Seminar on Set Theory

Hand-in exercise 7

November 6, 2015

Exercise 1.

(a) Suppose $p, q \in C(x, y)$ and $q \not\supseteq p$, so there exists an element $(a, b) \in p$ such that $(a, b) \notin q$. Now suppose that there is an element $c \in y \backslash \{b\}$ such that $(a, c) \in q$, then choose $p' = q$. If such an element c does not exist, choose an element $c \in y \backslash \{b\}$ at random, which is always possible because y contains at least two elements, and define $p' = q \cup \{(a, c)\}$. Now suppose that there is some element $r \in C(x, y)$ such that $r \supseteq p$ and $r \supseteq p'$, then $(a, b) \in r$ and $(a, c) \in r$, but $b \neq c$, so r is not well-defined. So such an r cannot exist, so $\neg\text{Comp}(p, p')$, and therefore we find that $\forall p, q \in C(x, y)(q \not\supseteq p \rightarrow \exists p' \supseteq q \neg\text{Comp}(p, p'))$. □

This exercise was worth 2 points. Students lost 1 point if they failed to notice the two different cases. Students lost $\frac{1}{2}$ point if they didn't mention the fact that y contains at least two elements where it is needed.

(b) First, we show that the image of N is in $\text{RO}(y^x)$. Since the topology on y is the discrete topology we know that $\{\{a\} \mid a \in y\}$ is a basis for y, and that all these sets are clopen. Now define for all $x_0 \in x, y_0 \in y$ the collection $S(x_0, y_0) := \{f \in y^x \mid f(x_0) = y_0\}$. Then $\{S(x_0, y_0) \mid x_0 \in x, y_0 \in y\}$ is a subbasis for the product topology on y^x, and it consists of clopen sets. This means that the collection of all finite non-empty intersections of elements of $\{S(x_0, y_0) \mid x_0 \in x, y_0 \in y\}$ is a basis for the product topology on y^x. For any set $\{S(x_0, y_0), ..., S(x_n, y_n)\}$ with nonempty intersection, define the function $p \in C(x, y)$ by $p(x_i) = y_i$ for $0 \leq i \leq n$. Then $S(x_0, y_0) \cap ... \cap S(x_n, y_n) = \{f \in y^x \mid p \subseteq f\} = N(p)$. So the $N(p)$ are a basis for the product topology on y^x. We also see that $N(p)$ is a finite intersection of clopen sets, therefore $N(p)$ is itself clopen. In particular, the $N(p)$ are contained in $\text{RO}(y^x)$.

Next, we show that $\langle \text{RO}(y^x), N \rangle$ is a Boolean completion of $C(x, y)$. Since y^x is a topological space we know from the first hand-in exercise that $\text{RO}(y^x)$ is a complete Boolean algebra. So we have to prove that N is an order-isomorphism of $C(x, y)$ onto a dense subset of $\text{RO}(y^x)$. We will first show that N is an injective map. Suppose that $p, q \in C(x, y)$ and $N(p) = N(q)$. Then $p \in N(q)$ so $q \subseteq p$, and $q \in N(p)$ so $p \subseteq q$. Because $(C(x, y), \supseteq)$ is a poset we find that $p = q$. So N is indeed injective, and in particular, N is a bijective map onto its image.
Now we show that N is an order-isomorphism. Suppose that $p, q \in C(x, y)$ and $p \leq q$, so $p \supseteq q$. Then if $f \in N(p)$ we see that $q \subseteq p \subseteq f$, so $f \in N(q)$. This means that $N(p) \subseteq N(q)$, so $N(p) \leq N(q)$. So N is order-preserving. On the other hand, if $N(p) \leq N(q)$, then $p \in N(p) \subseteq N(q)$, whence $p \supseteq q$. But this means precisely that $p \leq q$, so N is order-reflecting as well.
The last thing we have to show is that the image $\{N(p) \mid p \in C(x, y)\}$ of N is dense in $\text{RO}(y^x)$. We notice again that $p \in N(p)$ for every $p \in C(x, y)$, so $\emptyset \neq N(p)$ for all $p \in C(x, y)$. Suppose $X \neq \emptyset$ is some element of $\text{RO}(y^x)$. Then in particular, X is open, so because the $N(p)$ form a basis for the topology on y^x, the set X can be written as a

union of sets of the form $N(p)$. This means there must be an $N(p)$ such that $N(p) \subseteq X$. So the $N(p)$ are dense in $\mathrm{RO}(y^x)$.

This completes the proof that $\langle \mathrm{RO}(y^x), N\rangle$ is a Boolean completion of $C(x,y)$. $\square$

The first part was worth one point, which could only be obtained with a clear explanation of the topological ideas behind it. The second part was worth 2 points, and contained injectivity, order-preserving, order-reflecting, mentioning that $\mathrm{RO}(y^x)$ *is a complete Boolean algebra and showing that the image of* N *is dense in it. Showing that the image of* N *is dense was worth 1 point, students lost* $\frac{1}{2}$ *point if they forgot to show that* $\emptyset$ *is not in the image of* N*. The other parts where worth 1 point together. Students lost* $\frac{1}{2}$ *if they forgot one or made a mistake in one of them.*

Exercise 2. As always, we drop the superscript from $[\![\cdot]\!]^B$.

(a) First, suppose that $p \Vdash \sigma \to \tau$ and let $q \leq p$ such that $q \Vdash \sigma$. Then we have $q \leq p \leq [\![\sigma \to \tau]\!] = [\![\sigma]\!] \Rightarrow [\![\tau]\!]$ and $q \leq [\![\sigma]\!]$, whence $q \leq ([\![\sigma]\!] \Rightarrow [\![\tau]\!]) \wedge [\![\sigma]\!] \leq [\![\tau]\!]$. This means that $q \Vdash \tau$, which establishes the first direction.

Now suppose that for any $q \leq p$ such that $q \Vdash \sigma$, we also have $q \Vdash \tau$. Then for any such q, we also have $q \not\Vdash \neg\tau$ by property (vi) of the hand-out. That is,

$$\forall q \leq p \ \ (q \Vdash \sigma \to q \not\Vdash \neg\tau)\,.$$

This is equivalent to

$$\neg\exists q \leq p \ \ (q \Vdash \sigma \text{ and } q \Vdash \neg\tau)\,.$$

By properties (iii) and (v) from the hand-out, this means that $p \Vdash \neg(\sigma \wedge \neg\tau)$. But $\sigma \to \tau$ is equivalent to $\neg(\sigma \wedge \neg\tau)$, which means that $[\![\sigma \to \tau]\!] = [\![\neg(\sigma \wedge \neg\tau)]\!]$. We may conclude that $p \Vdash \sigma \to \tau$, which establishes the other direction. $\square$

It was also possible to use the definition of $[\![\sigma \to \tau]\!] = [\![\sigma]\!] \Rightarrow [\![\tau]\!]$ *directly, as many students in fact did. Each direction was worth 1 point, but* $\frac{1}{2}$ *point might be awarded if there was a non-essential mistake. (In practice, however, this turned out to be an all-or-nothing matter.)*

(b) Recall that $[\![\forall x\ \phi(x)]\!] = \bigwedge_{u \in V^{(B)}} [\![\phi(u)]\!]$. So

$$\begin{aligned} p \Vdash \forall x\ \phi(x) \qquad &\text{iff} \qquad p \leq \bigwedge_{u \in V^{(B)}} [\![\phi(u)]\!] \\ &\text{iff} \qquad \forall u \in V^{(B)}\ \ (p \leq [\![\phi(u)]\!]) \\ &\text{iff} \qquad \forall u \in V^{(B)}\ \ (p \Vdash \phi(u))\,, \end{aligned}$$

which was to shown. $\square$

Students received $\frac{1}{2}$ *point for observing that* $[\![\forall x\ \phi(x)]\!] = \bigwedge_{u \in V^{(B)}} [\![\phi(u)]\!]$ *and* $\frac{1}{2}$ *point for finishing the proof.*

(c) Recall that $\mathrm{dom}(\hat{a}) = \{\hat{x} \mid x \in a\}$ and that $\hat{a}$ takes the value 1 everywhere. So

$$[\![\forall x \in \hat{a}\ \phi(x)]\!] = \bigwedge_{u \in \mathrm{dom}\,\hat{a}} (\hat{a}(u) \Rightarrow [\![\phi(u)]\!]) = \bigwedge_{x \in a} (1 \Rightarrow [\![\phi(\hat{x})]\!]) = \bigwedge_{x \in a} [\![\phi(\hat{x})]\!].$$

We can now proceed as in the previous part. $\square$

Students received $\frac{1}{2}$ *point for observing that* $[\![\forall x \in \hat{a}\ \phi(x)]\!] = \bigwedge_{x \in a} [\![\phi(\hat{x})]\!]$ *and* $\frac{1}{2}$ *point for finishing the proof or noticing it to be analogous to the previous part.*

(d) Suppose that $[\![\sigma]\!] \neq 1$. Then $[\![\neg\sigma]\!] \neq 0$, so there is a $p \in P$ such that $p \Vdash \neg\sigma$, by property (iv) of the hand-out. In particular, we have $p \nVdash \sigma$ by property (vi) from the hand-out. So $[\![\sigma]\!] \neq 1$ implies that $\exists p \in P\ p \nVdash \sigma$, and the statement we had to prove follows. $\square$

Students received $\frac{1}{2}$ point for the strategy of applying property (vi) / the density of P to $\neg\sigma$ / $[\![\sigma]\!]^*$ and $\frac{1}{2}$ point for finishing the proof.