Hand-in Exercise 14 Solutions
Student Seminar on Set Theory

January 21, 2016

Exercise 1

- ***(i)*** Because P and Q commute, P and Q^c commute and thus PQ and PQ^c commute. We then have
  $P \wedge Q = P + Q - PQ$ and $P \vee Q = PQ$
  Then $(P \wedge Q) \vee (P \wedge Q^c) = PQ + PQ^c - PQPQ^c$ because P and Q and PQ and PQ^c commute*(1 point)*
  $= P(Q + Q^c) - P^2QQ^c$ because PQ and Q commute
  $= P$ because $Q + Q^c = 1$ and $QQ^c = 0$ *(1 point)*

- ***(ii)*** Proof of lemma:
  Let P, Q commute. Then $P = (P \wedge Q) \vee (P \wedge Q^c)$, with $P \wedge Q \leq Q$ and $P \wedge Q^c \leq Q^c$, which proves one part of the lemma.
  Now let there be P_1, P_2 such that $P = P_1 \vee P_2$, $P_1 \leq Q$ and $P_2 \leq Q^c$.
  Now as $P_1 \vee P_2 \geq P_1$ and $P_2^c \wedge Q = Q$(as $P_2^c \geq Q$) and as $P_1 \leq Q$, P_1 is compatable with Q:

  $$\begin{aligned} Q \geq (Q \wedge P) \vee (Q \wedge P^c) &= (Q \wedge (P_1 \vee P_2)) \vee (Q \wedge P_1^c \wedge P_2^c) \\ &\geq (Q \wedge P_1) \vee (Q \wedge P_1^c) = Q \end{aligned}$$

  Similarly:

  $$\begin{aligned} P \geq (Q \wedge P) \vee (Q^c \wedge P) &= (Q \wedge (P_1 \vee P_2)) \vee (Q^c \wedge (P_1 \vee P_2)) \\ &\geq (Q \wedge P_1) \vee (Q^c \wedge P_2) \\ &= P_1 \vee P_2 = P \end{aligned}$$

  which completes the proof of the lemma *(3 points for a correct proof of the lemma, or alternatively if a correct proof has been given without it)*
  Now set (using the lemma) $Q_{i,1}$ and $Q_{i,2}$ such that $Q_{i,1} \leq P$ and $Q_{i,2} \leq P^c$ with $Q_i = Q_{i,1} \vee Q_{i,2}$.
  Then $\bigvee_{i \in I} Q_i = \bigvee_{i \in I} Q_{i,1} \vee \bigvee_{i \in I} Q_{i,2}$ and $\bigvee_{i \in I} Q_{i,1} \leq P$, $\bigvee_{i \in I} Q_{i,2} \leq P^c$, so P is compatible with $\bigvee_{i \in I} Q_i$ by the proved lemma. Now as P is compatible with all Q_i, it is also compatible with all Q_i^c, so by the previous argument P is compatible with $\bigvee_{i \in I} Q_i^c$, so P is compatible with $(\bigvee_{i \in I} Q_i^c)^c = \bigwedge_{i \in I} Q_i$.*(1 point to finish the proof)*

Exercise 2

$[\![u \leq v]\!] = 1$, iff $[\![\forall x \in v x \in u]\!] = 1$ as u, v are defined by left Dedekind cut.
$[\![\forall x \in v x \in u]\!] = 1$ iff $[\![\forall x \in \mathbb{Q}(x \in v \to x \in u)]\!] = \bigwedge_{x\in\mathbb{Q}}[\![(x \in v \to x \in u)]\!] = 1$ as $[\![u, v \in \mathbb{Q}]\!] = 1$ *(1 point)*.
Now this is equivalent to $\forall x \in \mathbb{Q}[\![\hat{x} \in v \to \hat{x} \in u]\!] = 1$ *(1 point)*.
Writing this out gives $\forall x \in \mathbb{Q} E'_x \Rightarrow E_x = 1$
This is equivalent to $\forall x \in \mathbb{Q} E'_x \leq E_x$.
This is obviously implied by $\forall x \in \mathbb{R} E'_x \leq E_x$ and if $x \in \mathbb{R}$, then this implies $E'_x = \bigwedge_{x<q} E'_q \leq \bigwedge_{x<q} E_q = E_x$ *(2 points)*. This completes the proof.