Exercise 2

(a) There are several ways to do this. One is the following: let $\alpha \in \mathrm{ORD}$ with $\alpha \neq \emptyset$ and let $Z_\alpha = \{\langle x, 0_B\rangle \mid x \in V_\alpha^{(B)}\}$. Then

$$\begin{aligned}
[\![Z_\alpha = \emptyset]\!] &= \bigwedge_{x\in\mathrm{dom}(Z_\alpha)} [Z_\alpha(x) \Rightarrow [\![x \in \emptyset]\!]] \wedge \bigwedge_{y\in\mathrm{dom}(\emptyset)} [\emptyset(x) \Rightarrow [\![x \in Z_\alpha]\!]] \\
&= \bigwedge_{x\in\mathrm{dom}(Z_\alpha)} [0 \Rightarrow [\![x \in \emptyset]\!]] \wedge 1 \\
&= \bigwedge_{x\in\mathrm{dom}(Z_\alpha)} [1] \wedge 1 = 1.
\end{aligned}$$

Now consider $u = \{\langle Z_\alpha, 0\rangle,\ \langle \emptyset, 1\rangle\}$. We have

$$\begin{aligned}
[\![Z_\alpha \in u]\!] &= \bigvee_{x\in\mathrm{dom}(u)} [u(x) \wedge [\![Z_\alpha = x]\!]] \\
&= [u(Z_\alpha) \wedge [\![Z_\alpha = Z_\alpha]\!]] \vee [u(\emptyset) \wedge [\![Z_\alpha = \emptyset]\!]] \\
&= [u(Z_\alpha) \wedge [\![Z_\alpha = Z_\alpha]\!]] \vee [1 \wedge 1] = 1.
\end{aligned}$$

(b) Let u, v be as given. First we will show that v is extensional. We have for all $y \in \mathrm{dom}(v) = \mathrm{dom}(u)$

$$\begin{aligned}
v(y) \leq [\![y \in v]\!] &= \bigvee_{x\in\mathrm{dom}(v)} [v(x) \wedge [\![x = y]\!]] \\
&= \bigvee_{x\in\mathrm{dom}(u)} [[\![x \in u]\!] \wedge [\![x = y]\!]] \leq \bigvee_{x\in\mathrm{dom}(u)} [[\![y \in u]\!] = v(y),
\end{aligned}$$

and thus $[\![y \in u]\!] = v(y) = [\![y \in v]\!]$. Therefore

$$\begin{aligned}
[\![u = v]\!] &= \bigwedge_{x\in\mathrm{dom}(u)} [u(x) \Rightarrow [\![x \in v]\!]] \wedge \bigwedge_{y\in\mathrm{dom}(v)} [v(y) \Rightarrow [\![y \in u]\!]] \\
&= \bigwedge_{x\in\mathrm{dom}(u)} [u(x) \Rightarrow [\![x \in u]\!]] \wedge \bigwedge_{y\in\mathrm{dom}(v)} [v(y) \Rightarrow v(y)] \\
&= 1 \wedge 1 = 1,
\end{aligned}$$

as required.