mathbf{M}_{13,\lambda}$ will also model TCo. V_λ will be a subclass but not a member of $\mathbf{M}_{13,\lambda}$; $ON \cap \mathbf{M}_{13,\lambda} = \lambda$. V_λ will be definable over $\mathbf{M}_{13,\lambda}$ as the class of those sets which lie in the domain of an attempt at the rank function. The union of those attempts will be a class but not a set of $\mathbf{M}_{13,\lambda}$.

We show that $\mathbf{M}_{13,\lambda}$ will contain sets of all ranks. Let u be any member of $\mathbf{A}_{13,\lambda}$ which is not an ordinal. Define the sequence

$$u_0 = u; \quad u_{\nu+1} = u_\nu \cup \{u_\nu\}; \quad u_\eta = \bigcup_{\nu<\eta} u_\nu \quad \text{for } 0 < \eta = \bigcup \eta.$$

Then it is easily shown by induction on ν that no u_ν is an ordinal; that each u_ν is transitive; that each u_ν is a member of each $u_{\nu'}$ with $\nu < \nu'$; that $\varrho(u_\nu) = \varrho(u_0) + \nu$; that $u_\nu \cap ON = u_0 \cap ON$; and hence that each u_ν is in $\mathbf{A}_{13,\lambda}$ and therefore in $\mathbf{M}_{13,\lambda}$.

The case $\lambda = \omega$ gives us a model of Z which has infinite members but for which the axiom of infinity in the form $\omega \in V$ is false.

Variants of Model 13 will be studied in *Rudimentary Recursion* [M4].

PART II

8: Adding $\mathcal{S}(x) \in V$ to these systems

Devlin in his book [Dev] had the aim of finding a theory that would hold in all structures L_λ for λ a limit ordinal, and in all structures J_α for α an arbitrary non-zero ordinal, be strong enough for a unified development of both hierarchies, and yet not require the introduction of rudimentary functions at too early a stage; and proposed BS as such a theory. Alas, it proves to be too weak, as we shall see in Section 10 through the use of the models that we have built in earlier sections. Devlin's treatment is further flawed by other mistakes such as those mentioned by Stanley in his review (*Journal of Symbolic Logic* **53** pp 864–8) of Devlin's book *Constructibility*, where Solovay (unpublished) is quoted as declaring [Dev, I.9.5] to be false "as can be seen by a forcing argument," and [Dev I.9.3] to be refutable "by the use of Ehrenfeucht games."

Stanley concludes his review of [Dev] by asking whether such a theory might be found. We have three
DS candidates: our first proposal, which we call DS, for "Devlin strengthened", is to add to the axioms of DB the axioms

$$\omega \in V \text{ and } \mathcal{S}(x) \in V,$$

ReSs, GJs where $\mathcal{S}(x)$ is to mean the set of finite subsets of x. Call ReSs, GJs, fReRs the result of adding, to ReS, GJ,
fReRs and fReR respectively, the same two principles. Note that whereas BS had full foundation, we allow DS and our other systems to have only Π_1 foundation.

8·0 Proposition *The existence of Cartesian products is provable in* ReSs*: so* DS *is the same as* ReSs.

Proof : given a, $\mathcal{S}(a)$ will contain all 1- and 2-element subsets of a; hence $a \times a$ is a Δ_0 subclass of the set $\mathcal{S}(\mathcal{S}(a))$; to form $b \times c$, take $a = b \cup c$ and apply Δ_0 Separation. ⊣ (8·0)

At the stronger end of our lattice of theories, the enhancement amounts to no more than adding the axiom of infinity, since by Proposition 2·103, ReRI proves that $\forall x\, \mathcal{S}(x) \in V$.

8·1 Problem Is TCo derivable from the other axioms of ReR ?

8·2 Remark It is tempting to add a further axiom,

$$\mathbf{HF} \in V,$$

which in many ways makes life easier, because **HF** is a model of ZF − Infinity, and therefore a large number of functions become automatically available. But a feeling, that doing so does not address the chief problem with BS, is reinforced by the variant of Model 6 mentioned after Remark 5·25, in which **HF** exists but some ${}^3x$ not.

Our aim, in this section and the next, is to study these systems, and we shall begin by enlarging our syntax to treat a class of formulæ that is slightly more general than Δ_0 but still limited in a specific sense.

A syntactical enhancement

We examine the consequences of allowing limited quantifiers $\forall y_{\in \mathcal{S}(x)}$, $\exists y_{\in \mathcal{S}(x)}$. The paradigm for our discussion is section 6 of "The Strength of Mac Lane Set Theory" where the quantifiers $\forall y_{\in \mathcal{P}(x)}$, there written as $\forall y\colon\subseteq x$ and in the present paper as $\forall y_{\subseteq x}$, were discussed.

We call a formula $\Delta_{0,\mathcal{S}}$ if all its quantifiers are of the form $Qx_{\in \mathcal{S}(y)}$ or $Qx_{\in y}$ where Q is $\forall$ or $\exists$, and x and y are distinct variables. We preserve "restricted" as a description of the quantifiers $Qx_{\in y}$, and speak of the occurrences of y in $Qx_{\in \mathcal{S}(y)}$ or $Qx_{\in y}$ as *limiting* the range of the bound variable x. It is tempting, indeed, to adopt a different presentation of the language by declaring the class of atomic formulæ to consist of every formula of one of the three forms

$$x \in y \qquad x = y \qquad x \in \mathcal{S}(y)$$

and to have three kinds of quantifiers, $\forall x$, $\forall x_{\in y}$ and $\forall x_{\in \mathcal{S}(y)}$ in the language; but we shall not formally adopt this approach here. Gandy in his paper [G] suggests considering the ancestral $\in^*$ of $\in$, where $x \in^* y$ iff $x \in \mathrm{tcl}(y)$, which will become easily available in our system.

8·3 Proposition (DS) *"$x \in \mathcal{S}(y)$", "$x = \mathcal{S}(y)$" and "$\mathcal{S}(y) \in x$" are all $\Delta_{0,\mathcal{S}}$.*

Normal forms for $\Delta_{0,\mathcal{S}}$ formulæ

8·4 We sketch a method of rewriting a $\Delta_{0,\mathcal{S}}$ formula so that all variables are limited by terms constructed from the free variables of the original formula using only $\bigcup$; thus ultimately the terms limiting variables contain no variables that are themselves bound by other quantifiers.

Unlike $\in$, $\subseteq$ is transitive. Hence the following reduction is available:

$$\exists x_{\in\mathcal{S}(t)}\,\forall y_{\in\mathcal{S}(x)}\,\mathfrak{A} \iff \exists x_{\in\mathcal{S}(t)}\,\forall y_{\in\mathcal{S}(t)}\,[y \subseteq x \Longrightarrow \mathfrak{A}].$$

Note here that on the left hand side the x limiting y in the quantifier $\forall y_{\in\mathcal{S}(x)}$ is itself bound by the preceding quantifier $\exists x_{\in\mathcal{S}(t)}$, whereas on the right hand side the t that limits both quantifiers is itself free. We may speak of t in the above displayed formula or $\bigcup t$ in the next as a *free term*.

We thus obtain these reductions:

$$\begin{aligned}
\forall x_{\in a}\,\exists y_{\in x}\,\mathfrak{A} &\iff \forall x_{\in a}\,\exists y_{\in\cup a}\,[y \in x \;\&\; \mathfrak{A}];\\
\forall x_{\in\mathcal{S}(a)}\,\exists y_{\in x}\,\mathfrak{A} &\iff \forall x_{\in\mathcal{S}(a)}\,\exists y_{\in a}\,[y \in x \;\&\; \mathfrak{A}];\\
\forall x_{\in a}\,\exists y_{\in\mathcal{S}(x)}\,\mathfrak{A} &\iff \forall x_{\in a}\,\exists y_{\in\mathcal{S}(\cup a)}\,[y \subseteq x \;\&\; \mathfrak{A}]\\
&\iff \forall x_{\in a}\,\exists y_{\in\mathcal{S}(\cup a)}\,[\forall s_{1\in\cup a}\,(s_1 \in y \Longrightarrow y_1 \in x) \;\&\; \mathfrak{A}];\\
\forall x_{\in\mathcal{S}(a)}\,\exists y_{\in\mathcal{S}(x)}\,\mathfrak{A} &\iff \forall x_{\in\mathcal{S}(a)}\,\exists y_{\in\mathcal{S}(a)}\,[y \subseteq x \;\&\; \mathfrak{A}]\\
&\iff \forall x_{\in\mathcal{S}(a)}\,\exists y_{\in\mathcal{S}(a)}\,[\forall s_{2\in a}\,(s_2 \in y \Longrightarrow s_2 \in x) \;\&\; \mathfrak{A}].
\end{aligned}$$

Those equivalences, which are all valid in S_0, and, where applicable, preserve the stratifiability of the formula under consideration, show that one may progressively rewrite the formula to one in which all limitations are of the form $_{\in\mathcal{S}(\cup^{\mathfrak{k}} a)}$ or $_{\in\cup^{\mathfrak{k}} a}$ with a a free variable. We call such a formula one in *free form*. Our expansion of $y \subseteq x$ in the fourth and sixth lines, which would be unnecessary if we treated $y \subseteq x$ as atomic, helps to secure free form. We call the bound variables s, t introduced in those expansions *subsidiary variables*: we shall suppress mention of them in our discussion below, so that when we speak of "every quantifier", we mean "every quantifier binding other than a subsidiary variable".

Given a formula in free form, we replace each limiting free term by a new variable and add a clause expressing the equality of the term and the variable.

We have reached the

8·5 First Limited Normal Form *Let Φ be a $\Delta_{0,\mathcal{S}}$ formula with free variables $a_0, \ldots a_{\mathfrak{n}}$. Let $\mathfrak{m}+1$ be the number of quantifiers occurring in Φ. Then for $0 \leqslant j \leqslant \mathfrak{m}$, there are numbers $0 \leqslant \mathfrak{k}(j) \leqslant \mathfrak{n}$, $0 \leqslant \mathfrak{l}(j)$, determined by the quantifier structure of Φ, new variables $y_0, \ldots y_{\mathfrak{m}}$, and a $\Delta_{0,\mathcal{S}}$ formula Ψ_1 with free variables $a_0, \ldots a_{\mathfrak{n}}$, $y_0, \ldots y_{\mathfrak{m}}$, in which every quantifier is limited by one of the parameters y_i, such that, abbreviating $\forall y_0, \ldots, \forall y_{\mathfrak{m}}$ by $\forall\vec{y}$, we have*

$$\vdash_{\mathsf{DB}_0} \forall\vec{a}\forall\vec{y}\Big[\bigwedge_{0\leqslant j\leqslant\mathfrak{m}} y_j = \bigcup{}^{\mathfrak{l}(j)} a_{\mathfrak{k}(j)} \Longrightarrow \big[\Phi(\vec{a}) \iff \Psi_1(\vec{a},\vec{y})\big]\Big]$$

To take things to a second stage, if we know that we intend using the formula $\Phi(a)$ in a context where a_i will be constrained to be a member of b_i, we may replace the restriction $_{\in\cup^{\mathfrak{l}} a_i}$ by the restriction $_{\in\cup^{\mathfrak{l}+1} b_i}$; and each limitation $_{\in\mathcal{S}(\cup^{\mathfrak{l}} a_i)}$ by the limitation $_{\in\mathcal{S}(\cup^{\mathfrak{l}+1} b_i)}$, since if $a \in b$, $\bigcup^{\mathfrak{l}} a \subseteq \bigcup^{\mathfrak{l}+1} b$, and make a corresponding adjustment to the matrix.

We could also consider intended limitations $a_i \subseteq b_i$ instead of restrictions $a_i \in b_i$: the replacements to be made then would be $_{\in\cup^{\mathfrak{l}} a_i}$ by $_{\in\cup^{\mathfrak{l}} b_i}$ and $_{\in\mathcal{S}(\cup^{\mathfrak{l}} a_i)}$ by $_{\in\mathcal{S}(\cup^{\mathfrak{l}} b_i)}$, since if $a \subseteq b$ then $\bigcup^{\mathfrak{l}} a \subseteq \bigcup^{\mathfrak{l}} b$.

Further, we could mix our intentions, and also leave some a_i untouched, which is tantamount to saying $a_i = b_i$. We thus have the

8·6 SECOND LIMITED NORMAL FORM *Continuing the notation of the First Limited Normal Form, let R, S and U be disjoint sets partitioning $[0,\mathfrak{n}]$, and let $b_0,\dots,b_\mathfrak{n}$ be variables not occurring in Φ. Then for the same numbers $\mathfrak{k}(j)$, $\mathfrak{l}(j)$, there is a $\Delta_{0,\mathcal{S}}$ formula Ψ_2 with free variables $a_0,\dots a_\mathfrak{n}$, $y_0,\dots y_\mathfrak{m}$, in which every quantifier is limited to one of the parameters y_i, such that*

$$\vdash_{\mathsf{DB}_0} \forall\vec{b}\forall\vec{a}\forall\vec{y}\Bigg[\Big[\bigwedge_{i \text{ in } R} a_i \in b_i \;\&\; \bigwedge_{i \text{ in } S} a_i \subseteq b_i \;\&\; \bigwedge_{i \text{ in } U} a_i = b_i \;\&\; \bigwedge_{\mathfrak{k}(j) \text{ in } R} y_j = \bigcup{}^{\mathfrak{l}(j)+1} b_{\mathfrak{k}(j)} \;\&\; \bigwedge_{\substack{\mathfrak{k}(j) \text{ in}\\ S \text{ or } U}} y_j = \bigcup{}^{\mathfrak{l}(j)} b_{\mathfrak{k}(j)}\Big] \Longrightarrow$$
$$\Longrightarrow \Big[\Phi(\vec{a}) \Longleftrightarrow \Psi_2(\vec{a},\vec{y})\Big]\Bigg]$$

8·7 EXAMPLE Let $\mathfrak{A}$ be quantifier-free, with six variables a, b, x, y, z, w. Suppose we want to re-write the formula $\Phi(a,b) \iff_{\text{df}} \exists x_{\in a}\, \forall y_{\in \mathcal{S}(x)}\, \exists z_{\in x}\, \forall w_{\in \mathcal{S}(z)}\, \mathfrak{A}(a,b,x,y,z,w)$.

Let

$$\mathfrak{B}(a,b,x,y,z,w) \iff_{\text{df}} \big(y \subseteq x \Longrightarrow [z \in x \;\&\; (w \subseteq z \Longrightarrow \mathfrak{A}(a,b,x,y,z,w))]\big).$$

Notice that $\mathfrak{B}$ is Δ_0, or indeed quantifier-free if we count $s \subseteq t$ as atomic. Then

$$\exists x_{\in a}\, \forall y_{\in \mathcal{S}(x)}\, \exists z_{\in x}\, \forall w_{\in \mathcal{S}(z)}\, \mathfrak{A}(a,b) \iff$$
$$\iff \exists x_{\in a}\, \forall y_{\in \mathcal{S}(\cup a)}\, \exists z_{\in \cup a}\, \forall w_{\in \mathcal{S}(\cup\cup a)}\, \big[\mathfrak{B}(a,b,x,y,z,w)\big]$$

In order not to use $\mathcal{S}$ applied to a term that is not a variable, we introduce further variables z_j.

8·8 FIRST RESTRICTED NORMAL FORM *Continuing the notation of the First Limited Normal Form, for the same numbers $\mathfrak{k}(j)$, $\mathfrak{l}(j)$, there is a partition of $\{j \mid 0 \leqslant j \leqslant \mathfrak{m}\}$ into disjoint sets L_Φ, R_Φ; there are new variables y_j, z_j for $0 \leqslant j \leqslant \mathfrak{m}$; and there is a Δ_0 formula Ψ_3, with free variables the a's and the z's; such that every quantifier in Ψ_3 is restricted to one of the parameters z_i, and*

$$\vdash_{\mathsf{DB}_0} \forall\vec{a}\forall\vec{y}\forall\vec{z}\Big[\Big[\bigwedge_{j \text{ in } R_\Phi}\big(z_j = y_j \;\&\; y_j = \bigcup{}^{\mathfrak{l}(j)} a_{\mathfrak{k}(j)}\big) \;\&\; \bigwedge_{j \text{ in } L_\Phi}\big(z_j = \mathcal{S}(y_j) \;\&\; y_j = \bigcup{}^{\mathfrak{l}(j)} a_{\mathfrak{k}(j)}\big)\Big] \Rightarrow \Big[\Phi(\vec{a}) \Longleftrightarrow \Psi_3(\vec{a},\vec{z})\Big]\Big]$$

Taking that to the corresponding second stage, and noting that if $a \subseteq b$ then $\mathcal{S}(\bigcup^{\mathfrak{l}} a) \subseteq \mathcal{S}(\bigcup^{\mathfrak{l}} b)$, whereas if $a \in b$, $\mathcal{S}(\bigcup^{\mathfrak{l}} a) \subseteq \mathcal{S}(\bigcup^{\mathfrak{l}+1} b)$, we reach the

8·9 SECOND RESTRICTED NORMAL FORM *Let Φ be a $\Delta_{0,\mathcal{S}}$ formula with free variables $a_0,\dots a_\mathfrak{n}$. Let R, S and U be disjoint sets partitioning $[0,\mathfrak{n}]$, and let $b_0,\dots,b_\mathfrak{n}$ be variables not occurring in Φ. Let $\mathfrak{m}+1$ be the number of quantifiers occurring in Φ. Then there is a partition of $\{j \mid 0 \leqslant j \leqslant \mathfrak{m}\}$ into disjoint sets L_Φ, R_Φ; for $0 \leqslant j \leqslant \mathfrak{m}$, there are numbers $0 \leqslant \mathfrak{k}(j) \leqslant \mathfrak{n}$, $0 \leqslant \mathfrak{l}(j)$, determined by the quantifier structure of Φ, there are new variables y_j, z_j for $0 \leqslant j \leqslant \mathfrak{m}$; and there is a Δ_0 formula Ψ_4 with free variables the a's and the z's, in which every quantifier is restricted to one of the parameters z_i; such that,*

$$\vdash_{\mathsf{DB}_0} \forall\vec{b}\forall\vec{a}\forall\vec{y}\forall\vec{z}\Bigg[\Big[\bigwedge_{i \text{ in } R} a_i \in b_i \;\&\; \bigwedge_{i \text{ in } S} a_i \subseteq b_i \;\&\; \bigwedge_{i \text{ in } U} a_i = b_i \;\&$$
$$\& \bigwedge_{\substack{j \text{ in } R_\Phi,\\ \mathfrak{k}(j) \text{ in } R}} \big(z_j = y_j \;\&\; y_j = \bigcup{}^{\mathfrak{l}(j)+1} b_{\mathfrak{k}(j)}\big) \;\&\; \bigwedge_{\substack{j \text{ in } L_\Phi,\\ \mathfrak{k}(j) \text{ in } R}} \big(z_j = \mathcal{S}(y_j) \;\&\; y_j = \bigcup{}^{\mathfrak{l}(j)+1} b_{\mathfrak{k}(j)}\big) \;\&$$
$$\& \bigwedge_{\substack{j \text{ in } R_\Phi,\\ \mathfrak{k}(j) \text{ in } S \text{ or } U}} \big(z_j = y_j \;\&\; y_j = \bigcup{}^{\mathfrak{l}(j)} b_{\mathfrak{k}(j)}\big) \;\&\; \bigwedge_{\substack{j \text{ in } L_\Phi,\\ \mathfrak{k}(j) \text{ in } S \text{ or } U}} \big(z_j = \mathcal{S}(y_j) \;\&\; y_j = \bigcup{}^{\mathfrak{l}(j)} b_{\mathfrak{k}(j)}\big)\Big] \Longrightarrow$$
$$\Longrightarrow \Big[\Phi(\vec{a}) \Longleftrightarrow \Psi_4(\vec{a},\vec{z})\Big]\Bigg]$$

Self-strengthening of DS

We may now deduce the

8·10 METATHEOREM DS *proves all instances of the scheme of $\Delta_{0,\mathcal{S}}$ separation.*

Proof : Suppose that there are $\mathfrak{m}+1$ quantifiers in the $\Delta_{0,\mathcal{S}}$ formula $\Phi(x,a)$. By the Second Restricted Normal Form, we know that there are new variables $y_0,\ldots,y_{\mathfrak{m}}$, $z_0,\ldots,z_{\mathfrak{m}}$ and a Δ_0 formula $\Psi_4(x,\vec{a},\vec{z})$ with the free variables shown, such that

$$\mathsf{DB}_0 \vdash x\in d\ \&\ \Big\langle \text{conditions on } \vec{z},\ \vec{y},\ d \text{ and } \vec{a}\Big\rangle \Longrightarrow \big[\Phi(x,\vec{a}) \Longleftrightarrow \Psi_4(x,\vec{a},\vec{z})\big],$$

where there are $\mathfrak{m}+1$ conditions, each of one of the four following types, according to the quantifier structure of Φ:

$$[z=y\ \&\ y=\bigcup{}^{\mathfrak{l}+1}d];\ [z=\mathcal{S}(y)\ \&\ y=\bigcup{}^{\mathfrak{l}+1}d];\ [z=y\ \&\ y=\bigcup{}^{\mathfrak{l}}a];\ [z=\mathcal{S}(y)\ \&\ y=\bigcup{}^{\mathfrak{l}}a].$$

In DS we may prove that given d and $\vec{a}$ there are y's and z's satisfying the conditions, and for those z, we have $\forall x_{\in d}\big(\Phi(x,\vec{a}) \Longleftrightarrow \Psi_4(x,\vec{a},\vec{z})\big)$, whence

$$d\cap\{x \mid \Phi(x,\vec{a})\} = d\cap\{x \mid \Psi_4(x,\vec{a},\vec{z})\}\in V. \qquad \dashv (8{\cdot}10)$$

8·11 Proposition DS *proves that the graph $\mathcal{G}_+$ of integer addition, or indeed of any partial recursive function, is a set.*

Proof : To get the graph of addition, we would apply separation to $\omega\times(\omega\times\omega)$ to form the set of all triples such that there exists an attempt: *prima facie* Σ_1 or perhaps just Δ_1 separation, given that attempts are unique (a fact that we have not proved). But the attempts are all in $\mathcal{S}(\omega\times(\omega\times\omega))$, and so with that set as a parameter, only Δ_0 separation is needed. $\dashv$ (8·11)

The results following Definitions 2·16 and 2·17 can be improved:

8·12 Lemma *"$x\in\mathcal{S}(y)$" is Δ_1^{DS}.*

Proof : by Corollary 2·20 and Lemma 2·21.

8·13 Lemma (DS) $z\subseteq\mathcal{S}(y) \Longleftrightarrow \exists c[\forall w_{\in z}\, w\subseteq y\ \&\ \forall w_{\in z}\exists f_{\in c}\exists n_{\in\omega}\, f: n\longleftrightarrow w]$.

Proof : Take $c=\mathcal{S}(y\times\omega)$. $\dashv$ (8·13)

8·14 Corollary *"$z\subseteq\mathcal{S}(y)$" and "$z=\mathcal{S}(y)$" are Δ_1^{DS}.*

Proof : the first part by Lemmata 2·22 and 8·13; the second then follows by Lemma 2·23. $\dashv$ (8·14)

8·15 Remark The above discussion shows that the function $x\mapsto\mathcal{S}(x)$ is Σ_1 in ReR with $\omega\in V$ and Π_1 foundation.

8·16 Metatheorem *Every $\Pi_{1,\mathcal{S}}$ predicate is Π_1^{DS}.*

Proof : Consider a predicate of the form $\forall c\Phi(c,\vec{a})$ where Φ is $\Delta_{0,\mathcal{S}}$. We again use the Second Restricted Normal Form, which tells us that there is a Δ_0 predicate $\Psi_4(c,a,\vec{z})$ and further variables $\vec{b}$ and $\vec{y}$, such that $\Phi(c,a)$ is equivalent to $\Psi_4(c,a,\vec{z})$ provided finitely many conditions hold, of the form $z=\mathcal{S}(y)\ \&\ y=\bigcup^{\mathfrak{k}}b$ or $z=y\ \&\ y=\bigcup^{\ell}b$, and each a and c is either a member of or a subset of or equal to the corresponding b.

Thus, writing out a sample condition,

$$\forall c\Phi(c,a)\Longleftrightarrow \forall c\forall\vec{b}\forall\vec{z}\forall\vec{y}\Big[\big[[\underbrace{z=\mathcal{S}(y)}_{\Sigma_1}\ \&\ \underbrace{y=\textstyle\bigcup^{\mathfrak{k}}b\ \&\ a\subseteq b}_{\Delta_0}]\ \&\ \ldots\ \&\ \underbrace{[\ldots]}_{\Sigma_1}\big]\Longrightarrow \underbrace{\Psi_4(c,a,\vec{z})}_{\Delta_0}\Big],$$

which is Π_1, as required. $\dashv$ (8·16)

DS with TCo

8·17 Proposition ($\mathsf{DS}+\mathsf{TCo}$) $\mathrm{tcl}(x)\in V$.

Proof : fix x, and using TCo, let u be a transitive set of which x is a member. Using $\mathcal{S}(x)\in V$, let a be the set $\mathcal{S}(u\times\omega)$.

Say that f *descends from x to y* if

$$\mathrm{Fn}(f)\ \&\ \mathrm{Dom}\,f\in\omega\ \&\ 2\leqslant\mathrm{Dom}\,f\ \&\ f(0)=x\ \&\ \forall k:<\mathrm{Dom}\,(f)-1 f(k+1)\in f(k)\ \&\ f(\mathrm{Dom}\,(f)-1)=y.$$

That is a Δ_0 predicate of f, and each such f is in a, so the class

$$u\cap\big\{y\ \big|\ \exists f_{\in a}[f \text{ descends from } x \text{ to } y]\big\}$$

is a set and is the desired transitive closure of x. $\dashv$ (8·17)

Self-strengthening of GJs

8·18 LEMMA (GJs) $\{\mathcal{S}(x) \mid x \in a\} \in V$.

Proof : Fix the set a. If $x \in a$ then $x \subseteq \bigcup a$, so $\mathcal{S}(x) \subseteq \mathcal{S}(\bigcup a)$. The desired set is the class

$$\bigl\{\mathcal{S}(\bigcup a) \cap \{y \mid y \subseteq x\} \bigm|_x x \in a\bigr\},$$

which is a set by an application of RR$^+$. ⊣ (8·18)

8·19 COROLLARY (GJs) $\{\langle \mathcal{S}(\bigcup w), \mathcal{S}(w)\rangle \mid_w w \in b\} \in V$.

Proof : consider $\{\mathcal{S}(v) \mid_v v \in a\} \times \{\mathcal{S}(w) \mid_w w \in b\} \cap \{(c,d)_2 \mid_{c,d} \bigcup c = \bigcup\bigcup d\}$, taking $a = \{\bigcup w \mid_w w \in b\}$. ⊣ (8·19)

8·20 PROPOSITION GJs *proves* $\Delta_{0,\mathcal{S}}$ *rud replacement.*

Proof : Aiming, in fact, for the extended form corresponding to RR^+, defined in 2·88, we must show that

$$\forall x_2 \forall x_1 \exists w \forall \vec{v}_{\in x_1} \exists t_{\in w} \forall u (u \in t \iff u \in x_2 \ \&\ \Phi(u, \vec{v}),$$

where Φ is a $\Delta_{0,\mathcal{S}}$ formula with the free variables shown.

Suppose that there are $\mathfrak{m}+1$ quantifiers in Φ. By the Second Restricted Normal Form, we know that there are new variables $y_0, \dots, y_{\mathfrak{m}}$, $z_0, \dots, z_{\mathfrak{m}}$ and a Δ_0 formula $\Psi_4(u, \vec{v}, \vec{z})$ with the free variables shown, such that

$$\mathsf{DB}_0 \vdash u \in x_2 \ \&\ v \mathrel{\vec{\in}} x_1 \ \&\ \Bigl\langle \text{conditions on } \vec{z},\ \vec{y},\ x_1, \text{ and } x_2 \Bigr\rangle \implies \bigl[\Phi(u, \vec{v}) \iff \Psi_4(u, \vec{v}, \vec{z})\bigr],$$

where there are $\mathfrak{m}+1$ conditions, each of one of the four following types, according to the quantifier structure of Φ:

$$[z = y \ \&\ y = \textstyle\bigcup^{\mathfrak{l}+1} x_2];\ [z = \mathcal{S}(y) \ \&\ y = \bigcup^{\mathfrak{l}+1} x_2];\ [z = y \ \&\ y = \bigcup^{\mathfrak{l}+1} x_1];\ [z = \mathcal{S}(y) \ \&\ y = \bigcup^{\mathfrak{l}+1} x_1].$$

A slight extension of RR^+ would tell us that

$$\forall x_2 \forall x_1 \exists w \forall \vec{z}_{\in A} \forall \vec{v}_{\in x_1} \exists t_{\in w} \forall u \bigl(u \in t \iff u \in x_2 \ \&\ \Psi_4(u, \vec{v}, \vec{z})\bigr),$$

where A is a certain class, provably a set containing at most $\mathfrak{m}+1$ elements, namely the values of the form $\bigcup^{\mathfrak{l}+1} x_2$ or $\mathcal{S}\bigl(\bigcup^{\mathfrak{l}+1} x_2\bigr)$ given to the z's by the conditions.

To show that, fix x_2. If we write x_3 for $x_1 \cup A$, then by RR^+, we may deduce that

$$\exists w \forall \vec{v}_{\in x_3} \forall z_{\in x_3} \exists t_{\in w} \forall u \bigl(u \in t \iff u \in x_2 \ \&\ \Psi_4(u, \vec{v}, \vec{z})\bigr),$$

whence

$$\exists w \forall \vec{v}_{\in x_3} \forall z_{\in x_3} \exists t_{\in w} \forall u \bigl(u \in t \iff u \in x_2 \ \&\ \Phi(u, \vec{v})\bigr).$$

We may now cut this w down to exactly the one we want by applying $\Delta_{0,\mathcal{S}}$ separation. ⊣ (8·20)

Self-strengthening of fReRs

8·21 PROPOSITION fReRs *proves flat* $\Delta_{0,\mathcal{S}}$ *replacement.*

Proof : We must show that

$$\forall x_{\in u} \exists! d [\Phi(x, d) \ \&\ d \subseteq e] \implies \exists v \forall d \bigl[d \in v \iff \exists x_{\in u} [\Phi(x, d) \ \&\ d \subseteq e]\bigr],$$

where Φ is a $\Delta_{0,\mathcal{S}}$ formula with the two free variables shown.

Suppose that there are $\mathfrak{m}+1$ quantifiers in Φ. By the Second Restricted Normal Form, we know that there are new variables $y_0, \dots, y_{\mathfrak{m}}$, $z_0, \dots, z_{\mathfrak{m}}$ and a Δ_0 formula $\Psi_4(x, d, \vec{z})$ with $\mathfrak{m}+3$ free variables, such that

$$\mathsf{DB}_0 \vdash x \in u \ \&\ d \subseteq e \ \&\ \Bigl\langle \text{conditions on } \vec{z},\ \vec{y},\ u, \text{ and } e \Bigr\rangle \implies \bigl[\Phi(x, d) \iff \Psi_4(x, d, \vec{z})\bigr],$$

where there are $\mathfrak{m}+1$ conditions, each of one of the four following types, according to the quantifier structure of Φ:

$$[z = y \;\&\; y = \bigcup^{\mathfrak{l}+1} u];\; [z = \mathcal{S}(y) \;\&\; y = \bigcup^{\mathfrak{l}+1} u];\; [z = y \;\&\; y = \bigcup^{\mathfrak{l}} e];\; [z = \mathcal{S}(y) \;\&\; y = \bigcup^{\mathfrak{l}} e].$$

Fix u and e; then, using $\forall x\, \mathcal{S}(x) \in V$, the conditions will give fixed values to the y's and z's; for those values we shall have that for $x \in u$ and $d \subseteq e$, $\Phi(x, d) \iff \Psi_4(x, d, \vec{z})$.

Suppose now that $\forall x_{\in u}\, \exists! d[\Phi(x, d) \;\&\; d \subseteq e]$; then $\forall x_{\in u}\, \exists! d[\Psi_4(x, d, \vec{z}) \;\&\; d \subseteq e]$. We appeal to the extended form of (BdR) proved as Proposition 2·101, to deduce that

$$\exists v \forall d\big[d \in v \iff \exists x_{\in u}\, [\Psi_4(x, d, \vec{z}) \;\&\; d \subseteq e]\big],$$

whence

$$\exists v \forall d\big[d \in v \iff \exists x_{\in u}\, [\Phi(x, d) \;\&\; d \subseteq e]\big]. \qquad \dashv (8{\cdot}21)$$

8·22 REMARK In [M5] it will be seen that the system fReRs proves appropriate for the development of the definition of forcing, and that fReCs might be the weakest system persistent under set-generic extensions.

8·23 REMARK In [M4] we shall study rudimentary recursions on the ancestral and related relations.

Self-strengthening of ReR

8·24 LEMMA (ReR) *All instances of Δ_0 replacement where, as in Proposition 2·101, φ is allowed to have further free variables.*

Proof : Suppose that $\mathfrak{A}$ is Δ_0 and that $\forall x_{\in u}\, \exists! y \mathfrak{A}(x, y, w)$. Let $u_1 = u \times \{w\}$. Then

$$\forall x_{\in u_1}\, \exists! y\, \underbrace{\mathfrak{A}(\mathrm{left}(x), y, \mathrm{right}(x))}_{\Delta_0^{\mathsf{S}_0}}.$$

So applying Δ_0 replacement, we get $\exists v \forall y (y \in v$ iff $\exists x_{\in u_1}\, \mathfrak{A}(\mathrm{left}(x), y, \mathrm{right}(x))$, which in turn is equivalent to $\exists x_{\in u}\, \mathfrak{A}(x, y, w)$, as required.

8·25 PROPOSITION ReRI *proves each instance of $\Delta_{0,\mathcal{S}}$ replacement.*

Proof : The argument given for 8·21 adapts easily, using the Lemma. $\dashv$ (8·25)

8·26 PROBLEM Does ReR prove $\mathcal{S}(x) \in V$? the idea being that if there is an infinite set, then one ought to be able to prove that ω exists, and thence that $\mathcal{S}(x) \in V$; and if all sets are finite a proof of $\mathcal{S}(x) \in V$ will be provided by Proposition 2·13.

We pause to establish two results concerning the sets $Z(a)$ defined in [M1], whose definition was recalled in our discussion of Model 7.

8·27 DEFINITION We write "f attempts $Z(a)$ at n" for the $\Delta_{0,\mathcal{S}}$ formula

$$\mathrm{Fn}(f) \;\&\; \mathrm{Dom}\,(f) = n + 1 \;\&\; f(0) = \varnothing \;\&\; \forall k_{\in n}\, (f(k + 1) = \mathcal{S}(f(k)) \cup \{a\} \smallsetminus \{\varnothing\}).$$

8·28 PROPOSITION (ReRI) $\forall a : \omega \longrightarrow 2$, *$Z(a)$ exists.*

Proof : Fix a. Note that if $\mathrm{Fn}(f)$ then

$$x = \mathcal{S}(f(k)) \iff \underbrace{\exists y_{\in \bigcup\bigcup(f)}\, (y, k)_2 \in f \;\&\; x = \mathcal{S}(y)}_{\Delta_{0,\mathcal{S}}}.$$

Hence we may assert that

$$\forall n_{\in \omega}\, \exists f \big(f \text{ attempts } Z(a) \text{ at } n\big);$$

for the class of n for which the assertion fails is $\Pi_{1,\mathcal{S}}$ and therefore by Metatheorem 8·16 has, if non-empty, a minimal element, necessarily a successor; which can rapidly be refuted.

For each n, there can be at most one such f, so by $\Delta_{0,\mathcal{S}}$ replacement, the set of such f exists; its union will be a function, of which the class $Z(a)$ is the image and therefore a set. ⊣ (8·28)

8·29 Definition Let $\Psi(x,a)$ be the $\Delta_{0,\mathcal{S}}$ formula

$$a \in x \;\&\; \forall b_{\in x}\,[\{b\} \in x \;\&\; (b \in \mathcal{S}(x) \vee b = a) \;\&\; (b = \varnothing \Longrightarrow b = a)] \;\&\; \forall s_{\in \mathcal{S}(x)}\,[s \neq \varnothing \Longrightarrow s \in x].$$

8·30 Lemma (ReRI) $Z(a) \in V \Longrightarrow [x = Z(a) \Longleftrightarrow \Psi(x,a)]$.

Proof : It is readily checked that $x = Z(a) \Longrightarrow \Psi(x,a)$.

Suppose that $Z(a) \in V$ and that $\Psi(x,a)$. Let $c = \mathcal{S}(Z(a) \times \omega)$. Then

$$\{n \mid Z_n(a) \not\subseteq x\} = \{n \mid \underbrace{\exists f_{\in b}\, f \text{ attempts } Z(a) \text{ at } n \;\&\; f(n) \not\subseteq x}_{\Delta_{0,\mathcal{S}}}\};$$

Π_1 foundation would yield a minimal element of that class, if non-empty; but $Z_0(a) = \varnothing \subseteq x$, and it is easily checked that $\Psi(x,a) \;\&\; Z_n(a) \subseteq x \Longrightarrow Z_{n+1} \subseteq x$. Thus $Z(a) \subseteq x$.

If $x \not\subseteq Z(a)$, let y be an $\in$-minimal element of $x \smallsetminus Z(a)$. Then $y \neq \varnothing$, $y \in \mathcal{S}(x)$ and $y \subseteq Z(a)$. Hence $\forall z_{\in y}\, \exists! n_{\in\omega}\, (\underbrace{z \in Z_{n+1}(a) \;\&\; z \notin Z_n(a)}_{\Delta_{0,\mathcal{S}}})$; the class of such n's is therefore a set, which is finite and therefore bounded in ω; so $\exists m_{\in\omega}\, y \subseteq Z_m(a)$, whence $y \in Z_{m+1}(a)$, contradicting $y \notin Z(a)$. ⊣ (8·30)

8·31 Corollary (ReRI) *"$x = Z(a)$" is $\Delta_{0,\mathcal{S}}$.*

8·32 Proposition (ReRI) $\forall b_{\subseteq {}^{\omega}2}\, \{Z(a) \mid a \in b\} \in V$.

Proof : Fix b. Then $\forall a_{\in b}\, \exists! x\; \underbrace{x = Z(a)}_{\Delta_{0,\mathcal{S}}}$; apply $\Delta_{0,\mathcal{S}}$ replacement to complete the proof. ⊣ (8·32)

Self-strengthening of KPI

8·33 Proposition KPI *proves every instance of $\Delta_{0,\mathcal{S}}$ collection.*

Proof : We may either use Remark 8·30 or else Metatheorem 8·31, which implies that in the context of KPI, every $\Delta_{0,\mathcal{S}}$ formula is equivalent to a Σ_1 one; but it is well-known that KP is self-strengthening to Σ_1 collection. ⊣ (8·33)

8·34 Problem In Proposition 8.33, can KPI be reduced to KP ? In KP rank is definable and the rank of an infinite set must be at least ω; but with infinity $\mathcal{S}(x) \in V$ becomes provable.

9: The Gandy sequence

In this section we wish to assess the relative strength of the enhanced theories DS, etc.

9·0 Proposition *There is a model of* DS *plus* $\mathbf{HF} \in V$ *in which* GJ *is false.*

Proof : The model $\mathbf{M}_7$ will do. We have to prove that "$\mathcal{S}(x) \in V$" is true in $\mathbf{M}_7$. Note that any non-empty finite set must have successor rank. So if u is transitive and contains only finitely many transitive sets of limit rank, then $u \cup \mathcal{S}(u) \cup \{\mathcal{S}(u)\}$ will have the same property. That suffices. ⊣ (9·0)

GJs in L and J

Now we wish to verify that GJs is true in every L_λ ($\lambda = \bigcup \lambda > \omega$)and J_α ($\alpha > 1$).

9·1 Proposition *"$\mathcal{S}(x) \in V$" is true in every L_λ.*

Proof : evidently so for $\lambda = \omega$; thereafter we have languages. Given $x \in L_\zeta$, all its finite subsets will be in $L_{\zeta+1}$, and the set of them will be in $L_{\zeta+2}$. ⊣ (9·1)

9·2 Proposition *"$\mathcal{S}(x) \in V$" is true in every J_α.*

9·3 Lemma *The sequence $\langle [\zeta]^{<\omega} \mid \zeta < \omega\alpha \rangle$ is uniformly Σ_1 over every J_α.*

Proof : by a rudimentary recursion, as discussed in [M4]. ⊣ (9·3)

The $S_{\omega\beta+k}$ used in the next proof may be defined as in Dodd's book, or one might use the sets corresponding to the T_n defined in the proof of Proposition 9·7.

9·4 Lemma *In each J_α, to every set x there is an ordinal λ and a surjection $f : \lambda \xrightarrow{\text{onto}} x$.*

Proof : In J_α each set is a member of some $S_{\omega\beta+k}$, with $\beta < \alpha$, so we may derive the lemma from [Do], chapter 1, section 2, Lemma 2.42 on page 20, which Dodd proves within his theory R^+_ω that he introduces on page 12. In our terms that is the theory GJ plus TCo (in view of his Lemma 2.6) plus a version of "V = L" plus certain instances of the scheme of full foundation. He shows though that each J_α models this theory: see his Lemma 2.21 on page 14. ⊣ (9·4)

Proof of the proposition: let $f \in J_\alpha$ be a surjection from ζ to x. Then $\mathcal{S}(x) = \{f``a \mid a \in \mathcal{S}(\zeta)\}$. ⊣ (9·2)

9·5 Proposition *Let λ be a limit ordinal. Then L_λ models (RR).*

Proof : For if x is in L_ζ each of the $x \cap \{u|\phi(u, \vec{v})\}$ is in $L_{\zeta+1}$ and the set of them is in $L_{\zeta+2}$. ⊣ (9·5)

9·6 Proposition $\mathbf{HF} = L_\omega = J_1$*, and hence is a member of $L_{\omega+\nu}$ and of $J_{1+\nu}$ for each $\nu > 0$.*

Model 14: of GJs without fReR

9·7 Theorem *There is a model of* GJs *plus* $\mathbf{HF} \in V$ *in which* fReR *is false.*

Proof : Such a model is J_2. Here we shall use the existence of our single rudimentary function $\mathbb{T}$ of Definition 2·73 that for any transitive set u generates the rudimentary closure of $u \cup \{u\}$. It has these properties: the elements of $\mathbb{T}(u)$ are subsets of u and, for non-empty u, are precisely the sets of the form $S(u; x, y)$, where S is one of our list $S_0, \dots S_{11}$ of twelve rudimentary functions, and $x, y \in u$. Similarly the elements of $\mathbb{T}(\mathbb{T}(u))$ are the sets $S(\mathbb{T}(u); x, y)$, where x and y are members of $\mathbb{T}(u)$, and are subsets of $\mathbb{T}(u)$.

Our function $\mathbb{T}$ differs slightly from those used by Jensen, Devlin and Dodd, and so we make a corresponding change of notation. We write T_0 for J_1, and successively T_{n+1} for $\mathbb{T}(T_n)$. Then $T_0 \subseteq T_1 \subseteq \dots$ and $J_2 = \bigcup_{n\in\omega} T_n$.

Our intention is to build a calculus of terms, using names $\dot{S}_i$ for S_i in that finite list, and allowing as arguments names for the various T_n and their members. We define the class of terms recursively. $\mathcal{W}_0$ is to comprise symbols for the members of J_1. Having formed $\mathcal{W}_n$, we take a new symbol τ_n for T_n, and let $\mathcal{W}_{n+1}$ be the set of words of the form $\dot{S}_i \dot{(} \tau_n; v, w \dot{)}$ where v and w are words in $\mathcal{W}_n$, $0 \leqslant i \leqslant 9$, and $\dot{(}$ and $\dot{)}$ are the parentheses of the formal language we are developing.

Thus $\mathcal{W}_1$ comprises words of the form $\dot{S} \dot{(} \tau_1; x, y \dot{)}$ where x and y are in $\mathcal{W}_0$.

We suppose that our symbols are coded so that $\mathcal{W}_n \subseteq \omega \subseteq J_1 = \mathbf{HF}$, and that the $\mathcal{W}_n$ are pairwise disjoint, and that the coding has been done in some reasonable recursive way, so that in particular the map

$k \mapsto \ulcorner k \urcorner$ is recursive with recursive inverse, and that there is a recursive map $(n,k)_2 \mapsto w^n_k$ such that for each n, $(w^n_k)_k$ is a recursive enumeration of the words in $\mathcal{W}_n$.

Let $\mathcal{E}_n$ be the evaluation function of these words: so that the set of evaluations, $\mathcal{E}_n[\mathcal{W}_n]$, is precisely the set T_n just defined.

Let $\mathcal{M}_n$ be the relation on ω defined by

$$\mathcal{M}_n(w,v) \iff w \in \mathcal{W}_n \ \&\ v \in \mathcal{W}_n \ \&\ \mathcal{E}_n(w) \in \mathcal{E}_n(v).$$

Let $\mathcal{Q}_n$ be the relation on $\mathcal{W}_n$ defined by

$$\mathcal{Q}_n(w,v) \iff w \in \mathcal{W}_n \ \&\ v \in \mathcal{W}_n \ \&\ \mathcal{E}_n(w) = \mathcal{E}_n(v).$$

9·8 Remark In our context, of full extensionality, $\mathcal{Q}_n$ will of course be rudimentary in $\mathcal{M}_n$, and might therefore be dropped from this discussion; but with possible applications of the present argument in a non-extensional context in mind, we keep both predicates in play.

9·9 Proposition *There are rudimentary functions G and H such that*

$$\mathcal{M}_{n+1} = G(\mathcal{M}_n, \mathcal{Q}_n) \ \&\ \mathcal{Q}_{n+1} = H(\mathcal{M}_n, \mathcal{Q}_n)$$

Proof : We examine the passage from one stage to the next in greater detail. We have a non-empty set W of words and an evaluation $\mathcal{E}$ for those words, such that $\mathcal{E}[W] = U$, a non-empty transitive set. We add a term τ to the language to denote U. We define a new set of words thus:

$$W^+ = \{\dot{S}_i\dot{(}\tau; v, w\dot{)} \mid 0 \leqslant i \leqslant 11, v \in W, w \in W\}.$$

We define an evaluation $\mathcal{E}^+$ of the words in W^+ thus:

$$\mathcal{E}^+(\dot{S}_i(\tau; v, w)) = S_i(U; \mathcal{E}(v), \mathcal{E}(w)).$$

The evaluation of course takes place in the set theoretical universe. We wish to show that it can be carried out at a more formal level.

We define relations $\mathcal{M}$, $\mathcal{Q}$ on W, and $\mathcal{M}^+$, $\mathcal{Q}^+$ on W^+, and we shall show that the second pair are uniformly rudimentary in the first pair.

9·10 Definition
$$\begin{aligned} \mathcal{M}(v,w) &\iff_{\mathrm{df}} \mathcal{E}(v) \in \mathcal{E}(w) \\ \mathcal{Q}(v,w) &\iff_{\mathrm{df}} \mathcal{E}(v) = \mathcal{E}(w) \end{aligned}$$

and similarly

$$\begin{aligned} \mathcal{M}^+(v^+,w^+) &\iff_{\mathrm{df}} \mathcal{E}^+(v^+) \in \mathcal{E}^+(w^+) \\ \mathcal{Q}^+(v^+,w^+) &\iff_{\mathrm{df}} \mathcal{E}^+(v^+) = \mathcal{E}^+(w^+) \end{aligned}$$

9·11 Remark Let $U^+ = \mathcal{E}^+(W^+)$: then $U^+ = \mathbb{T}(U)$. Thus each evaluation $\mathcal{E}^+(v^+)$ of a word in W^+ will be a subset of U, and therefore quantification over U suffices for comparing one evaluation with another; the finitely many functions involved being rudimentary, describing the evaluations will always be Δ_0.

9·12 Lemma *For $z \in W$ and w^+ a word in W^+, the relation $\mathcal{E}(z) \in \mathcal{E}^+(w^+)$ is (uniformly) rudimentary in W, $\mathcal{M}$ and $\mathcal{Q}$.*

Proof : essentially because the class of rudimentary relations is closed under definition by rudimentarily distinguishable cases. Let w^+ be $\dot{S}_p(\tau; w_1, w_2)$. If, say, $p = 2$, we shall have

$$\mathcal{E}(z) \in \mathcal{E}^+(w^+) \iff \exists w_{3 \in W}\, (\mathcal{M}(z, w_3) \ \&\ \mathcal{M}(w_3, w_1)).$$

For the general case, the function S_i being rudimentary, the predicate $z \in S_i(u; x, y)$ will be a Δ_0 predicate of z, u, x and y; rewrite that predicate by requiring all bound variables to be restricted to members of W, and as for atomic formulæ, replace $a = b$ by $\mathcal{Q}(a,b)$ and $a \in b$ by $\mathcal{M}(a,b)$. Note that for $i = 0$, 1, 2, 3, 5, 9 and 11, u does not occur; otherwise u only occurs in contexts such as $u \cap R_i(x)$ (for $i = 6$ or 7), $u \cap R_j(x, y)$ (for $i = 4$ or 8), and $z \in u$ (for $i = 10$); and so in all cases when the formula is written out, u will occur only in atomic formulæ of the form $a \in u$; of which the formal counterparts will always be evaluated as **true**, as τ denotes U, the set of evaluations of the variables. ⊣ (9·12)

Given that lemma, the relation $\mathcal{Q}^+(v^+, w^+)$ being equivalent to $\forall z_{\in W}\, (\mathcal{E}(z) \in \mathcal{E}^+(v^+) \iff \mathcal{E}(z) \in \mathcal{E}^+(w^+)$, will be rudimentary in W, $\mathcal{M}$ and $\mathcal{Q}$.

Now for $\mathcal{M}^+$.

9·13 LEMMA *For $z \in W$ and w^+ a word in W^+, the relation $\mathcal{E}(z) = \mathcal{E}^+(w^+)$ is (uniformly) rudimentary in W, $\mathcal{M}$ and $\mathcal{Q}$.*

Proof : With Remark 9·11 in mind, we see that $\mathcal{E}(z) = \mathcal{E}^+(w^+) \iff \forall y_{\in W} \left[\mathcal{E}(y) \in \mathcal{E}^+(w^+) \iff \mathcal{M}(y, z)\right]$, since $\mathcal{M}(y, z) \iff \mathcal{E}(y) \in \mathcal{E}(z)$. ⊣ (9·13)

Now $\mathcal{M}^+(v^+, w^+) \iff \exists z_{\in W}\, \mathcal{E}^+(v^+) = \mathcal{E}(z)\ \&\ \mathcal{E}(z) \in \mathcal{E}^+(w^+)$, and so $\mathcal{M}^+$ is rudimentary in W, $\mathcal{M}$ and $\mathcal{Q}$ by the last two lemmata.

Our Proposition is now established by the uniformity of the above discussion. ⊣ (9·9)

Hence we may write a Δ_0 formula $\Phi(n, Z)$ which says that Z, a subset of ω, codes the sequences $\langle \mathcal{M}_m \mid 0 \leqslant m \leqslant n\rangle$ and $\langle \mathcal{Q}_m \mid 0 \leqslant m \leqslant n\rangle$; once we have fixed our coding, there will be a unique Z, call it Z_n, that does that.

$\mathcal{M}_0$ and $\mathcal{Q}_0$ will be in J_2, since $J_1 \in J_2$ and J_1 is an admissible set, and hence terms for the members of J_1, and the corresponding evaluation function, can be set up very easily in a way that is definable over J_1. Thus $\mathcal{M}_0$ and $\mathcal{Q}_0$ can be obtained by applying Δ_0 separation (with J_1 as a parameter) inside J_2.

Then repeated application of the Proposition, together with the fact that J_2 is rud closed, will show that each $\mathcal{M}_n$ and $\mathcal{Q}_n$ is in J_2; and by the uniformity of the progression, J_2 will model the statement that $\forall n \exists! Z \Phi(n, Z)$.

Suppose that fReR were true in J_2. Then there would be a set containing all the Z_n's, and therefore a set containing all the $\mathcal{M}_n$'s. But uniformly from $\mathcal{M}_n$ we can form the set X_n defined by

$$X_n =_{\mathrm{df}} \{k \in \omega \mid \neg \mathcal{M}_n(\ulcorner k \urcorner, w^n_k)\},$$

where $\ulcorner k \urcorner$ is our canonical symbol for k (so that $\mathcal{E}_n(\ulcorner k \urcorner) = k$ for every n) and $(w^n_k)_k$ is our recursive enumeration of $\mathcal{W}_n$. Hence there will be some ℓ such that T_ℓ contains all the X_n's. We now get a contradiction, for X_ℓ itself cannot be a member of T_ℓ. If it were, it would for some k be the evaluation $\mathcal{E}_\ell(w^\ell_k)$ of some word w^ℓ_k. But then for that k,

$$k \in X_\ell \iff \mathcal{M}_\ell(\ulcorner k \urcorner, w^\ell_k) \iff k \notin X_\ell. \qquad ⊣ (9·7)$$

9·14 PROPOSITION *There is a model of* fReCs *in which* ReR *is false.*

Proof : $V_{\omega+\omega}$; alternatively, $V_{\omega+\omega} \cap \mathbf{HC}$. ⊣ (9·14)

Model 15: of Z without restricted rank-bounded replacement

We apply the pivotal idea of Zarach [Z] to the model-building of [M1, section 4]. We have above recalled the definition of $Z(a)$; we shall use these further definitions from [M1]:

9·15 DEFINITION $b_0(n) = n; b_{k+1}(n) = 2^{b_k(n)}$; $\mathcal{F}$ is the family of functions from ω to ω that are dominated by some b_k; for u transitive, $f^a_u(n) = \overline{\overline{u \cap Z_n(a)}}$; $\mathcal{T}^a = \{u \mid \bigcup u \subseteq u\ \&\ f^a_u \in \mathcal{F}\}$. $T(a) = \mathrm{tcl}(a) \cup Z(a) \cup \{Z(a)\}$.

9·16 LEMMA *(i) If $Z(b)$ is in u, transitive, then f^b_u is not in $\mathcal{F}$, so u is not in $\mathcal{T}^b$.*

(ii) For $a \neq b$, $Z(b) \in T(b) \in \mathcal{T}^a$.

Proof : as in the proof of [M1, Theorem 4·8], but note that (ii) of the present lemma corrects a slip in the last sentence of the first paragraph of that proof. ⊣ (9·16)

Now let A be an infinite subset of ${}^\omega 2$. Let I be a proper ideal on A extending the Fréchet ideal of all finite subsets of A. For $s \in I$, let $A^s = \bigcap\{\mathcal{T}^a \mid a \in A \setminus s\}$, and let $M^s = \bigcup A^s$. Finally, set $\mathbf{M}_{15} = \bigcup_s M^s$.

9·17 Now $M^s \cup M^t \subseteq M^{s \cup t}$, since $s_1 \subseteq s_2 \implies A^{s_1} \subseteq A^{s_2}$, so AxPair will hold in $\mathbf{M}$. Further, $b \in s \implies T(b) \in A^s$, so $Z(b) \in M^s$, and so each $Z(b)$ is in $\mathcal{M} = \bigcup_s M^s$.

Indeed, $\mathbf{M}_{15}$ is a supertransitive model of Z containing all ordinals, in which full flat collection holds, and TCo; and in which every set has a rank.

But $\{Z(b) \mid b \in A\}$ is not in $\mathbf{M}_{15}$; if it were a member of u, transitive and in A^s, take $a \in A \setminus s$; then f_u^a is not in $\mathcal{F}$ so $u \notin \mathcal{T}^a$ and therefore not in A^s. Hence by Proposition 8·32, $\mathbf{M}_{15}$ is not a model of ReRI; and indeed the failure is one of rank-bounded replacement in that all the $Z(a)$ are of rank $\omega + \omega$. $\dashv$ (9·17)

9·18 As ReRI proves $\mathcal{S}(x) \in V$ and $\mathbf{HF} \in V$, Zarach's model suffices to show that that theory does not prove restricted collection.

10: Mending the flaws in Devlin's book

We turn now to a discussion of the flaws in Devlin's book *Constructibility* to which attention was drawn in Stanley's review mentioned in Section 8.

We begin with some notes on Devlin's notation, which is not always identical with ours; in this section unexplained notation will be as defined in [Dev]. We then mention a general problem, not, alas, confined to Devlin's book; then we work through Section 9 of Chapter I, where the system BS is introduced as the intended vehicle for the stream of thought in that section: we point out places where BS is inadequate, and places where, with some correction, it suffices; as we go, we suggest various revisions of Devlin's definitions; we mention passages in Chapters II and VI that are affected by those errors in Chapter I; then we introduce a system, which we call MW, that forms a mild strengthening of DBI and furnishes a framework within which the desired Σ_1 definition of the satisfaction relation $\models_u \varphi$ can be given; finally we suggest that the systems DS and GJI, each of them a strengthening of MW, offer possibly smoother treatments than that available in MW itself.

Some notes on Devlin's notation

On page 9: an n-tuple is introduced as a Wiener-Kuratowski one. In a familiar tradition, a function is treated as a subset of its image $\times$ its domain. On page 11: a *sequence* is defined as a function whose domain is an ordinal; so a finite sequence is one whose domain is a finite ordinal; a natural number is a finite ordinal.

Thus an n-sequence is an object of cardinality n consisting of ordered pairs of which the second elements form a finite initial segment of the ordinals. The 4-sequence $\langle 5, 1, 4, 2\rangle$ is written thus to distinguish it from the (WK) 4-tuple $(5, 1, 4, 2)_4$.

We maintain our policy of writing ${}^3X$ for the set of 3-sequences of members of X; X^3 for the set of WK 3-tuples of members of X; thus $\omega^3 = \omega \times (\omega \times \omega)$.

10·0 Remark Devlin makes no distinction between $(X \times X) \times X$ and $X \times (X \times X)$, writing both as X^3. With weak systems that is scarcely satisfactory, since the variant given of Model 4, using weak right WK-rank, is a model of ReS_0 which contains $(\omega \times \omega) \times \omega$ but not $\omega \times (\omega \times \omega)$; and, following the lead of Model 9, we can get models of BS containing either, but not both, of ${}^3(\omega \times (\omega \times \omega))$ and ${}^3((\omega \times \omega) \times \omega)$.

As for abbreviations of lists of variables, Devlin follows the useful convention that $\vec{x} \in A$ abbreviates $x_1 \in A \;\&\; \ldots \;\&\; x_n \in A$, whereas $(\vec{x}) \in A$ indicates that the corresponding WK n-tuple is in A.

We shall make a slight change to his notation: we shall use the letters φ, ϑ and χ for formal formulæ, ψ and θ for building sequences, or similar sequences of formulæ, and α, β and γ for (finite) attempts at addition. The reader will be able to distinguish a reference to his Lemma 9.4 from one to our Lemma 9·4 by the position of the dot.

The problem of levels of language

There is an ambiguity over the meaning of Δ_0 (which Devlin calls Σ_0). Devlin on page 230 writes:

> "In class terms a function is Σ_0 if of the form $\{(y, \vec{x}) \mid \Phi(y, \vec{x})\}$ where Φ is a Σ_0 formula of LST. In set-theoretic terms a function f is said to be Σ_0 if there is a Σ_0 formula φ of $\mathcal{L}$ such that for any $\vec{x}$, y, if M is a transitive set such that $\vec{x}$, $y \in M$, then
>
> $$f(\vec{x}) = y \iff \models_M \varphi(\mathring{y}, \mathring{\vec{x}})."$$

10·1 Remark The second definition has the advantage that one can then legitimately quantify over all φ; but the disadvantage that the definition collapses if TCo is false; whereas the first definition is still operational. Thus Devlin's remark that the two definitions are "equivalent" is dangerous.

Definition of Finseq

10·2 REMARK The definition, on page 33 of [Dev], of Finseq might not be as intended; what is written is that members of Finseq are functions with domain a non-empty bounded subset of ω (possibly not a proper initial segment of ω).

We shall suppose that the definition has been corrected to mean that members of Finseq are functions with domain a non-empty bounded initial segment of ω; that is still Δ_0, so no harm has been done.

Lemmata 9.1 and 9.2 are correct.

The trouble starts on page 34, with the formula $F_\wedge(\theta, \phi, \psi)$: in its definition the clause "$\mathrm{Dom}\,(\theta) = \mathrm{Dom}\,(\phi) + \mathrm{Dom}\,(\psi) + 3$" occurs, and thus addition of natural numbers is being used to define concatenation.

Lemma 9.3: "$F_\wedge$ is Δ_0"

Though the other parts of Lemma 9.3 are correct as stated, that statement is false—Solovay has remarked that that can be seen by Ehrenfeucht-Fraissé games.

Its falsehood may indeed be established by arguments from Gandy's paper, where he proves (by a quantifier elimination argument, which is what, presumably, Solovay had in mind) that every Δ_0 subset of ω is finite or cofinite; from that he shows that the graph of addition is not Δ_0, and further deduces that the graph of concatenation is not Δ_0.

Suppose we consider a language which accepts as atomic formulæ all finite constant sequences of $*$'s. Note that each such sequence is expressible as $\{*\} \times n$ for some n.

Let $\tau_{n,k}$ be the term

$$\big(\{*\} \times (k+3) \smallsetminus \{(*,0)_2, (*, n+1)_2, (*, k+2)_2\}\big) \cup \big\{(\dot{(}, 0)_2, (\wedge, n+1)_2, (\dot{)}, k+2)_2\big\},$$

where $\dot{(}$, $\wedge$ and $\dot{)}$ code the left parenthesis, conjunctive connective and right parenthesis of the formal language. Then $k = n + m \iff F_\wedge(\tau_{n,k}, \{*\} \times n, \{*\} \times m)$, and thus $F_\wedge$ cannot be Δ_0 as the graph of addition is not.

Complexity of $F_\wedge$

10·3 However, the Lemma is nearly correct in that one might say that $F_\wedge$ is Δ_0 in any sufficiently long attempt at integer addition. We therefore propose to revise the definition of $F_\wedge$, by making explicit the attempt at integer addition that is being used, as follows:

$$\begin{aligned}
\mathrm{At}_+(\vartheta;\alpha) \iff_{\mathrm{df}} & \big[\mathrm{Fn}(\alpha)\ \&\ \mathrm{Dom}\,(\alpha) \supseteq \mathrm{Dom}\,(\vartheta) \times \mathrm{Dom}\,(\vartheta)\ \&\ \alpha \text{ is an attempt at integer addition }\big];\\
F^0_\wedge(\vartheta,\varphi,\chi;\alpha) \iff_{\mathrm{df}} & \big[[\mathrm{Dom}\,(\varphi) < \mathrm{Dom}\,(\vartheta)]\ \&\ [\mathrm{Dom}\,(\chi) < \mathrm{Dom}\,(\vartheta)]\\
& \quad \&\ [\mathrm{Dom}\,(\vartheta) = \alpha(\mathrm{Dom}\,(\varphi)+1, \mathrm{Dom}\,(\chi)+1)+1]\\
& \quad \&\ [\vartheta(0) = 0]\ \&\ [\vartheta(1) = 6]\ \&\ [\vartheta(\|\vartheta\|) = 1]\\
& \quad \&\ \forall i_{\in \mathrm{Dom}(\varphi)}\,[\vartheta((i+1)+1) = \varphi(i)]\\
& \quad \&\ \forall i_{\in \mathrm{Dom}(\chi)}\,[\vartheta(\alpha(\mathrm{Dom}\,(\varphi)+1, i+1)) = \chi(i)]\big];\\
F_\wedge(\vartheta,\varphi,\chi) \iff_{\mathrm{df}} & \mathrm{Finseq}(\vartheta)\ \&\ \mathrm{Finseq}(\varphi)\ \&\ \mathrm{Finseq}(\chi)\ \&\ \exists\alpha\Big[\mathrm{At}_+(\vartheta;\alpha)\ \&\ F^0_\wedge(\vartheta,\varphi,\chi;\alpha)\Big].
\end{aligned}$$

PROPOSITION At_+ *and* $F^0_\wedge$ *are* Δ_0^{ReS}; $F_\wedge$ *is* Δ_1^{ReS}.

Proof: At_+ and $F^0_\wedge$ are composed entirely of S_0-suitable terms; therefore $F_\wedge$ is Σ_1^{ReS}; with Propositions 2·14 and 2·57 in mind, and because there is no disagreement between two attempts at addition where both are defined, we see that $F_\wedge$ is equivalent in ReS to the formula

$$\mathrm{Finseq}(\vartheta)\ \&\ \mathrm{Finseq}(\varphi)\ \&\ \mathrm{Finseq}(\chi)\ \&\ \forall\alpha\Big[\mathrm{At}_+(\vartheta;\alpha) \Longrightarrow F^0_\wedge(\vartheta,\varphi,\chi;\alpha)\Big]$$

which is Π_1^{ReS}. ⊣ (10·3)

The definition of Build

The trouble caused by $F_\wedge$ continues with the next Lemma:

Lemma 9.4 "Build(φ,ψ) is Δ_0."

The proof is certainly invalid since it uses 9.3. The statement is suspect: suppose we add to the definition of Build extra clauses admitting the "formulæ" $\{*\}\times n$, as atomic: that would not change the Δ_0 character of Build, as those clauses would be Δ_0, even (by Gandy's proof that ω is S_0-semi-suitable) when quantified over $n\in\omega$. Then for $\tau_{n,k}$ the term defined above,

$$k=n+m \Longleftrightarrow \mathrm{Build}(\tau_{n,k},\langle\{*\}\times n,\{*\}\times m,\tau_{n,k}\rangle),$$

and therefore Build (in the form modified to allow atomic formulæ of the form $\{*\}\times n$) cannot be Δ_0 as the graph of addition is not.

Complexity of Build

As one might again say that Build is Δ_0 in any sufficiently long attempt at addition, we shall make a similar revision of its definition by introducing a name, β, for the attempt at addition on which the formula implicitly relies; but first there is a further danger to be noted. Suppose that Build(φ,ψ). Now let θ result from ψ by adding various formulæ to the sequence, keeping φ always the last and observing the other rules of Build ; for example one might add many atomic formulæ and build up long conjunctions of atomic formulæ or one might interpolate the terms of some ψ' that builds some other formula, subject only to the condition on variables, which is that the only variables with bound occurrences are those with such occurrences in φ. Then θ also builds φ according to Devlin's definition of Build, but might easily list formulæ that contain free variables not occurring in φ or that are actually longer than φ and therefore beyond the domain of competence of the attempt at addition being used. Ideally one would like to require every formula listed to be actually a subformula of the formula being built, but we have not yet defined the notion of formula, let alone subformula. We shall therefore, in our reformulation of the definition of Build, impose the milder requirement that no finite sequence listed by ψ is strictly longer than φ.

10·4 Here is our revised definition:

$$\begin{aligned}
\mathrm{Build}^0(\varphi,\psi) &\Longleftrightarrow_{\mathrm{df}} \mathrm{Finseq}(\varphi)\ \&\ \mathrm{Finseq}(\psi)\ \&\ [\psi_{\|\psi\|}=\varphi]\\
&\qquad\qquad \&\ \forall i_{\in \mathrm{Dom}\,(\psi)}\,[\mathrm{Finseq}(\psi_i)\ \&\ \mathrm{Dom}\,(\psi_i)\leqslant \mathrm{Dom}\,(\varphi)]\\
\mathrm{Build}^1(\varphi,\psi;\beta) &\Longleftrightarrow_{\mathrm{df}} \forall i_{\in \mathrm{Dom}\,(\psi)}\big[\mathrm{PFml}(\psi_i)\vee \exists j,k_{\in i}\,F^0_\wedge(\psi_i,\psi_j,\psi_k;\beta)\vee \exists j_{\in i}\,F_\neg(\psi_i,\psi_j)\\
&\qquad\qquad \vee\ \exists j_{\in i}\,\exists u_{\in \mathrm{ran}(\varphi)}\,(\mathrm{Vbl}(u)\ \&\ F_\exists(\psi_i,u,\psi_j))\big];\\
\mathrm{Build}(\varphi,\psi) &\Longleftrightarrow_{\mathrm{df}} \mathrm{Build}^0(\varphi,\psi)\ \&\ \exists\beta\,[\mathrm{At}_+(\varphi;\beta)\ \&\ \mathrm{Build}^1(\varphi,\psi;\beta)]
\end{aligned}$$

PROPOSITION Build$^0(\varphi,\psi)$ *and* Build$^1(\varphi,\psi;\beta)$ *are* Δ_0^{ReS}; *Build(φ,ψ) is* Δ_1^{ReS}.

Proof : the first part by inspection; for the second, note that Proposition 2·57 implies that

$$\vdash_{\mathsf{ReS}} \mathrm{Build}(\varphi,\psi) \Longleftrightarrow \big[\mathrm{Build}^0(\varphi,\psi)\ \&\ \forall\beta\,[\mathrm{At}_+(\varphi;\beta)\Longrightarrow \mathrm{Build}^1(\varphi,\psi;\beta)]\big] \qquad \dashv (10{\cdot}4)$$

10·5 PROBLEM Does the absence of uniqueness matter ? One might try for a minimality condition of the form "ψ builds φ and no proper subsequence of ψ does". But that hardly seems worth the effort, as the redundancy in such formulæ as $\varphi\wedge(\vartheta\wedge\varphi)$ is liable to reappear in the corresponding building functions.

The formula Seq

At the bottom of page 36 of [Dev] a formula Seq(u,a,n) is defined which expresses the statement that u is the set of all finite sequences, of length less than n, of elements of a, and is correctly stated to be Σ_1. But this formula gives trouble in the proof of the next Lemma.

Lemma 9.5 "Seq is Δ_1^{BS}"

According to Solovay, the statement is false, "as may be seen using a forcing argument". I have been unable to demonstrate the falsity of the assertion using my present methods, but the model-building of Section 5 will pin-point flaws in the argument as printed.

In Model 6, there is no u such that $\mathrm{Seq}(u, \omega, 4)$; so in that model the proposed Π_1 form of the definition is true of everything, and the proposed Σ_1 form is false of everything. So the equivalence is not a theorem of BS, and the proposed proof of I.9.5 cannot succeed.

In greater detail:

10·6 The first displayed formula in the proof of 9.5 asserts that

"it is clear from the definition of BS that:

$$\mathrm{BS} \vdash (\forall a)(\forall n \in \omega)(\exists u)\mathrm{Seq}(u, a, n)."$$

But that statement, on lines 5 and 6 of page 37, is not a theorem of BS, as is shown by Model 9, in which there is no u with $\mathrm{Seq}(u, \{\omega\} \times \omega, 4)$, or, indeed, by Model 6, in which for no infinite a is there a u with $\mathrm{Seq}(u, a, 4)$.

10·7 Devlin wishes to bound the quantifier f by the set of n-sequences of finite sequences from a.

First problem: is it a set ? No, even if a has only two members: if A is the class of n-sequences of finite sequences of members of a, the class B of finite sequences of members of a is a subclass of $\bigcup\bigcup\bigcup A$; and Model 5 is a supertransitive model of BS not containing the set BIN of finite binary sequences, the reason being that $\overline{\overline{BIN \cap V_n}} = 2^{n-3}$ for all $n \geqslant 3$; and hence in Model 5, the class A is not a set.

Second problem: would B be a bounding class for the quantifier $\exists f$? No; it is the wrong type. The values of f are not finite sequences but sets of finite sequences.

However, the faulty proof of [Dev] Lemma I.9.5 becomes true if we confine a to being finite. First, a general lemma:

10·8 Proposition *Let G be a Δ_0 class. Then*

$$\vdash_{\mathsf{ReS}} \mathrm{Fn}(G) \;\&\; \mathrm{Dom}\,(G) = V \Longrightarrow \forall a\big(a \textit{ finite } \Longrightarrow G``a \in V\big).$$

Proof : Let $f : n \longleftrightarrow a$. Consider the class $n \cap \{k \mid G``\{f(i) \mid_i i < k\} \notin V\}$. That is $\Pi_1$, and so if not empty, a minimal element exists, which, trivially, is $> 0$, and hence equals $k+1$ for some $k$. Thus $G``\{f(i) \mid_i i < k\} \in V$; to that we must add $\{G(f(k))\}$. ⊣ (10·8)

10·9 Remark Under the hypotheses of the Proposition, $G``a$ will be finite.

10·10 Lemma (ReS) *If a is finite, then for each n there is a u such that $Seq(u, a, n)$. Hence for a finite, $Seq(u, a, n) \Longleftrightarrow \forall u' \neq u\, \neg Seq(u', a, n)$.*

Proof : an induction on n. The induction step will require us to form $\{x \cup y \mid_{x,y} x \in A \;\&\; y \in B\}$, where A and B are finite; but that is of the form $g``(A \times B)$ where g is rudimentary and provably total in ReS, and thus satisfies the hypotheses of Proposition 10·7. $A \times B$ will be finite by Proposition 2·14. ⊣ (10·10)

Lemma 9.6 "Fml(x) is Δ_1^{BS}"

This result is actually true, indeed it can be sharpened to "Fml(x) is Δ_1^{ReS}", but the proof given is seriously flawed.

There is a slight error in the definition of $A(x)$; replace the third occurrence of 'n' by 'm'.

At the bottom of page 37, in the proof of Lemma 9.6, the claim, said to be "easily checked", that

$$\text{"BS} \vdash \forall x \exists y[y = A(x)]."$$

is untrue, as is shown by Model 9 for appropriate infinite x.

However, this claim is needed only in the case that x is a finite sequence, when the result is indeed provable in the following form:

10·11 Lemma (ReS) *If x is a finite sequence, then $A(x)$ is a finite set.*

Proof : Let x be a finite set, and k a finite ordinal. Then the set $B(k,x)$ of functions from k to x is a Δ_0 subclass of $\mathcal{P}(x\times k)$, which as we have seen is, provably in ReS, a finite set.

This principle, applied twice, will yield the Lemma. $\dashv$ (10·11)

To complete the proof of 9.6, we appeal twice to our Proposition 10·4, that Build is Δ_1^{ReS}: first, it implies that Fml(x), being of the form $\exists f\mathrm{Build}(x,f)$, is Σ_1; and secondly, in view of our Metatheorem 2·24, it implies that, v being the finite set $A(x)$, the subformula $(\exists f_{\in v})\mathrm{Build}(x,f)$ is (taking the Π_1 form of Build), Π_1^{ReS}, and thus that the given alternative form of Fml is indeed Π_1.

Lemma 9.7

The above arguments, appropriately modified, will prove Lemma 9.7, with Δ_1^{BS} sharpened to Δ_1^{ReS}. The restriction in Fml(x,u) of the formal constants to those for members of u is Δ_0 and causes no difficulty.

The definition of Fr

Devlin now writes

"Our next task is to write down an LST formula Fr(φ,x) such that

$$\mathrm{Fr}(\varphi,x)\leftrightarrow\mathrm{Fml}(\varphi)\wedge[x\text{ is the set of variables occurring free in }\varphi]."$$

But the formula that he proposes does not work: given the fact that a ψ with Build(φ,ψ) may contain many formulæ with free variables not among those of φ, the truth of his formula Fr(φ,x) only guarantees that x *contains* all the variables with at least one free occurrence in φ. That invalidates the proof of his Lemma 9.8.

But really one wishes to know whether a particular occurrence is free or not. So it would be better to aim at achieving that. We shall be able to do so by using the relation Sub that Devlin is, without using Fr, about to define; so let us go on to that and postpone the present definition.

The definition of Sub

First, two minor points: in the fifth line from the bottom of page 39 of [Dev], for $F_\in$ one should read $F_\exists$; and in the build-up to Lemma 9.9, the phrase "the scope of this quantifier" is used but not defined.

Lemma 9.9: "Sub is Δ_1^{BS}"

The Lemma is essentially correct, and indeed admits a sharpening of Δ_1^{BS} to Δ_1^{ReS}, but there is a problem with Devlin's suggestion for Sub: as $F_\wedge$ is used, it is not immediately clear that Sub will be Σ_1. We could appeal to Metatheorem 2·24 since the domain of ψ is a finite set, but it will be better to follow the style of our earlier revisions and first formulate a Δ_0 version of Sub with explicit names for the various supporting characters. Here it is, where $S(\cdot,\cdot,\cdot,\cdot)$ is the Δ_0 formula given by Devlin on his page 39.

$$\begin{aligned}
&\mathrm{Sub}^0(\varphi',\varphi,v,t;\,\psi,\theta;\,\beta)\iff_{\mathrm{df}}\\
&\quad\underbrace{\mathrm{Vbl}(v)\ \&\ \mathrm{Const}(t)}_{\mathfrak{A}(v,t)}\ \&\\
&\quad\underbrace{\mathrm{At}_+(\varphi;\beta)\ \&\ \mathrm{Build}^0(\varphi,\psi)\ \&\ \mathrm{Build}^1(\varphi,\psi;\beta)}_{\mathfrak{B}(\varphi,\psi;\beta)}\ \&\\
&\quad\underbrace{\mathrm{At}_+(\varphi';\beta)\ \&\ \mathrm{Build}^0(\varphi',\theta)\ \&\ \mathrm{Dom}\,(\theta)=\mathrm{Dom}\,(\psi)}_{\mathfrak{C}(\varphi';\psi,\theta;\beta)}\ \&\ \underbrace{\theta_{\|\theta\|}=\varphi'}_{\mathfrak{D}(\varphi';\theta)}\ \&\\
&\quad\underbrace{\begin{aligned}\forall i_{\in\mathrm{Dom}\,(\psi)}\big[&\exists j,k_{\in i}\,(F^0_\wedge(\psi_i,\psi_j,\psi_k;\beta)\ \&\ F^0_\wedge(\theta_i,\theta_j,\theta_k;\beta))\\
&\vee\ \exists j_{\in i}\,(F_\neg(\psi_i,\psi_j)\ \&\ F_\neg(\theta_i,\theta_j))\\
&\vee\ \exists j_{\in i}\,\exists u_{\in\mathrm{ran}(\varphi)}\,(\mathrm{Vbl}(u)\ \&\ u\neq v\ \&\ F_\exists(\psi_i,u,\psi_j)\ \&\ F_\exists(\theta_i,u,\theta_j))\\
&\vee\ \exists j_{\in i}\,(F_\exists(\psi_i,v,\psi_j)\ \&\ (\theta_i=\psi_i))\\
&\vee\ S(\theta_i,\psi_i,v,t)\big]\end{aligned}}_{\mathfrak{E}(\varphi,v,t;\psi,\theta;\beta)}
\end{aligned}$$

Then we define, omitting the listing of free variables given in the underbraces to the above display,

$$\mathrm{Sub}(\varphi, \varphi', v, t) \iff_{\mathrm{df}} \mathfrak{A} \;\&\; \exists\psi\exists\theta\exists\beta \Big[\mathfrak{B} \;\&\; \mathfrak{C} \;\&\; \mathfrak{E} \;\&\; (\theta_{\|\theta\|} = \varphi')\Big]$$

and prove in ReS that a φ' with $\mathrm{Sub}(\varphi, \varphi', v, t)$ always exists (by a recursion of finite length); whence

$$\vdash_{\mathsf{ReS}} \mathrm{Sub}(\varphi, \varphi', v, t) \iff \mathfrak{A} \;\&\; \forall\psi\forall\beta\forall\theta\Big[[\mathfrak{B} \;\&\; \mathfrak{C} \;\&\; \mathfrak{E}] \implies (\theta_{\|\theta\|} = \varphi')\Big]$$

10·12 Proposition *$\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ and $\mathfrak{E}$ are all Δ_0; Sub is Δ_1^{ReS}.*

10·13 Remark We should (but won't) prove that ϕ' is a formula, by modifying θ to give a building sequence for it, and that the outcome of these tests is independent of the building sequence used.

We may now characterise *bound occurrences* of a given variable in a formula as those for which no change results in the formula when the above procedure is followed for substituting some constant for that variable, and then we may define *sentences* to be those formulæ whose every occurrence of a variable is bound.

With trifling loss of generality we take that constant to be $\mathring{\varnothing}$, the constant denoting the empty set, which will usually be a member of the sets in which we shall wish to interpret formulae, and may now give our definition of Sen^0 and Sen.

10·14 Definition i) $\mathrm{Sen}^0(\varphi; v; \psi, \theta, \gamma) \iff_{\mathrm{df}} \mathrm{Sub}^0(\varphi, \varphi; v, \mathring{\varnothing}; \psi, \theta, \gamma)$.

ii) $\mathrm{Sen}(\varphi) \iff_{\mathrm{df}} \mathrm{Fml}(\varphi) \;\&\; \forall v_{\in \mathrm{ran}(\varphi)} \big[\mathrm{Vbl(v)} \implies \exists\psi\exists\theta\exists\gamma \mathrm{Sen}^0(\varphi; v; \psi, \theta, \gamma)\big]$.

iii) Let v be a formal variable. If $\varphi(i) = v$, that occurrence of v at i in φ is *bound* $\iff_{\mathrm{df}}$ whenever $\mathrm{Sub}(\varphi, \varphi', v, \mathring{\varnothing})$, $\varphi'(i) = v$.

10·15 Remark It is necessary to include $\mathrm{Fml}(\varphi)$ in the definition of $\mathrm{Sen}(\varphi)$, lest φ have no variables at all in its range.

10·16 Remark In a manner to which we have become accustomed, the above concepts will be Δ_0 in any appropriate parameter, and Δ_1^{ReS} if no parameters are mentioned.

10·17 Definition $\mathrm{Sen}(\varphi, u) \iff_{\mathrm{df}} \mathrm{Fml}(\varphi, u) \;\&\;$ all occurrences of its variables are bound.

10·18 Lemma *$\mathrm{Sen}(\varphi, u)$ is Δ_1^{ReS}.*

The definition of Fr reconsidered

10·19 Definition $\mathrm{Fr}(\varphi, x) \iff_{\mathrm{df}} x = \mathrm{Vbl} \cap \{\varphi(i) \mid \text{ that occurrence is bound}\}$

10·20 Remark Such an x will be a Δ_1^{ReS} subclass of a bounded subset of ω, and therefore can be proved in ReS to be a set, by an argument reminiscent of the proof of Lemma 2·52.

The above wffs are Δ_0 in any w containing sufficiently many building sequences (and their attendant attempts), so we could give an alternative prove of the existence of an x with $\mathrm{Fr}(\varphi, x)$ by using Δ_0 separation with w as a parameter.

Lemma 9.8: "Fr is Δ_1^{BS}"

The Lemma is true, and can be sharpened to "Fr is Δ_1^{ReS}".

The definition of the precursor $S(u, \varphi)$ to Sat

At the bottom of page 40, Devlin introduces a formula $S(u, \varphi)$ and alleges that it defines the satisfaction relation. There is a minor slip in the last line of page 40: for $F_\in$ read $F_\exists$; but there is a more substantial error in the formula. Devlin's strategy is to build two finite sequences f and g of sets of formulæ; roughly at stage i, $f(i)$ is to comprise all formulæ of $\mathcal{L}_u$ built up within i steps from atomic formulae; and $g(i)$ is to comprise the sentences of $\mathcal{L}_u$ which are both members of $f(i)$ and true in u. But let ϑ be a member of $f(i)$ which has a free occurrence of a variable, and therefore is not a sentence; then $\vartheta \notin g(i)$; let χ be $\neg\vartheta$; then according to his definition χ should be placed in $g(i+1)$; but it is not a sentence. Thus his definition should be amended by adding the requirement that the members of each $g(i)$ are sentences.

We shall also require a bound for the length of formulæ to be considered when evaluating the truth of φ. Atomic formulæ are of length 5; by inspection, the length of formulæ in $f(i+1)$ will be at most three times the length of the longest formula in $f(i)$; if φ is of length ℓ it will be in $f(\ell)$; thus a bound for the length of any other formula in $f(\ell)$ is 5.3^ℓ, and we should therefore establish in ReS that every integer is in the domain of an attempt at the function $n \mapsto 3^n$. Arguments similar to those we have given for attempts at addition will suffice for that, and will show in addition that the property of being such an attempt is Δ_0.

Let us now revise the definition of $S(u,\varphi)$ in the light of these remarks and our previous revisions. The predicate E used in the definition of S^3 is that defined in [Dev] in the lower half of page 40.

$$\begin{aligned}
S^0(u,\varphi) &\iff_{\mathrm{df}} u \neq \varnothing \;\&\; \mathrm{Sen}(\varphi,u);\\
S^1(\varphi;f,g) &\iff_{\mathrm{df}} \mathrm{Finseq}(f) \;\&\; \mathrm{Finseq}(g) \;\&\; \mathrm{Dom}\,(f) = \mathrm{Dom}\,(g)\\
&\qquad\qquad \& \;\forall i_{\in \mathrm{Dom}\,(f)}\, \forall x_{\in f(i)\cup g(i)}\, [\mathrm{Finseq}(x) \;\&\; \mathrm{Dom}\,(x) \leqslant \mathrm{Dom}\,(\varphi)]\\
S^2(u;\chi;f;\alpha) &\iff_{\mathrm{df}} \mathrm{At}_+(\chi;\alpha) \;\&\; \Big[\big[\chi \in f(0) \iff \mathrm{PFml}(\chi,u)\big] \;\&\\
&\qquad \forall j_{\in \mathrm{Dom}\,(f)}\, \forall i_{\in j}\, \big[\chi \in f(i+1) \iff\\
&\qquad\quad \big(\chi \in f(i) \vee \exists \vartheta,\vartheta'_{\in f(i)}\, F^0_\wedge(\chi,\vartheta,\vartheta';\alpha)\\
&\qquad\qquad \vee \exists \vartheta_{\in f(i)}\, F_\neg(\chi,\vartheta)\\
&\qquad\qquad \vee \exists \vartheta_{\in f(i)}\, \exists v_{\in \mathrm{ran}(\chi)}\, [\mathrm{Vbl}(v) \;\&\; F_\exists(\chi,v,\vartheta)]\big)\big]\Big];\\
S^3(u;\chi;f,g;\alpha;\psi,\theta) &\iff_{\mathrm{df}} \mathrm{At}_+(\chi;\alpha) \;\&\; \Big[\big[\chi \in g(0) \iff E(\chi,u)\big] \;\&\\
&\qquad \forall j_{\in \mathrm{Dom}\,(f)}\, \forall i_{\in j}\, \big[\chi \in g(i+1) \iff\\
&\qquad\quad \big(\chi \in g(i) \vee \exists \vartheta,\vartheta'_{\in g(i)}\, F^0_\wedge(\chi,\vartheta,\vartheta';\alpha)\\
&\qquad\qquad \vee \exists \vartheta_{\in f(i)}\, \mathrm{Sen}^0(\vartheta;v;\psi,\theta;\alpha) \;\&\; (\vartheta \notin g(i) \;\&\; F_\neg(\chi,\theta))\\
&\qquad\qquad \vee \exists \vartheta_{\in f(i)}\, \exists v_{\in \mathrm{ran}(\chi)}\, \exists x_{\in u}\, \exists \vartheta'_{\in g(i)}\\
&\qquad\qquad\qquad [\mathrm{Vbl}(v) \;\&\; F_\exists(\chi,v,\vartheta) \;\&\; \mathrm{Sub}^0(\vartheta',\vartheta;v,\mathring{x};\psi,\theta;\alpha)]\big)\big]\Big]\\
S^4(\varphi;g) &\iff_{\mathrm{df}} \varphi \in g(\|g\|)
\end{aligned}$$

10·21 PROPOSITION *Each $S^{\mathfrak{k}}$ is Δ_0.*

We are getting warm: we may now show that

$$\begin{aligned}
\models_u \varphi \iff & S^0(u,\varphi) \;\&\\
&\exists f, \exists g \Big[S^1(\varphi;f,g)\\
&\qquad \& \;\big[\text{for all appropriate } \chi \text{ and for all sufficiently long } \alpha \; S^2(u;\chi;f;\alpha)\big]\\
&\qquad \& \;\big[\text{for all appropriate } \chi \text{ and for all sufficiently long } \alpha,\psi,\theta,\; S^3(u;\chi;f,g;\alpha;\psi,\theta)\big]\\
&\qquad \& \; S^4(\varphi;g)\Big]
\end{aligned}$$

Here "appropriate" is to mean the Δ_0 requirement that χ is a finite sequence all of whose terms are either symbols of our formal language or constants for members of u, and whose length is at most $p =_{\mathrm{df}} 5.3^{\mathrm{Dom}(\varphi)}$; and "sufficiently long" is, in the case of α, an attempt at integer addition, to mean that its domain includes $p \times p$.

We might remark here that a further restraint on the possible values of χ is possible whilst preserving the above equivalence, namely by requiring the formal variables occurring in χ to be among those occurring in φ.

The definition of Sat

At this point, Devlin's strategy (in our revised context) is to convert the above universal quantifications, which we have qualified with phrases such as "appropriate" and "sufficiently long", to restricted ones by finding a set w which will contain sufficiently many possible values of the variables χ, α, ψ, θ to preserve the intended meaning of $S(u,\varphi)$ and, as his candidate for w, defines, on his page 41, a class $w(u,\varphi)$. But there is a final problem: as is shown by Model 9, the class $w(u,\phi)$ is not provable in BS to be a set. Even if we adopt the further restraint on variables mentioned above, and give a correspondingly restrained definition of a class we might call $w^*(u,\varphi)$, its set-hood, for arbitrary u, would not be provable in BS.

Lemma 9.10 "the LST formula Sat(u,ϕ) is Δ_1^{BS}"

The statement is false, so this time there is no hope of saving the proof. In Model 6, for no infinite set x does there exist a y with $\mathrm{Seq}(y,x,4)$; for u infinite, the set a of names of members of u will be infinite, and so the given Σ_1 formula for $\mathrm{Sat}(u,\varphi)$ will always be false; but then so is the Σ_1 version of $\mathrm{Sat}(u,\neg\varphi)$; but one of them ought to be true !

10·22 Remark We have just used the axiom of infinity to build our counterexample, and necessarily so, for we could indeed, without invoking the axiom of infinity, give a Σ_1^{ReS} definition of $\models_u \varphi$ for *finite* u by adopting the above restraint, so that the set-hood (and finiteness) of the correspondingly restrained class, $w^*(u,\varphi)$, would be provable in ReS. Thus Lemma 9.10 holds in sharpened form for u finite.

But as we wish to use and to define truth in infinite sets, we must seek a set theory, including the axiom of infinity, sufficiently strong to prove that Devlin's classes $w(u,\varphi)$ are indeed sets, even when u is infinite; for if they are, the rest of his argument is correct and we shall finally have reached a Δ_1 definition of Sat.

Before discussing possible candidates for such a theory, we comment briefly on some other passages in Chapters I, II and VI of Devlin's book.

Lemma 9.12

The amended proofs of Lemmata 9.6 and 9.7 will now yield Lemma 9.12, with Δ_1^{BS} sharpened to Δ_1^{ReS}.

Lemma 9.14 is in error named Lemma 9.4.

Errors in Chapter II

Amenability

On page 45, in section 10, a set M is defined to be *amenable* if it is transitive and satisfies five conditions: closed under pairing, sumsets, and cartesian products; contains ω; and closed under $\dot{\Delta}_0(M)$ separators, though Devlin writes "Σ_0."

Given the ambiguity in the meaning of Δ_0 discussed in Remark 10·1, I would suggest defining an amenable set as a transitive set containing ω and closed under the functions in the finite set $R_0,\dots R_7$, listed in paragraph 2·61, of generators of the class $\mathcal{B}$.

On page 65, in section 2 of Chapter 2, Devlin writes

> "by repeating the proof of I.9.10 for $\mathcal{L}$ in place of LST, we obtain a proof of the fact that the class Sat $(=\{(u,\varphi)\mid \mathrm{Sat}(u,\varphi)\})$ is uniformly Δ_1^M for amenable sets M. That is, there is a Σ_1 formula $\psi(x,y)$ of $\mathcal{L}$ and a Π_1 formula $\theta(x,y)$ of $\mathcal{L}$ such that for any amenable set M, if $u,\varphi\in M$ then
>
> $$\mathrm{Sat}(u,\varphi)\iff\models_M \psi(\mathring{u},\mathring{\varphi})\iff\models_M \theta(\mathring{u},\mathring{\varphi}).$$
>
> (The formulas ψ and θ are just the $\mathcal{L}$ analogues of the LST formulas described in I.9.10.)"

With Model $\mathbf{M}_{6,5}$ in mind, we give a counterexample to the alleged uniformity for the specific formulation of Sat given by Devlin.

Let u be an infinite transitive set containing only finitely many sets of cardinality 5. Let M be the rud closure of $u\cup\{u\}$. Let N be the union of the class of all transitive members of M which have only finitely many sets of cardinality 5. So $u \in N$ and N is amenable. Suppose we wish to evaluate the truth in u of the sentence $\bigwedge x \bigvee y\ x \epsilon y$: readers will recognise that that is true in many u and also false in many others. M can correctly make that evaluation; so the Π_1 form holds in M; therefore in N; therefore, if Devlin's assertion were correct, the Σ_1 form would hold in N. But it is false in N, because all atomic formulae such as $(x \epsilon y)$ are sequences of length 5, and therefore, u being infinite, the set of atomic sentences of $\mathcal{L}_u$ is infinite and therefore not a member of N; and therefore not available to be the $f(0)$ of Devlin's formulation.

10·23 REMARK This argument suggests that no other pair of Π_1 and Σ_1 formulæ will work for amenable sets such as N, as information concerning the infinitely many atomic formulæ must be coded in some way into any truth-evaluation, which cannot therefore lie in N if the said information can be recovered by some rudimentary function.

If one calls a set M *$\mathcal{S}$-amenable* if it is amenable and for each $x \in M$ $\mathcal{S}(x) \in M$, then Sat will indeed be uniformly Δ_1^M for $\mathcal{S}$-amenable sets M. By the remark following Proposition 10·26, the same will be true for amenable sets M that are *weakly $\mathcal{S}$-amenable* in the sense that for each x in M and each k in ω, $[x]^k$ is in M.

Errors in Chapter VI

Lemma VI.1.13 "Sat^A is Δ_1^{BS}"

The statement is false, being a generalisation of the false Lemma I.9.10.

Lemma VI.1.14 "truth for Δ_0 wffs is uniformly Σ_1 for transitive rud-closed structures $\langle M, A\rangle$."

This ought to be correct, and it is of the greatest importance. We make some minor comments, but defer to a sequel, *Rudimentary Recursion*, a full discussion of the proof.

On page 242, in the proof of Lemma VI.1.14, the displayed formula in the middle of the page is incomplete as 't' does not occur on the right-hand side. I suggest that the clause $f(\mathrm{Dom}\,(f) - 1) = t$ should be added.

There is a delicate visual confusion of the meaning of brackets in the following subformula of that same displayed formula:

$$\big(f(i) = \mathring{F}_0 \dot{\big(} f(j), f(k) \dot{\big)} \Longrightarrow g(i) = F_0(g(j), g(k))\big)$$

where the two parentheses that I have dotted are part of the syntax of the object language, not the language of discourse; but in Devlin's text no visual difference is made between them. Normally of course such confusion would cause no trouble, but in this particular context, greater exactitude might be desirable.

Lower on page 242, in line -7, there is a typo: t^φ should be t_φ.

Finally on page 243, some correction will be needed as the troublemaker $F_\wedge$ recurs here and appeal is made to the false Lemma I.9.3.

The definition of $G_\exists$ oscillates between two and three variables.

On page 243, line -5, reference to 1.7 should perhaps be to 1.8.

Taking stock

Much of the problem with Chapter I Section 9 has now been repaired, but the proposed definition of Sat is not possible in BS, and no other seems likely to succeed.

In the Introduction we spoke of three systems that might work in place of BS. One is our suggestion DS; the second is GJI; and we now introduce the third system, which we call MW, for "Middle Way":

MW $\quad \mathsf{DBI} + \forall a \forall k_{\in\omega}\, [a]^k \in V$

Of those, GJI is the longest established candidate, being, apart from the restriction to Π_1 foundation, the system RUD discussed in Stanley's review; DS emerged as the present author's first response to Stanley's call for a replacement for BS that does not use the theory of rudimentary functions; and then at a late stage in the writing of the present paper, the system MW, which is a proper subsystem both of GJI and of DS, revealed itself, and might now be thought to be the "right" answer to Stanley's call.

The system MW proves Theorem 2·93, and is a proper extension of DBI. Model 5 provides an example of a structure where MW is true but both DS and GJI fail; in Model 7, DS is true but not GJI, and in Gandy's model $\mathbf{G}_2$, GJI is true but not DS.

We shall discuss the definition of Sat first in the system MW and then in GJI and in DS. The reader might wonder what is to be gained by considering the problem of defining Sat within the latter two systems, once one knows that a definition of Sat in MW is possible and that MW is a subsystem of both.

Our answer would be that defining Sat in MW is laborious, whereas it seems possible that each of the other two systems can supply a more elegant treatment, the one drawing on the theory of rudimentary functions, and the other on the enhanced logic of limited quantitifers discussed in Section 8.

The cure in MW

A first step is to collect the $[a]^k$ for given a and a bounded set of k's. The proof seems not to be trivial: Model 6 provides examples where the existence of $[a]^{\mathfrak{k}}$ for one value of $\mathfrak{k}$ does not imply its existence for another.

10·24 Definition $P(g,k,a) \iff_{\mathrm{df}}$ g is a function with domain $k+1$ and $g(0)=\varnothing$ and for all $i<k$, $g(i+1)=\{x\cup\{p\} \mid x\in g(i) \mathbin{\&} p\in a\smallsetminus x\}$. P is Δ_0.

10·25 Lemma (MW) *(i) If $P(g,k,a)$, then for each $i\leqslant k$, $g(i)=[a]^i$.*

(ii) $\forall a\forall k_{\in\omega}\exists g\, P(g,k,a))$.

Proof of (ii): Fix a; use Π_1 foundation to find the least k such that there is no g with $P(g,k,a)$; show that k is not 0; $[a]^k$ exists, so if $\ell+1=k$ and $P(h,\ell,a)$ we can create g with $g\restriction k=h$ and $g(k)=[a]^k$, after all. So no failure k exists. ⊣ (10·25)

10·26 Proposition (MW) $\forall a\forall n_{\in\omega}\exists t(t=[a]^{\leqslant n})$.

Proof : Lemma 10·24 shows that $x=[a]^i$ is a Σ_1^{MW} predicate. Since $\forall i_{\in n}\exists x\, x=[a]^i$, Metatheorem 2·24 coupled with Remark 2·25 implies that there is a w such that $\forall i_{\in n}\exists x_{\in w}\, x=[a]^i$. Then the desired t is a Δ_0 subclass of $\bigcup w$ and therefore a set. ⊣ (10·26)

That argument readily extends to give the set-hood of the classes $w(u,\varphi)$. We may now implement Devlin's definition of Sat and show that it is Δ_1^{MW}; by working with the restrained versions $w^*(u,\varphi)$, we could avoid appeal to the axiom of infinity in defining Sat; though of course if we want our languages to be sets we must use it.

The cure in GJ

10·27 Lemma (GJ) $\forall n_{\in\omega}\forall a\exists u\,\mathrm{Seq}(u,a,n)$.

Proof : fix a; least failed n is given by Π_1 foundation. then piece things together using appropriate rudimentary functions. ⊣ (10·27)

10·28 Proposition *The LST formula Seq(u,a,n) is Δ_1^{GJ}.*

10·29 Lemma (GJ) *$w^*(u,\varphi)\in V$; if $\omega\in V$, then $w(u,\varphi)\in V$.*

Proof : use the result and reasoning behind Theorem 2·93. ⊣ (10·29)

10·30 Remark The natural proof of Devlin I.9.6 would use Π_2 foundation to reduce the problem to showing that $\{\bigcup x \mid x\in a\}$ is a set, which is possible in GJ, but, by Model $\mathbf{M}_7$, not in DB.

With the existence of $w(u,\varphi)$ and $w^*(u,\varphi)$ now established, we could follow the structure of Devlin's argument; but the present author's inclination would now be to adopt a slightly different approach to the definition of Sat. Fix u and φ. For any sentence ϑ of $\mathcal{L}_u$, let $B(\vartheta)$ be the set of "simpler sentences" to which the computation of $\models_u \vartheta$ is naturally referred; thus if ϑ is atomic, $B(\vartheta)$ will be empty; if ϑ is $\vartheta_1\wedge\vartheta_2$ then $B(\vartheta)=\{\vartheta_1,\vartheta_2\}$; if ϑ is $\neg\vartheta_1$, then $B(\vartheta)=\{\vartheta_1\}$; and if ϑ is $\forall x\vartheta_1(x)$, then $B(\vartheta)$ will be the set of all substitution instances $\vartheta_1(\mathring{a})$ for $a\in u$.

The first step would then be to define the function that unfolds the formula φ as a tree $\mathcal{T}_\phi$ with φ as its top point; immediately below φ one would place all the members of $B(\varphi)$; immediately below each such formula ϑ one would place all the members of $B(\vartheta)$, and so on, so that the bottom points of the tree are atomic sentences of $\mathcal{L}_u$. GJI is strong enough to do that, for the length of φ gives an upper bound to the (finite) number of steps required.

Then by recursion on the tree $\mathcal{T}_\varphi$ one can compute the truth of $\models_u \vartheta$ for each node ϑ of the tree, culminating with the computation of the truth of $\models_u \varphi$. Thus we would arrive at a proof of

10·31 PROPOSITION *The LST formula* $Sat(u, \phi)$ *is* Δ_1^{GJ}.

We should remark that Gandy in developing (his variant of) the system GJI was specifically aiming at an elegant framework for treating the syntax of formalised languages.

The cure in DS

We recall that DS is the theory $\mathsf{S}_0 + \Delta_0$ separation $+\ \Pi_1$ foundation $+\ \omega \in V + \mathcal{S}(x) \in V$. The following remarks are intended to suggest that in DS, given a greater knowledge of the behaviour of limited quantifiers with respect to rudimentary substitution, we might arrive at a third proof.

$F_\wedge$ is Δ_0 in the parameter $\mathcal{S}(\omega \times \omega)$, by the result, given as Proposition 8·11, that in DS the graph of each partial recursive function is a set. Further,

corrected Lemma I.9.3:

10·32 LEMMA $F_\wedge$ *is* $\Delta_{0,\mathcal{S}}^{\mathsf{DS}}$

corrected Lemma I.9.4:

10·33 LEMMA *Build is* $\Delta_{0,\mathcal{S}}^{\mathsf{DS}}$.

10·34 LEMMA (DS) *(i)* $\forall a\ {}^{<\omega}a \in V$.
(ii) $\forall n_{\in\omega} \forall a \exists u \mathrm{Seq}(u, a, n)$.

Proof : by two applications of Δ_0 separation, as

$$^{<\omega}a = \mathcal{S}(a \times \omega) \cap \{x \mid \underbrace{\mathrm{Fn}(x)\ \&\ \mathrm{Dom}\,(x) \in \omega}_{\Delta_0}\}$$

and the desired u with $\mathrm{Seq}(a, u, n)$ is

$$^{<\omega}a \cap \{x \mid \underbrace{\mathrm{Fn}(x)\ \&\ \mathrm{Dom}\,(x) \in n}_{\Delta_0}\}. \qquad \dashv (10·34)$$

corrected Lemma I.9.5:

10·35 PROPOSITION *The LST formula* $Seq(u, a, n)$ *is* Δ_1^{DS}.

Corrected I.9.10:

10·36 LEMMA (DS) $w(u, \varphi) \in V$.

Proof : by arguments similar to those of Lemma 10·33. ⊣ (10·36)

10·37 PROPOSITION *The LST formula* $Sat(u, \phi)$ *is* Δ_1^{DS}

Proof : apply Lemma 10·35 and Proposition 10·34. ⊣ (10·37)

Conclusion

As each of the three systems holds in all J_ν and L_λ with λ a limit ordinal $> \omega$ each might be claimed to be a good replacement for BS.

Each of the three is open to criticism: whilst DS is perhaps closest to Devlin's original conception, and the enhancement of its logic studied in Section 8 gives it a certain smoothness, it might be felt that the axiom $\mathcal{S}(x) \in V$ is too strong for its intended use; MW avoids that problem, but at the cost of a certain austerity; whether it will lend itself to an enhancement of its logic of the kind studied in Section 8 and enjoyed by DS must remain a question for another time. GJI is open to the pedagogical criticism that it relies on too early an introduction of the notion of rudimentary function.

The proof of VI.1.14 rests on a different idea, unrelated to the problems of defining Sat. The proof given by Devlin is tainted by its appeal to the false Lemma I.9.3, and therefore I intend in [M4] to rework the proof.

I cannot claim to have checked through the whole book, but my remarks reassure me, if no-one else, that the errors are not catastrophic. A modest strengthening of the meaning of BS and all seems to be well.

PART III

11: Gandy's inexact remarks

Gandy in [G] says of his four weak set theories PZ, BST', BRT and PZF, that were one to drop the requirement of Δ_0 the four would stretch from Zermelo to Zermelo–Fraenkel, and continues "presumably these are also all distinct". His first remark is *prima facie* false as he makes no mention of the power set axiom (nor of the axiom of foundation) and the power set axiom is certainly independent of the others as (working say in ZFC) **HC** satisfies all other axioms of ZF.

We insert BS in the sequence and comment on the effect on the five of dropping the restriction to Δ_0, of adding the power set axiom, and of doing both.

The full systems without power set

The first system will have axioms of extensionality, pairset, sumset and infinity, and the full separation scheme. The second system will add Cartesian product to that.

The model $\mathbf{M}_2$ satisfies full separation but not Cartesian product.

Corresponding to GJ, we have the "full rudimentary" replacement scheme:

(full RR) $$\forall x \exists w \forall v_{\in x} \exists t_{\in w} \forall u (u \in w \iff .u \in x \;\&\; \phi[u, v]).$$

for ϕ any formula.

The model $\mathbf{M}_7$ satisfies full separation and Cartesian product, but witnesses a failure of (restricted) rudimentary replacement.

Corresponding to fReR we have the full flat replacement axiom: namely, for any ϕ,

(full flat repl.) $$\forall x_{\in u} \exists! y (\phi(x, y) \;\&\; y \subseteq z) \Longrightarrow \exists u \forall y [y \in v \iff \exists x_{\in u} (\phi(x, y) \;\&\; y \subseteq z)]$$

But full flat replacement is derivable from "full rudimentary" replacement, using the self-strengthening of full RR corresponding to that noted in Proposition 2·88 for RR, by remarking that the set promised by an instance of full flat replacement is of the form

$$\{Z \cap \{y \mid \exists Y \Phi(X, Y) \;\&\; y \in Y\} \mid X \in U\}.$$

So in fact the distinction between the two systems will collapse already at Σ_1.

As for full flat collection, full replacement and full collection, Gandy's choice $\mathbf{G}_3 = V_{\omega+\omega}$ gives a model of full flat collection in which replacement fails—but since gfReR is a subsystem of Z, we may also find a model for it in which **HF** does not exist— and Zarach's model, [Z] Theorem 6.4, gives a model of full replacement in which collection, possibly even flat collection, fails.

Gandy's systems with added power set

PZ + P is the system M_0, in which Cartesian product is provable, as are Rudimentary Replacement, and flat Δ_0 Replacement and Collection. PZF + P is strictly stronger, as it builds $\omega + \omega$.

11·0 Problem Is KPI + P the same as ReR + P ?

The full systems with foundation and power set added

We have just Z in the first case; and the first four cases now coincide, for full flat replacement is provable in Z, just as fReR is provable in M_0 using power set plus Δ_0 separation. The fifth is ZF.

12: A model of Z plus full Foundation in which TCo fails

Boffa [B1] [B2] has constructed two other models of $\mathsf{Z} + \neg\mathsf{TCo}$; ours appears to be a third.

12·0 Definition $\iota^0(x) =_{\rm df} x$; $\iota^{n+1}(x) =_{\rm df} \{\iota^n(x)\}$.

12·1 Definition ϱ is the set-theoretical rank of x.

12·2 Definition $V_n =_{\rm df} \{x \mid \varrho(x) < n\}$; $b_n =_{\rm df} \iota^n(V_n)$.

12·3 Definition For each $n \in \omega$, set $c_n =_{\rm df} \{\bigcup^n b_m \mid n \leqslant m < \omega\}$.

12·4 EXAMPLE $c_0 = \{V_0, \{V_1\}, \{\{V_2\}\}, \ldots\}$; $c_1 = \{V_1, \{V_2\}, \{\{V_3\}\}, \ldots\}$; $c_2 = \{V_2, \{V_3\}, \{\{V_4\}\}, \ldots\}$.

12·5 PROPOSITION $\bigcup c_n = V_n \cup c_{n+1}$.

12·6 DEFINITION $K_0 =_{\text{df}} \omega \cup \{c_0\}$; $K_{n+1} =_{\text{df}} \mathcal{P}(K_n) \cup K_n \cup c_n$; $K =_{\text{df}} \bigcup_{n\in\omega} K_n$.

12·7 THEOREM *K is a supertransitive model of Zermelo set theory Z in which some set is a member of no transitive set.*

12·8 LEMMA *$K_n \subseteq K_{n+1}$, and $K_n \in K_{n+1} \subseteq K$, so that each $K_n \in K$.*

12·9 COROLLARY *K models Pairing.*

12·10 LEMMA *$V_n \subseteq K_n$.*

Proof : induction on n. $V_0 = \varnothing$; if $V_n \subseteq K_n$, $V_{n+1} = \mathcal{P}(V_n) \subseteq \mathcal{P}(K_n) \subseteq K_{n+1}$. ⊣ (12·10)

12·11 COROLLARY *K includes all of $V_\omega = \mathbf{HF}$; in particular K contains all finite ordinals. Moreover $\omega \in K_1 \subseteq K$.*

12·12 LEMMA *K is transitive:*

Proof : Let $x \in y \in K_0$. Then either $y \in \omega$ when $x \in K$ or $y = c_0$ when $x \in \mathbf{HF} \subseteq K$.

Let $x \in y \in K_{n+1}$. then either $y \subseteq K_n$, when $x \in K_n$, or $y \in K_n$, when inductively we have already shown that $x \in K$; or $y \in c_n \subseteq \mathbf{HF}$, when $x \in \mathbf{HF} \subseteq K$. ⊣ (12·12)

12·13 COROLLARY *K models Extensionality, Null Set, Infinity and (full) Foundation.*

12·14 LEMMA *K is supertransitive,*

Proof : $x \subseteq y \in K_n \Longrightarrow x \in K_{n+1} \in K$. ⊣ (12·14)

12·15 COROLLARY *K is a model of full Separation.*

12·16 LEMMA *Each $\bigcup K_n$ is a subset of K_{n+1} and thus is in K by supertransitivity.*

Proof : $\bigcup K_0 = \omega \cup c_0 \subseteq K_1 \in K$. If $\bigcup K_n \subseteq K_{n+1}$, $\bigcup K_{n+1} = K_n \cup \bigcup K_n \cup V_n \cup c_{n+1} \subseteq K_{n+2}$. ⊣ (12·16)

12·17 COROLLARY *K models Union.*

Proof : if $y \in K_n$, then $y \subseteq \bigcup K_n \subseteq K_{n+1}$, so $\bigcup y \subseteq \bigcup K_{n+1} \in K$, so $\bigcup y$ is in K. ⊣ (12·17)

12·18 LEMMA *K models Power set.*

Proof : If $x \in K_n$, $x \subseteq K_{n+1}$ so $\mathcal{P}(x) \subseteq \mathcal{P}(K_{n+1}) \subseteq K_{n+2}$. ⊣ (12·18)

Thus we have shown that K models Z.

12·19 PROPOSITION $\forall n \forall m[m \geqslant n+3 \Longrightarrow V_m \notin K_n]$.

Proof : $V_0 = 0$; $V_1 = 1$, $V_2 = 2$ but for $m \geqslant 3$, V_m is not an ordinal and is therefore not in ω, nor is it, a finite set, equal to c_0, an infinite set. Hence $V_3 \notin K_0$.

Suppose that $V_m \notin K_n$, for any $m \geqslant n+3$. If $V_{m+1} \in K_{n+1}$, then either $V_{m+1} \subseteq K_n$, so that $V_m \in K_n$, contradicting the inductive hypothesis; or $V_{m+1} \in K_n$, again contrary to the inductive hypothesis; or $V_{m+1} \in c_n = \{V_n, \{V_{n+1}\}, \{\{V_{n+2}\}\} \ldots\}$, again impossible by inspection. ⊣ (12·19)

12·20 PROPOSITION *TCo fails in K.*

Proof : $c_0 \in K$. Suppose that $c_0 \in u \in K$ with u transitive. Then $\mathbf{HF} \subseteq u$, so $\mathbf{HF} \in K$, and hence $HF \in K_n$ say, so that $\mathbf{HF} \subseteq K_{n+1}$. But K_{n+1} contains at most $n+4$ of the sets V_m. ⊣ (12·20)

Other constructions of models of Zermelo are given in *Slim Models*. The constructions there furnish an entertaining independence argument for the axiom of pairing, which we shall give in the next section.

Let Z be Zermelo set theory, including the axioms of infinity and foundation. Let TCo be the assertion that every set is a member of a transitive set. Let TIn be the assertion that every set is a subset of a transitive set. Let AxSing be the assertion that for each set x, $\{x\}$ is a set. Let AxPair be the assertion that for all sets x and y, $\{x, y\}$ is a set.

13·0 REMARK TCo trivially (in the strict sense) implies TIn; TIn + AxSing implies TCo. AxSing is usually derived from AxPair, either by taking $x = y$ or if AxPair is confined to the strict case, by using separation. Indeed AxSing is provable using separation and power set, since each set x is a member of its power set, should the latter exist.

We shall exhibit a model of almost all of Zermelo, in which AxSing is true but AxPair is false, and a model of a substantial amount of set theory in which TIn holds but AxSing and TCo fail.

It is amusing to note that in the system of Bourbaki, the pairing axiom has been proved to be redundant. see Sonner [S]. That it is not redundant in Z was first shown by Boffa [B3].

Failure of AxPair

Let T be the theory Z + TCo + WO,—WO being the statement "every set has a well-ordering"—and let T^- be the theory T with the axiom of pairing replaced by its negation: $\exists x \exists y \{x, y\} \notin V$, and with the addition of AxSing.

13·1 REMARK The scheme of foundation for all classes is provable in T^-.

We find a model for T^-: indeed we show that if Consis(Z) then Consis(T^-).

It follows from the last part of Theorem 5 of *The Strength of Mac Lane Set Theory* [M2], proved in Section 5 of that paper, that if Z is consistent, so is Z + KP + WO.

A set or class $\mathbf{M}$ is said to be *supertransitive* if it is transitive and, further, $x \subseteq y \in \mathbf{M} \Longrightarrow x \in \mathbf{M}$.

As in the proof of Theorem 4·8 of *Slim Models of Zermelo Set Theory* [M1] one can, working in the theory Z + KP + WO, build two supertransitive models $\mathbf{M}$ and $\mathbf{N}$ of Z + TCo + WO, with neither a subset of the other: e.g. take $\mathbf{M}$ to contain $Z(0)$ but not $Z(\omega)$ and $\mathbf{N}$ to contain $Z(\omega)$ but not $Z(0)$, in the notation of that paper.

THEOREM *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be supertransitive models of* T, *neither included in the other; then* $\mathbf{M} \cup \mathbf{N}$ *is a model of* T^-, *and* $\mathbf{M} \cap \mathbf{N}$ *is a model of* T.

Proof : Note first that $\mathbf{M} \cup \mathbf{N}$ is supertransitive, and hence absolute for most of the set-theoretical concepts used in the axioms; therefore it will be a model of Extensionality, Sum Set, Power Set, full Separation, Foundation, TCo (whence also Foundation for all classes), and WO.

[For power set, use supertransitivity; otherwise there would be a risk of $\mathbf{N}$ containing subsets of some element of $\mathbf{M}$ which were not in $\mathbf{M}$. Supertransitivity also gives the truth of full separation in $\mathbf{P}$. For the other axioms the transitivity of $\mathbf{P}$ is enough.]

Pairing fails, for if $a \in \mathbf{M} \setminus \mathbf{N}$ and $b \in \mathbf{N} \setminus \mathbf{M}$, then $\{a, b\} \notin \mathbf{M} \cup \mathbf{N}$. But AxSing holds.

The verification of the second assertion is straightforward. ⊣ (13·1)

METACOROLLARY *If* Z *is consistent so is* T'.

13·2 REMARK The $\mathbf{M}$ and $\mathbf{N}$ just used can be chosen to contain all ordinals, all sequences of ordinals and all sets of sequences of ordinals, and to be such that for all limit ordinals $\lambda > \omega$, neither of $\mathbf{P}_\lambda =_{\mathrm{df}} \mathbf{M} \cap V_\lambda$ nor $\mathbf{Q}_\lambda =_{\mathrm{df}} \mathbf{N} \cap V_\lambda$ is contained in the other. In such a case, each of $\mathbf{P}_\lambda$, $\mathbf{Q}_\lambda$ and $\mathbf{R}_\lambda =_{\mathrm{df}} \mathbf{P}_\lambda \cap \mathbf{Q}_\lambda$ will be a supertransitive model of Z+ WO, each being the intersection of two such. If $p \in \mathbf{P}_\lambda \setminus \mathbf{R}_\lambda$, then for $x \in \mathbf{R}_\lambda$ $\{p, x\}$ will be in $\mathbf{P}_\lambda \setminus \mathbf{R}_\lambda$; so the three sets $\mathbf{P}_\lambda \setminus \mathbf{R}_\lambda$, $\mathbf{Q}_\lambda \setminus \mathbf{R}_\lambda$ and $\mathbf{R}_\lambda$ will all be of cardinal $\beth_\lambda$.

13·3 REMARK Boffa in [B3] shows of every member a of $\mathbf{HF}$ that it is provable in Z that for any x, the pair $\{a, x\}$ exists: for example both the empty set and x are in $\mathcal{P}(x)$, and therefore the pair $\{\varnothing, x\}$ can be recovered using Separation. Thus a set which might not form a pair with something must be of rank at least ω, and Boffa shows that the set $\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\} \ldots\}$ of Zermelo integers indeed has that property.

Failure of AxSing

Consider, working in some suitable theory such as ZF, the class $\mathbf{C}$ of all sets x such that $\mathrm{tcl}(x)$ contains at most one strict pair, that is, a set of the form $\{b, c\}$ with $b \neq c$. $\mathbf{C}$ is supertransitive, and models "much" of Z: namely Extensionality, full separation, sum set, and infinity; and it contains all the ordinals, of which $2 = \{0, 1\}$ is the only strict pair. AxSing fails since $\{5, 6\}$ is a member of $\mathbf{C}$ but $\{\{5, 6\}\}$ is not.

Moreover TIn holds in $\mathbf{C}$, since the transitive closure of an element of $\mathbf{C}$ is itself an element of $\mathbf{C}$; but TCo is false, since for example $\{5, 6\}$ cannot be a member of any transitive element of $\mathbf{C}$.

Inadequate axioms in a French textbook

The well-established textbook *Tome 1, Algèbre,* of the *Cours de mathématiques* by Jacqueline Lelong-Ferrand and Jean-Marie Arnaudiès, [L-F,A], in its opening chapter gives some axioms for what is in effect a subsystem of Z. They follow Bourbaki in giving axioms for ordered pairs, but not for unordered pairs. But a model for the axioms that they state is furnished by any $\mathbf{P}_\lambda \cup \mathbf{Q}_\lambda$ as discussed in Remark 13·2; for let $f : \mathbf{R}_\lambda \longleftrightarrow \mathbf{R}_\lambda \times \{0\}$, $g : \mathbf{P}_\lambda \smallsetminus \mathbf{R}_\lambda \longleftrightarrow \mathbf{R}_\lambda \times \{1\}$ and $h : \mathbf{Q}_\lambda \smallsetminus \mathbf{R}_\lambda \longleftrightarrow \mathbf{R}_\lambda \times \{2\}$, let f_x be f if $x \in \mathbf{R}_\lambda$, g if $x \in \mathbf{P}_\lambda \smallsetminus \mathbf{R}_\lambda$ and h if $x \in \mathbf{Q}_\lambda \smallsetminus \mathbf{R}_\lambda$, and interpret the formal ordered pair of x and y as $(f_x(x), f_y(y))_2$; and in that model whenever $x \in \mathbf{P}_\lambda \smallsetminus \mathbf{R}_\lambda$ and $y \in \mathbf{Q}_\lambda \smallsetminus \mathbf{R}_\lambda$, their union $x \cup y$ will not be a set.

The reader will find in [M6] a more detailed scrutiny of the account of logic and set theory in [L-F,A].

14: A remark on rud closure answering a question of MacAloon

Let $\mathbb{T}$ be the rudimentary function of Definition 2·73.

14·0 Lemma *Let $\langle u_n \mid n \in \omega\rangle$ be any sequence of transitive sets. Define*

$$K_0 = u_0; \quad K_{n+1} = \mathbb{T}^5(K_n) \cup u_n \quad K_\omega = \bigcup_{n<\omega} K_n.$$

Then K_ω is rud closed.

Proof : We show that K_ω is closed under each of the functions R_0 to R_8. By the properties of $\mathbb{T}$ established in Section 2, x, y in u implies $R_i(x) \in \mathbb{T}(u)$ for i = 2,3,5; and x, y, in u implies $R_i(x, y) \in \mathbb{T}(u)$ for i = 0,1; x in u implies $R_i(x) \in \mathbb{T}^5(u)$ for i = 6, 7; x, y in u implies $R_4(x, y) \in \mathbb{T}^3(u)$; and x, y in u implies $R_8(x, y) \in \mathbb{T}^2(u)$. As $u \subset \mathbb{T}(u) \subset \mathbb{T}^2(u) \ldots$, it follows that for each n, $K_n \subseteq \mathbb{T}^5(K_n) \subseteq K_{n+1}$. ⊣ (14·0)

14·1 Definition $\iota(x) =_{\mathrm{df}} \{x\}$

14·2 Lemma *If $x \notin u$ then $\iota(x) \notin \mathbb{T}(u)$; and hence $\iota^4(x) \notin \mathbb{T}^4(u)$.*

Proof : every member of $\mathbb{T}(u)$ is a subset of u. ⊣ (14·2)

14·3 Proposition *Suppose that u is a transitive set closed under pairing. Then whenever w is a transitive set of which u is not a subset, u is not a member of the rud closure of $u \cup w$.*

Proof : u must be of limit rank λ say. Suppose first that u is countable, so that λ is of cofinality ω. Let $\lambda_n \nearrow_n \lambda$. We fix an enumeration of x and use it to make the following choices.

Pick $x_0 \in u \smallsetminus w$. Let $u_0 = u \cap V_{\max\{\lambda_0, \varrho(x_0)+1\}}$.

Pick $x_1 \in u \smallsetminus u_0$, with $\iota^4 x_0 \in x_1$. Let $u_1 = u \cap V_{\max\{\lambda_1, \varrho(x_1)+1\}}$.

Pick $x_{n+1} \in u \smallsetminus u_n$, with $\iota^4 x_n \in x_{n+1}$. Let $u_{n+1} = u \cap V_{\max\{\lambda_{n+1}, \varrho(x_{n+1})+1\}}$.

Finally let $K_0 = w; K_{n+1} = \mathbb{T}^5(K_n) \cup u_n; K_\omega = \bigcup_n K_n$.

Then every K_n is transitive and by the Lemma, K_ω is rud closed, and includes $w \cup \{w\} \cup u$. If $x_{n+1} \in K_{n+1}$, it cannot, by construction, be a member of u_n and so must be a subset of $\mathbb{T}^4(K_n)$, so $\iota^4(x_n) \in \mathbb{T}^4(K_n)$, which by Lemma 14·2 implies $x_n \in K_n$. But $x_0 \notin K_0$; so by induction no $x_n \in K_n$. Hence no superset of $\{x_n \mid n \in \omega\}$ can be a member of K_ω. In particular, u cannot be.

The Proposition is now proved for the case that u is countable. In the general case, go to a generic extension of the universe in which u is countable; the hypotheses will still hold; hence in the generic extension, u is not in the rud closure of $u \cup w \cup \{w\}$; but that latter statement is absolute and therefore true in the ground model. ⊣ (14·3)

14·4 Corollary *Let u be transitive and closed under pairing; then u is not in the rud closure of $ON \cup u$.*

A particular case answers a question posed by McAloon in the 1970's:

14·5 COROLLARY *For any $\alpha > 0$, $J_\alpha \notin$ rud cl$\big(J_\alpha \cup \{\omega\alpha\}\big)$*

I thank Lee Stanley for telling me of McAloon's question.

14·6 REMARK So far as the definition of K_ω goes, other functions T could be used instead of $\mathbb{T}$, provided they had the property that the members of $T(u)$ are subsets of u: for example, if we instead use $u \mapsto \mathcal{P}(u)$, K_ω will be a model of Zermelo set theory, probably including the axiom of infinity, though possibly not in the form $\omega \in V$: we adopt this strategy in the following variant.

14·7 PROPOSITION *Suppose that $(x_n)_n$ and $(u_n)_n$ are two sequences of sets such that for each $n < \omega$:*

(14·7·0) $x_n \in u_n$;
(14·7·1) $u_n \subseteq u_{n+1}$;
(14·7·2) u_n *is transitive;*
(14·7·3) $x_n \in \mathrm{tcl}(x_{n+1})$;
(14·7·4) $x_{n+1} \notin u_n$.

Then $\bar{u} =_{\mathrm{df}} \bigcup_n u_n$ is transitive and if w is a transitive set with $x_0 \notin w$, the set $\bar{x} =_{\mathrm{df}} \{x_n \mid n \in \omega\}$ is not a member of the rud closure of $\bar{u} \cup w \cup \{w\}$. If in addition $\omega \subseteq w$, then there is a supertransitive model of Zermelo set theory of which $\bar{u} \cup w \cup \{w\}$ is a subset but $\bar{x}$ and $\bar{u}$ are not members.

Proof : Let K be the model formed as follows:

$$K_0 = w; \qquad K_{n+1} = \mathcal{P}(K_n) \cup u_n; \qquad K = \bigcup_n K_n.$$

Then each K_n is transitive.

14·8 LEMMA *Each K_n is a member of K_{n+1}.*

14·9 LEMMA *$K_0 \subseteq K_1$; if $K_n \subseteq K_{n+1}$ then $K_{n+1} \subseteq K_{n+2}$.*

Proof : As K_0 is transitive, its members are also subsets of it and therefore members of K_1. Under the hypotheses of the second statement, $\mathcal{P}(K_n) \subseteq \mathcal{P}(K_{n+1}) \subseteq K_{n+2}$ and $u_n \subseteq u_{n+1} \subseteq K_{n+2}$. ⊣ (14·9)

14·10 LEMMA *$\bigcup K_0 \subseteq K_0$; $\bigcup K_{n+1} = K_n \cup \bigcup u_n$.*

14·11 LEMMA *If $x \in K$, then for some ℓ, $x \subseteq K_\ell$.*

14·12 LEMMA *K is transitive.*

Proof : If $y \in x \in K$ then for some ℓ, $y \in x \subseteq K_\ell$, so $y \in K_\ell \subseteq K$. ⊣ (14·12)

14·13 LEMMA *K is supertransitive.*

*Proof :*If $y \subseteq x \in K$ then for some ℓ, $y \subseteq x \subseteq K_\ell$, so $y \in \mathcal{P}(K_\ell) \subseteq K_{\ell+1} \subseteq K$. ⊣ (14·13)

14·14 COROLLARY *K models the full separation scheme.*

14·15 LEMMA *$x \in K \Longrightarrow \mathcal{P}(x) \in K$.*

Proof : by Lemma 14·11, x is a subset of some K_ℓ; by the proof of Lemma 14·13, any subset of x is in $K_{\ell+1}$, and so $\mathcal{P}(x)$ is a subset of $K_{\ell+1}$ and therefore a member of $K_{\ell+2}$. ⊣ (14·15)

14·16 LEMMA *Each $\bigcup K_n$ is in K.*

Proof : by supertransitivity, as each $K_n \in K$.

14·17 LEMMA *$x \in K \Longrightarrow \bigcup x \in K$.*

Proof : If $x \subseteq K_\ell$, then $\bigcup x \subseteq \bigcup K_\ell$, which is in K; as K is supertransitive, $\bigcup x \in K$. ⊣ (14·17)

14·18 LEMMA *For no n is x_n a member of K_n; hence $\bar{x}$ is a subset of no K_n; hence neither it not $\bar{u}$ can be a member of K.*

Proof : $x_0 \notin K_0$ by hypothesis. Suppose that $x_{n+1} \in K_{n+1}$, then either $x_{n+1} \subseteq K_n$, giving $x_n \in K_n$, (since K_n is transitive) or else $x_{n+1} \in u_n$, contrary to hypothesis.

So $x_n \notin K_n \Longrightarrow x_{n+1} \notin K_{n+1}$; by induction, for no n is x_n a member of K_n; as $x_n \in \bar{x}$, $\bar{x} \not\subseteq K_n$. Lemma 14·11 now implies that $\bar{x}$ is not a member of K; as it is a subset of $\bar{u}$ and K is supertransitive, $\bar{u}$ cannot be a member of K. ⊣ (14·18)

14·19 LEMMA $\bar{u} \cup w \cup \{w\} \subseteq K$.

14·20 LEMMA *If* $x \in K_m$ *and* $y \in K_n$ *then for* $\ell = \max(m, n)$, $\{x, y\} \subseteq K_\ell$ *and so is in* K.

14·21 PROPOSITION $\omega \in K \iff \omega \subseteq w$.

14·22 PROPOSITION *K is a model of all axioms of Zermelo set theory except possibly the axiom of infinity.*

14·23 COROLLARY *K is rud closed.*

14·24 REMARK If we take $u = \mathbf{HF}$ and $w = \omega$, K_ω will be a set model of Zermelo of which **HF** is not a member. Thus our argument generalises constructions to be found in the texts of Moschovakis and Enderton.

A third possibility is in the proof of the next remark.

14·25 PROPOSITION *Let u be transitive and be the strictly increasing union of a sequence u_n of transitive sets with u_0 not an ordinal and $u_n \in u_{n+1}$. Let $\zeta = ON \cap u$. Then the rud closure of $u \cup \{\zeta\}$ is a proper subset of the rud closure of $u \cup \{u\}$.*

Proof : define $K_0 = \zeta$; $K_{n+1} = \mathrm{Def}(K_n) \cup u_n$; $K = \bigcup_n K_n$.

K is rud closed and includes $u \cup \{\zeta\}$; but one may show that each $u_n \notin K_n$; hence $u \notin K$. ⊣ (14·25)

15: An application to Gandy numerals

The method of Section 14 casts some light on the proposal made by Gandy in [G] for discarding the von Neumann ordinals as numerals for the purpose of developing formal syntax. Their problem is that the rank of n is n. His method makes use of ideas of Smullyan [Sm].

First step: $\hat{\omega}$

15·0 DEFINITION We assign to each $n \in \omega$ a hereditarily finite set $\hat{n}$ and a level $\lambda(n) \in \omega$.

$\hat{0} = 0; \hat{1} = \{0\}; \lambda(0) = \lambda(1) = 0$.

For $n > 0$ let $n - 1 = \Sigma_{\ell<k} a_\ell 2^\ell$, where $a_\ell \in \{1, 2\}$. Then put

$$\hat{n} = \{\{\hat{\ell} \mid \ell < k \ \&\ a_\ell = 2\}, \{\hat{\ell} \mid \ell < k\}\}; \lambda(n) = k.$$

15·1 EXAMPLE $\hat{2} = \{0, \{0\}\}$; $\hat{3} = \{\{0\}\}$; $\lambda(2) = \lambda(3) = 1$

$\hat{4} = \{0, \{0, \{0\}\}\}$; $\hat{5} = \{\{0\}, \{0, \{0\}\}\}$; $\hat{6} = \{\{\{0\}\}, \{0, \{0\}\}\}$; $\hat{7} = \{\{0, \{0\}\}\}$; $\lambda(4) = \lambda(5) = \lambda(6) = \lambda(7) = 2$.

Set $\hat{\omega} = \{\hat{n} \mid n \in \omega\}$.

To get λ we need the graph of exponentiation.

Second step: $\bar{\omega}$

Then set $\bar{n} =_{\mathrm{df}} \{\hat{m} \mid m < n\}$ and $\bar{\omega} =_{\mathrm{df}} \{\bar{n} \mid n \in \omega\}$.

It is the members of $\bar{\omega}$ that Gandy proposes, and which we shall call Gandy numerals. He proves that

> the predicate $x \in \bar{\omega}$ is Δ_0; addition and multiplication of Gandy numerals are rudimentary; concatenation of sequences of Gandy numerals is rudimentary; but exponentiation of Gandy numerals is not rudimentary.

His reason for not remaining with $\hat{\omega}$ is that he was unable to prove that $x \in \hat{\omega}$ is Δ_0, and he speculated that $x \in \hat{\omega}$ is in fact not.

15·2 PROPOSITION *Neither $\hat{\omega}$ nor $\bar{\omega}$ is in* rud cl$(\{\omega\})$.

Proof : we apply Proposition 14·7. $\hat{0} = 0$, $\hat{1} = 1$, $\hat{2} = 2$ but $\hat{3} = \{1\}$ which is not an ordinal. Therefore let $x_0 = \hat{3}$, and $u_0 = \mathrm{tcl}(\{x_0\})$. Let x_{n+1} be $\hat{k}$ for k the least such that $\hat{k} \notin u_n$ and $x_n \in \mathrm{tcl}\hat{k}$; take $u_{n+1} = u_n \cup \mathrm{tcl}(\{x_{n+1}\})$. The resulting supertransitive model K is rud closed and does not contain $\bar{x}$; therefore it does not contain $\hat{\omega}$, of which $\bar{x}$ is a subset. But it does include the rudimentary closure of $\{\omega\}$.

Since $\bigcup \bar{\omega} = \hat{\omega}$, $\bar{\omega}$, too, cannot be in K. ⊣ (15·2)

15·3 REMARK We can define a version, $\widehat{\mathrm{ACK}}$, of the Ackermann relation by $\hat{m}\widehat{\mathrm{ACK}}\hat{n} =_{\mathrm{df}} \hat{m} \in \bigcap \hat{n}$.

By the Proposition, $\hat{\omega}$ is not provably a set in GJ. But in GJ, we can show that if $\hat{\omega}$ is a set, then so is the relation $\widehat{\mathrm{ACK}}$, and therefore the set of all finite subsets of $\hat{\omega}$ will be obtainable as $\{\widehat{\mathrm{ACK}}``\{x\} \mid x \in \hat{\omega}\}$.

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