Seminar on Set Theory

Hand-in exercise 7

November 6, 2015 (due Nov. 13)

Exercise 1. Let x and y be nonempty sets such that y contains at least two elements. Recall that $C(x,y)$ is the set of all maps $z \to y$, with z a finite subset of x. We know that $(C(x,y), \supseteq)$ is a poset.

(a) Prove that $(C(x,y), \supseteq)$ is refined. *(2 pt.)*

Now let y^x be the topological space of all maps from x to y, where y^x is assigned the product topology and y is assigned the discrete topology. Let $N: \ C(x,y) \to \mathrm{RO}(y^x)$ be the map sending $p \in C(x,y)$ to $N(p) = \{f \in y^x \mid p \subseteq f\}$.

(b) Prove that for each $p \in C(x,y)$, the set $N(p)$ is indeed an element of $\mathrm{RO}(y^x)$ (i.e. that the map N is well defined) *(1 pt.)* and that $\langle \mathrm{RO}(y^x), N \rangle$ is a Boolean completion of $C(x,y)$. *(2 pt.)*

Exercise 2. Suppose that (B, e) is a Boolean completion of a certain refined poset P. Furthermore, let σ and τ be B-sentences and let $\phi(x)$ be a B-formula. Show the following:

(a) For all $p \in P$, we have: $p \Vdash \sigma \to \tau$ if and only if $\forall q \leq p \ \ (q \Vdash \sigma \to q \Vdash \tau)$. *(2 pt.)*

(b) For all $p \in P$, we have: $p \Vdash \forall x \ \phi(x)$ if and only if $\forall u \in V^{(B)} \ \ (p \Vdash \phi(u))$. *(1 pt.)*

(c) For all $p \in P$ and $a \in V$, we have: $p \Vdash \forall x \in \hat{a} \ \phi(x)$ if and only if $\forall x \in a \ \ (p \Vdash \phi(\hat{x}))$. *(1 pt.)*

(d) If $p \Vdash \sigma$ for all $p \in P$, then $[\![\sigma]\!]^B = 1$. *(1 pt.)*

For this exercise, you are allowed to use all properties of forcing that were proven during the lecture. In particular, you may use properties (i)-(vi) from the hand-out.