Seminar on Set Theory

Model solution exercise 8

December 1, 2015

---

3 Some details about an action on $V^{(B)}$

In this exercise you will prove some omitted details from the proof of Theorem 3.3. Define the map $\langle g, u\rangle \mapsto gu : G \times V^{(B)} \to V^{(B)}$ by recursion on the well-founded relation $y \in \mathrm{dom}(x)$ via

$$gu = \{\langle gx, g \cdot u(x)\rangle : x \in \mathrm{dom}(u)\}.$$

a)

Prove that

$$g \cdot [\![u \in v]\!] = [\![gu \in gv]\!]$$

and

$$g \cdot [\![u = v]\!] = [\![gu = gv]\!].$$

(2 pt.)

Solution:

Use induction on the well-founded relation defined on page 23. Assume that $g \cdot [\![x \in y]\!] = [\![gx \in gy]\!]$ and $g \cdot [\![x = y]\!] = [\![gx = gy]\!]$ for all $\langle x, y\rangle < \langle u, v\rangle$. Then

$$\begin{aligned}
[\![gu \in gv]\!] &= \bigvee_{x \in \mathrm{dom}(gv)} (gv(x) \wedge [\![gu = x]\!]) \\
&= \bigvee_{y \in \mathrm{dom}(v)} (gv(gy) \wedge [\![gu = gy]\!]) \\
\text{(by Thm 3.3 (i) and IH)} \quad &= \bigvee_{y \in \mathrm{dom}(v)} (g \cdot v(y) \wedge g \cdot [\![u = y]\!]) \\
&= g \cdot \bigvee_{y \in \mathrm{dom}(v)} (v(y) \wedge [\![u = y]\!]) \\
&= g \cdot [\![u \in v]\!].
\end{aligned}$$

Also we have that

$$\begin{aligned}
[\![gu = gv]\!] &= \bigwedge_{x' \in \mathrm{dom}(gu)} (gu(x') \Rightarrow [\![x' \in gv]\!]) \wedge \bigwedge_{y' \in \mathrm{dom}(gv)} (gv(y') \Rightarrow [\![y' \in gu]\!]) \\
&= \bigwedge_{x \in \mathrm{dom}(u)} (gu(gx) \Rightarrow [\![gx \in gv]\!]) \wedge \bigwedge_{y \in \mathrm{dom}(v)} (gv(gy) \Rightarrow [\![gy \in gu]\!]) \\
\text{(by Thm 3.3 (i) and IH)} &= \bigwedge_{x \in \mathrm{dom}(u)} (g \cdot u(x) \Rightarrow g \cdot [\![x \in v]\!]) \wedge \bigwedge_{y \in \mathrm{dom}(v)} (g \cdot v(y) \Rightarrow g \cdot [\![y \in u]\!]) \\
&= g \cdot \Big(\bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow [\![x \in v]\!]) \wedge \bigwedge_{y \in \mathrm{dom}(v)} (v(y) \Rightarrow [\![y \in u]\!]) \Big) \\
&= g \cdot [\![u = v]\!].
\end{aligned}$$

b)

Prove Theorem 3.3 (ii): $g\hat{v} = \hat{v}$ for any $v \in V$. *(3 pt.)*

Solution:

By induction on the well-founded relation $\in$. Suppose $g\hat{y} = \hat{y}$ for all $y \in v$. Then

$$\begin{aligned}
g\hat{v} &= \{\langle gx, g \cdot \hat{v}(x) \rangle | x \in \mathrm{dom}(\hat{v})\} \\
&= \{\langle g\hat{y}, g \cdot \hat{v}(\hat{y}) \rangle | y \in v\} \\
&= \{\langle g\hat{y}, g \cdot 1 \rangle | y \in v\} \\
&= \{\langle \hat{y}, 1 \rangle | y \in v\} \\
&= \hat{v}.
\end{aligned}$$