Seminar Set Theory Handout

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Definition 1 (Stabilizer)**.** Let G act on $V^{(B)}$(or alternatively B by thm 3.3). Then for $x \in V^{(B)}$ the stabilizer of x is:

$$\text{stab}(x) = \{g \in G | gx = x\}$$

Definition 2 (Filter of subgroups)**.** Let G be a group. Then $\Gamma \subseteq \{H \subseteq G | H \text{ is a subgroup of } G\}$ is a filter of subgroups of G if the following two conditions hold:
For all $H, K \in \Gamma$ the intersection $H \cap K \in \Gamma$
For all $H \in \Gamma$, if $H \subseteq K$ with K a subgroup of G, then $K \in \Gamma$.

Definition 3 (Normal filter of subgroups)**.** If Γ is a filter of subgroups of G, then Γ is normal if for all $g \in G$, $H \in \Gamma$: $gHg^{-1} \in \Gamma$.

Definition 4 ($V^{(\Gamma)}$)**.** Let G act on B, and Γ be a filter of subgroups of G. Then define the sets $V_\alpha^{(\Gamma)}$ recursively:

$$V_\alpha^{(\Gamma)} = \{x | \text{Fun}(x) \wedge \text{ran}(x) \subseteq B \wedge \text{stab}(x) \in \Gamma \wedge \exists \xi < \alpha [\text{dom}(x) \subseteq V_\xi^{(\Gamma)}]\}$$

Now write:

$$V^{(\Gamma)} = \{x | \exists \alpha (x \in V_\alpha^{(\Gamma)})\}$$

We turn $V^{(\Gamma)}$ into a B-valued structure by defining for $u, v \in V^{(\Gamma)}$ $[\![u \in v]\!]^\Gamma$ and $[\![u = v]\!]^\Gamma$ recursively(recursion on $V_\alpha^{(\Gamma)}$):

$$[\![u \in v]\!]^\Gamma = \bigvee_{x \in \text{dom}(v)} [v(x) \wedge [\![x = u]\!]^\Gamma]$$
$$[\![u = v]\!]^\Gamma = \bigwedge_{x \in \text{dom}(u)} [u(x) \Rightarrow [\![x \in v]\!]^\Gamma] \wedge \bigwedge_{y \in \text{dom}(v)} [v(y) \Rightarrow [\![y \in u]\!]^\Gamma]$$

and by defining for $\mathcal{L}^{(\Gamma)}$-sentences σ, τ($\mathcal{L}^{(\Gamma)}$ is $\mathcal{L}^{(B)}$ without constants that are not in $V^{(\Gamma)}$), and $\phi(x)$ a $\mathcal{L}^{(\Gamma)}$-formula.

$$[\![\sigma \wedge \tau]\!]^\Gamma = [\![\sigma]\!]^\Gamma \wedge [\![\tau]\!]^\Gamma$$
$$[\![\neg\sigma]\!]^\Gamma = ([\![\sigma]\!]^\Gamma)^*$$
$$[\![\exists x \phi(x)]\!]^\Gamma = \bigwedge_{u \in v^{(\Gamma)}} [\![\phi(u)]\!]^\Gamma$$

Lemma 5 (Lemma 3.14)**.** *For every $x \in V$ $\hat{x} \in V^{(\Gamma)}$.*

From now on, Γ is assumed to be a **normal** filter of subgroups of G.

Lemma 6 (Lemma 3.15)**.** *G acts on $V^{(\Gamma)}$.*

Definition 7 (Truth and forcing in $V^{(\Gamma)}$)**.** If P is a basis for B, then $p \in P$ $p\Gamma$*-forces* the $\mathcal{L}^{(\Gamma)}$-sentence σ by

$$p \Vdash_\Gamma \sigma \leftrightarrow p \leq [\![\sigma]\!]^\Gamma$$

Any $\mathcal{L}^{(\Gamma)}$-sentence σ is called true in $V^{(\Gamma)}$(we write $V^{(\Gamma)} \vDash \sigma$) if $[\![\sigma]\!]^\Gamma = 1$

Theorem 8 (Theorem 3.18)**.** *Theorem 1.17(from Bell) holds when B is replaced by Γ.*

Theorem 9 (Theorem 3.19)**.** *All the axioms - and hence all the theorems - of ZF are true in $V^{(\Gamma)}$*