Category Theory and Topos Theory,
Spring 2014
Hand-In Exercises
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Exercise 1 (To be handed in February 17) Recall that a topological space is normal if every one-point subset is closed and for every pair $A, B$ of disjoint closed subsets, there exist disjoint open subsets $U, V$ with $A \subset U$, $B \subset V$. We denote by $\mathcal{N}$ the full subcategory of Top on the normal topological spaces.

a) Characterise the epimorphisms in $\mathcal{N}$. Hint: you may find it useful to invoke Urysohn’s Lemma.

b) Show that for two morphisms $f, g : A \to B$ in $\mathcal{N}$ we have: $f = g$ if and only if for every morphism $h : B \to \mathbb{R}$, $hf = hg$ holds (this property of $\mathbb{R}$ in $\mathcal{N}$ is sometimes called a coseparator).

Exercise 2 (To be handed in March 10) Let $\mathcal{C}$ be a regular category.

a) Suppose that

\[
\begin{array}{ccc}
A & \xrightarrow{h} & B \\
\downarrow{g} & & \downarrow{f} \\
C & \xrightarrow{e} & D
\end{array}
\]

is a pullback diagram in $\mathcal{C}$ with $e$ regular epi. Prove: if $g$ is mono, then so is $f$.

b) Prove that the composition of two regular epis in $\mathcal{C}$ is again regular epi in $\mathcal{C}$.

Exercise 3 (To be handed in March 24) We are given an adjunction $\mathcal{E} \xleftarrow{R} \xrightarrow{I} \mathcal{S}$ with $R \dashv I$, unit $\eta$ and counit $\varepsilon$.

a) Prove: $I$ is faithful if and only if every component of $\varepsilon$ is epi; and $I$ is full if and only if every component of $\varepsilon$ is split mono. Hint: you may use the fact that for an arrow $A \xrightarrow{f} B$ in $\mathcal{E}$, the composite arrow $RIA \xrightarrow{\varepsilon_A} A \xrightarrow{f} B$ transposes under the adjunction to the arrow $I(f) : I(A) \to I(B)$.
Exercise 4 (To be handed in April 7) Let \( \Omega \) be a frame, as in Definition 4.13 of the Category Theory lecture notes. We consider the category \( \mathcal{C}_\Omega \) defined there, and also the presheaf category \( \text{Set}^{\Omega^\text{op}} \).

We have the Yoneda embedding \( y : \Omega \to \text{Set}^{\Omega^\text{op}} \) and we have a functor \( H : \Omega \to \mathcal{C}_\Omega \), which sends \( p \in \Omega \) to the object \((X, E_X)\) where \( X = \{ * \} \) and \( E_X(*) = p \).

a) Show that there is an essentially unique functor \( F : \mathcal{C}_\Omega \to \text{Set}^{\Omega^\text{op}} \) which preserves all small coproducts and moreover makes the diagram

\[
\begin{array}{ccc}
\Omega & \xrightarrow{H} & \mathcal{C}_\Omega \\
\downarrow & & \downarrow \quad \downarrow \\
\text{Set}^{\Omega^\text{op}} & \xrightarrow{y} & \text{Set}^{\Omega^\text{op}}
\end{array}
\]

commute. Give a concrete description of \( F(X, E_X) \) as a presheaf on \( \Omega \).

b) Suppose \( \Omega \) has a \textbf{(nonempty!)-correction added later} subset \( B \) with the property that \( \bigvee B \notin B \). Show that the functor \( F \) does not preserve regular epis.

c) Show that the functor \( F \) has a left adjoint \( L \).

d) Show that the functor \( L \) from part c) does not preserve equalizers.

Exercise 5 (To be handed in April 28) We consider the category \( \mathcal{C} \) whose objects are subsets of \( \mathbb{N} \), and arrows \( A \to B \) are \textit{finite-to-one} functions, i.e. functions \( f \) satisfying the requirement that for every \( b \in B \), the set \( \{ a \in A \mid f(a) = b \} \) is finite.

a) Show that \( \mathcal{C} \) has pullbacks.

b) Define for every object \( A \) of \( \mathcal{C} \) a set \( \text{Cov}(A) \) of sieves on \( A \) as follows: \( R \in \text{Cov}(A) \) if and only if \( R \) contains a finite family \( \{ f_1, \ldots, f_n \} \) of functions into \( A \), which is jointly almost surjective, that is: the set

\[
A - \bigcup_{i=1}^n \text{Im}(f_i)
\]

is finite.

Show that \( \text{Cov} \) is a Grothendieck topology.

c) Show that if \( R \in \text{Cov}(A) \), then \( R \) contains a family \( \{ f_1, \ldots, f_n \} \) which is jointly almost surjective and moreover, every \( f_i \) is injective.
d) Given a (nonempty!–correction added later) set \( X \) and an object \( A \) of \( \mathcal{C} \), we define \( F_X(A) \) as the set of equivalence classes of functions \( \xi : A \to X \), where \( \xi \sim \eta \) if \( \xi(n) = \eta(n) \) for all but finitely many \( n \in A \).

Show that this definition can be extended to the definition of a presheaf \( F_X \) on \( \mathcal{C} \).

e) Show that \( F_X \) is a sheaf for \( \text{Cov} \).

Exercise 6 (To be handed in May 12)  This exercise is about interpreting Logic in the category of sheaves on a site. There is a ‘forcing’ definition similar to the one for presheaves; it is explained on p. 32 of the lecture notes, with one regrettable inaccuracy. The definition of \( C \models J \neg \varphi(a_1, \ldots, a_n) \) should be:

- \( C \models J \neg \varphi(a_1, \ldots, a_n) \) if and only if for every arrow \( g : D \to C \), if \( D \models J \varphi(a_1g, \ldots, a_ng) \) then \( \emptyset \in \text{Cov}(D) \)

Now the exercise. We assume that we have a site \( (\mathcal{C}, \text{Cov}) \) and an object \( I \) of \( \mathcal{C} \) which satisfy the following conditions:

i) \( \emptyset \not\in \text{Cov}(I) \)

ii) If there is no arrow \( I \to A \) then \( \emptyset \in \text{Cov}(A) \)

iii) If there is an arrow \( I \to A \) then every arrow \( A \to I \) is split epi

We call a sheaf \( F \) in \( \text{Sh}(\mathcal{C}, \text{Cov}) \) \( \neg\neg \)-separated if for every object \( A \) of \( \mathcal{C} \) and all \( x, y \in F(A) \),

\[ A \models J \neg \neg (x = y) \to x = y \]

Prove that the following two assertions are equivalent, for a sheaf \( F \):

a) \( F \) is \( \neg\neg \)-separated

b) For every object \( A \) of \( \mathcal{C} \) and all \( x, y \in F(A) \) the following holds: if for every arrow \( \phi : I \to A \) we have \( x\phi = y\phi \) in \( F(I) \), then \( x = y \)

Solution to Exercise 1.

a) An arrow \( f : X \to Y \) in \( \mathcal{N} \) is epi if and only if the image of \( f \) is dense in \( Y \).

The ‘if’ part is easy since normal spaces are Hausdorff and a continuous map between Hausdorff spaces is completely determined by its restriction to a dense subset of its domain. For the ‘only if’ part, suppose \( f \) does not have dense image. Pick \( y_0 \notin f(X) \). By Urysohn’s Lemma there is a continuous function \( g : Y \to \mathbb{R} \) satisfying: \( g(y) = 0 \) for every \( y \in f(X) \), and \( g(y_0) = 1 \). Let \( h : Y \to \mathbb{R} \) be the function constant 0. Then \( g \) and \( h \) agree on \( f(X) \) yet \( g \neq h \), so \( f \) is not epi.

b) Clearly, ‘only if’ is trivial. For the ‘if’ part, suppose \( f \neq g \). Pick \( a \in A \) with \( f(a) \neq g(a) \). Again by Urysohn, there is a continuous \( h : B \to \mathbb{R} \) with \( h(f(a)) = 0, h(g(a)) = 1 \). So \( hf \neq hg \).
Solution to Exercise 2.

a) Suppose $E \xrightarrow{p_0} B$ is a parallel pair for which $fp_0 = fp_1$. Let

$$
\begin{array}{c}
F \xrightarrow{h'} E \\
\downarrow g' \\
C \xrightarrow{e} D \\
\end{array}
$$

be a pullback. Then by the pullback property of the original diagram there are arrows $q_0, q_1 : F \to A$ such that $gq_0 = g', hq_0 = p_0h'$ and $gq_1 = g', hq_1 = p_1h'$:

$$
\begin{array}{c}
F \xrightarrow{h'} E \\
\downarrow q_0 \\
\downarrow q_1 \\
A \xrightarrow{h} \downarrow f \\
\downarrow e \\
B \xrightarrow{g} \downarrow D \\
\end{array}
$$

From $gq_0 = g' = gq_1$ and the assumption that $g$ is mono, we get $q_0 = q_1$. Therefore $p_0h' = hq_0 = hq_1 = p_1h'$. Since $h'$, being a pullback of the regular epi $e$, is regular epi (hence epi), we find $p_0 = p_1$. We conclude that $f$ is mono.

b) Suppose in $A \xrightarrow{e_1} B \xrightarrow{e_2} C$ the arrows $e_1, e_2$ are both regular epi. In order to show that the composite $e_2e_1$ is regular epi, we factor this composite as $me$ with $m$ mono and $e$ regular epi:

$$
\begin{array}{c}
A \xrightarrow{e_1} B \xrightarrow{e_2} C \\
\downarrow e \\
D \\
\end{array}
$$

If $E \xrightarrow{p_0} A$ is the kernel pair of $e_1$ then $mep_0 = e_2e_1p_0 = e_2e_1p_1 = mep_1$ so since $m$ in mono, $ep_0 = ep_1$. Therefore, since $e_1$ is the coequalizer of $p_0, p_1$ we have a unique map $n : B \to D$ satisfying $ne_1 = e$. Then we also have: $mne_1 = me = e_2e_1$, so since $e_1$ is epi, $mn = e_2$ and the following diagram commutes:

$$
\begin{array}{c}
A \xrightarrow{e_1} B \xrightarrow{e_2} C \\
\downarrow e \\
D \\
\end{array}
$$

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Repeating the argument for the kernel pair \( q_0, q_1 \) of \( e_2 \), we get that \( nq_0 = nq_1 \); so since \( e_2 \) is the coequalizer of its kernel pair, we get a unique arrow \( k : C \to D \) such that \( ke_2 = n \).

Then \( mke_2 = mn = e_2 \) so since \( e_2 \) is epi, \( mk = \text{id}_C \); and \( kme_1 = ne_1 = e \), so since \( e \) is epi, \( km = \text{id}_D \). We find that \( k \) is a two-sided inverse for \( m \), which is therefore an isomorphism. We conclude that \( e_2e_1 \) is regular epi.

**Solution to Exercise 3.**

a) By the hint we have for every parallel pair \( f, g : A \to B \), that \( I(f) = I(g) \) if and only if \( f\varepsilon_A = g\varepsilon_A \). From this it follows easily that \( I \) is faithful if and only if \( \varepsilon \) is epi.

Suppose \( I \) is full. Take \( \alpha : A \to RIA \) such that \( I(\alpha) = \eta_A : IA \to IRIA \). Then both \( \text{id}_{RIA} \) and \( \alpha \varepsilon_A \) are transposes of \( \eta_A \), so \( \alpha \varepsilon_A = \text{id}_{RIA} \) and \( \varepsilon \) is split monic.

Conversely, suppose \( \varepsilon_A \) is split monic, with retraction \( \alpha \). Any map \( h : IA \to IB \) transposes to

\[ RIA \xrightarrow{R(h)} RIB \xrightarrow{\varepsilon_B} B \]

which is equal to the composite

\[ RIA \xrightarrow{\varepsilon_A} A \xrightarrow{\alpha} RIA \xrightarrow{R(h)} RIB \xrightarrow{\varepsilon_B} B \]

which is the transpose of \( I(\varepsilon_B R(h) \alpha) \). Therefore \( h = I(\varepsilon_B R(h) \alpha) \), and \( I \) is full.

b) Let \( \mathcal{I} \) be an index category and \( M : \mathcal{I} \to \mathcal{E} \) be a diagram. Suppose \( \nu : \Delta_L \Rightarrow IM \) is a limiting cone for \( IM \) in \( \mathcal{S} \), with vertex \( L \). Then we have a cone \( \Delta_{RL} \xRightarrow{\varepsilon_l(R(\nu))} IM \) in \( \mathcal{E} \), and therefore a cone \( I(\varepsilon_l(R(\nu))) : \Delta_{IRL} \Rightarrow IM \) in \( \mathcal{S} \). Since \( \nu \) is limiting we have a unique map of cones \( d : IRL \to L \).

Moreover, for each object \( i \) of \( \mathcal{I} \) we have, by naturality of \( \eta \) and the triangle identities, a commutative diagram

\[
\begin{array}{ccc}
IRL & \xrightarrow{IR(\nu)} & IRIM(i) & \xrightarrow{I(\varepsilon)} & IM(i) \\
\downarrow{\eta} & & \downarrow{\eta} & \searrow{id} \\
L & \xrightarrow{\nu_i} & IM(i)
\end{array}
\]

which means that \( \eta \) is a map of cones from \( \nu \) to \( I(\varepsilon \circ (R(\nu))) \). Since \( \nu \) is limiting, we have \( d\eta = \text{id}_L \).

Now consider \( \eta d : IRL \to IRL \). Since \( I \) is full, this composition is of the form \( I(e) \) for some \( e : RL \to RL \). Let \( \tilde{e} : L \to IRL \) be the transpose of
e. Then \( \hat{e} = I(e)\eta = \eta d\eta = \eta \), which is the transpose of \( id_{RL} \). Therefore \( e = id_{RL} \) and \( \eta_L \) is an isomorphism with inverse \( d \).

We also see that the cone \( \nu \) is isomorphic to the cone \( I(\varepsilon \circ R(\nu)) : \Delta_{IRL} \Rightarrow IM \), which is therefore limiting. It now follows readily from the full and faithfulness of \( I \) that the cone \( \varepsilon \circ R(\nu) : RL \rightarrow M \) is limiting in \( E \).

c) Another proof of part b) is: prove that \( I \) is monadic and invoke the theorem (exercise 114) in the lecture notes that a monadic functor creates limits. So, let \( h : IRX \rightarrow X \) be an \( IR \)-algebra. Then \( h\eta_X = id_X \) and just as in the last part of the proof given above, one proves that \( h \) is an isomorphism with inverse \( \eta \).

Moreover, any object of the form \( IX \) has the structure of an \( IR \)-algebra:
\[ \text{IRIX} \xrightarrow{\iota(e)} IX. \]

We see that the category \( IR \text{-Alg} \) is equivalent to the full subcategory of \( S \) on objects in the image of \( I \). Since \( I \) is full and faithful, this subcategory is equivalent to \( E \) via \( I \). So \( I \) is indeed monadic.

**Solution to Exercise 4.**

a) The first thing to recognize is that in \( C_{\Omega} \), every object \((X, E_X)\) is the coproduct of the family \( \{H(E_X(x)) \mid x \in X\} \). Therefore, if the functor \( F \) is to preserve coproducts and make the given diagram commute, there is no choice but to put
\[ F(X, E_X) = \coprod \{y(E_X(x)) \mid x \in X\} \]

As a presheaf, \( F(X, E_X) \) can be described like this: it is the \( P \)-indexed collection of sets \((A_p)_{p \in P}\) where
\[ A_p = \{(x, p) \mid p \leq E_X(x)\} \]

and for \( q \leq p \) the transition map \( Aqp : A_p \rightarrow A_q \) sends \((x, p)\) to \((x, q)\).

For a morphism \( f : (X, E_X) \rightarrow (Y, E_Y) \) we have \( E_X(x) \leq E_Y(f(x)) \) so if the presheaf \( F(Y, E_Y) \) is \((B_p)_{p \in \Omega}\), then \((x, p) \in A_p \) implies \((f(x), p) \in B_p \), so we have an arrow \( F(f) : F(X, E_X) \rightarrow F(Y, E_Y) \) and this makes \( F \) a functor.

b) Here, we must know what regular epimorph look like in \( C_{\Omega} \). We have: \( f : (X, E_X) \rightarrow (Y, E_Y) \) is regular epi if and only if \( f \) is a surjective function and moreover, for each \( y \in Y \), \( E_Y(y) = \{E_X(x) \mid f(x) = y\} \).

Now suppose \( B \subset \Omega \) and \( \bigvee B \notin B \), so for all \( b \in B, b < \bigvee B \). We consider the objects \((B, id)\) and \( H(\bigvee B) \) of \( C_{\Omega} \). The unique map \( \pi : B \rightarrow \{*\} \) is a morphism from \((B, id)\) to \( H(\bigvee B) \) and it is regular epi (for this, it has to be assumed that \( B \) is nonempty! This was a slight inaccuracy in the formulation of the exercise).

However, the morphism \( F(\pi) \) is not epi in \( \text{Set}^{\text{op}} \), since \( FH(\bigvee B) = y(\bigvee B) \) has an element at level \( \bigvee B \), whereas \( F(B, id) \) has no such element. Hence the component of \( F(\pi) \) at \( \bigvee B \) is not surjective.
Solution to Exercise 5.

a) Given \( B \xrightarrow{f} A, C \xrightarrow{g} A \) in \( \mathcal{C} \), let

\[
\begin{array}{ccc}
X & \xrightarrow{g'} & B \\
\downarrow{f'} & & \downarrow{f} \\
C & \xrightarrow{g} & A
\end{array}
\]

a pullback diagram in Set. Then \( X \subseteq \mathbb{N} \). Because \( f, g \) are finite-to-one, so are \( f', g' \) and the diagram lives in \( \mathcal{C} \); and it is a pullback in \( \mathcal{C} \) because whenever we have arrows \( Y \xrightarrow{a} A, Y \xrightarrow{b} B \) in \( \mathcal{C} \) with \( fa = gb \), then the unique factorization \( Y \to X \) must be finite-to-one, and therefore in \( \mathcal{C} \).

c) Let \( (A_p)_{p \in \Omega} \) be a presheaf on \( \Omega \), with maps \( A_{qp} : A_p \to A_q \) for \( q \leq p \). Let \( \perp \) denote the bottom element of \( \Omega \). Consider a morphism \( f : (A_p)_{p \in \Omega} \to F(X, E_X) \). Suppose \( \xi \in A_p \) and \( \eta \in A_q \). By naturality of \( f \), if \( A_{\perp p}(\xi) = A_{\perp q}(\eta) \) and \( f_p(\xi) = (x, p), f_q(\eta) = (y, q) \), then \( x = y \). We see therefore, that \( f \) determines a function \( \hat{f} : A_{\perp} \to X \) with the property that for every element \( \xi \in A_p \),

\[
f_p(\xi) = (\hat{f}(A_{\perp p}(\xi)), p)
\]

Moreover, we must have for \( \xi \in A_p \) that \( p \leq E_X(\hat{f}(A_{\perp p}(\xi))) \). This gives us the idea to define \( L : \text{define } L((A_p)_{p \in P}) \) as \( (A_{\perp}, E) \) where

\[
E(\xi) = \bigvee\{p \in P \mid \text{for some } x \in A_p, A_{\perp p}(x) = \xi\}
\]

We now see that the map \( \hat{f} : A_{\perp} \to X \) is a morphism \( L((A_p)_{p \in P}) \to (X, E_X) \) in \( \mathcal{C}_\Omega \). Conversely, given a map \( g : L((A_p)_{p \in P}) \to (X, E_X) \) we have a map \( \tilde{g} : (A_p)_{p \in P} \to F(X, E_X) \) by putting

\[
\tilde{g}_p(\xi) = (g(A_{\perp p}(\xi)), p)
\]

You can check yourself that \( \tilde{g} \) is well-defined and that the operations \( (\tilde{\cdot}) \) and \( (\tilde{\cdot}) \) are each other’s inverse. So, \( L \) is left adjoint to \( F \).

d) For a concrete example we have to fix \( \Omega \). So let \( \Omega = \{0 < 1\} \). Consider the presheaves \( A \) and \( B \) on \( \Omega \), where \( A_1 = A_0 = \{\ast\}, B_0 = \{\ast\}, B_1 = \{a, b\} \) with \( a \neq b \). We have two arrows, \( f_a \) and \( f_b \), from \( A \) to \( B \) and their equalizer is the inclusion \( E \subseteq A \) where \( E_0 = \{\ast\}, E_1 = \emptyset \). Applying the functor \( L \), we see that \( L(A) = L(B) = H(1) \), and that \( L(f_a) = L(f_b) \) is the identity map. So the equalizer of \( L(f_a) \) and \( L(f_b) \) is an isomorphism. However, \( L(E) = H(0) \) and \( L(E) \to L(A) \) is not an isomorphism. So \( L \) does not preserve equalizers.
b) Certainly the maximal sieve is in $\text{Cov}(A)$ since it contains the one-element family consisting of the identity on $A$.

For stability, suppose $R \in \text{Cov}(A)$ and $g : B \to A$ is an arrow in $\mathcal{C}$. We have to prove that $g^*(R) \in \text{Cov}(B)$. Let $\{f_1, \ldots, f_n\}$ a finite subfamily of $R$ which is jointly almost surjective. It is enough to show that the sieve on $B$ generated by $\{f'_1, \ldots, f'_n\}$ is in $\text{Cov}(B)$, where each $f'_i$ is such that

$$
\begin{array}{ccc}
B & \xrightarrow{g} & A \\
\downarrow & & \downarrow \\
B & \xrightarrow{f'_i} & f_i
\end{array}
$$

is a pullback. This is because this sieve is a subsieve of $f^*(R)$. Now the set

$$A - \bigcup_{i=1}^{n} \text{Im}(f_i)$$

is a finite set, call it $E$. Since $g$ is an arrow in $\mathcal{C}$, hence a finite-to-one function, its preimage under $g$, $g^{-1}(E)$, is finite. Hence we have that

$$B - \bigcup_{i=1}^{n} \text{Im}(f'_i)$$

is also finite, which shows that the sieve generated by $\{f'_1, \ldots, f'_n\}$ is in $\text{Cov}(B)$, as desired.

For local character, suppose $R, S$ are sieves on $A$, $R \in \text{Cov}(A)$ and for every $f : D \to A$ in $R$ we have $f^*(S) \in \text{Cov}(D)$. We have to prove that $S \in \text{Cov}(A)$. Now if $R$ contains the jointly almost surjective family $\{f_1, \ldots, f_n\}$ and for every $i$ the sieve $f_i^*(S)$ contains the jointly almost surjective family $\{g_{i1}^j, \ldots, g_{ik_i}^j\}$, then the family

$$\{f_i g_{ij}^i | 1 \leq i \leq n, 1 \leq j \leq k_i \}$$

is a jointly almost surjective family of arrows into $A$, and this family is contained in $S$. So $S \in \text{Cov}(A)$, as desired.

c) Suppose $\{f_1, \ldots, f_n\} \subset R$ is jointly almost surjective. For each $i$ let $e_i : \text{Im}(f_i) \to \text{dom}(f_i)$ be a section of $f_i$. Then $R$ contains the family $\{f_1 e_1, \ldots, f_n e_n\}$ since $R$ is a sieve. Moreover, every composition $f_i e_i$ is injective; and the joint image of the maps $f_i e_i$ is the same as the joint image of the maps $f_i$.

d) Again, we need the set $X$ to be nonempty. For, if $A \subset \mathbb{N}$ is finite and nonempty, then $\emptyset \in \text{Cov}(A)$ because the empty family is jointly almost surjective. However, if $X = \emptyset$ then there are no equivalence classes of functions $A \to X$. 

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Provided $X$ is nonempty we define $F_X(A)$ as given. For an arrow $f : B \to A$ and $[\xi] \in F_X(A)$ we put: $[\xi] f = [\xi \circ f]$. This is well-defined, for if $\xi \sim \eta$ in $F_X(A)$ then $\xi \circ f \sim \eta \circ f$ in $F_X(B)$. Clearly, we have a presheaf structure on $F_X$.

c) Suppose $\xi, \eta : A \to X$ are two functions such that for all $f : B \to A$ in some $R \in \text{Cov}(A)$ we have $[\xi] f = [\eta] f$ in $F_X(B)$. Then in particular this holds for a finite, jointly almost surjective subfamily $\{f_1, \ldots, f_n\}$ of $R$. So for each $i$, the compositions $\xi \circ f_i$ and $\eta \circ f_i$ agree on all but finitely elements of their domain. Since the family is finite, $\xi$ and $\eta$ agree on all but finitely elements of $A$. So $F_X$ is separated.

Now suppose we have a compatible family

$$\{[\xi_f] \in F_X(\text{dom}(f)) \mid f \in R\}$$

indexed by some $R \in \text{Cov}(A)$. We must produce an amalgamation. Now $R$ contains a finite, jointly almost surjective subfamily $\{f_1, \ldots, f_n\}$ consisting of injective functions. Let $A_i$ be the image of $f_i$. Clearly we have a unique function $\eta_i : A_i \to X$ such that $\eta_i \circ f_i = \xi_i$. For different indices $i$ and $j$, there can be at most finitely many elements $x \in A_i \cap A_j$ for which $\eta_i(x) \neq \eta_j(x)$, by the compatibility of the family. So in the whole of $A$ there are at most finitely many $x$ such that either $x \notin \bigcup_{i=1}^n \text{Im}(f_i)$, or for some $i \neq j$, $x \in A_i \cap A_j$ and $\eta_i(x) \neq \eta_j(x)$. Let the finite set of such $x$’s be $E$. Then define $\eta : A \to X$ by: $\eta(x) = \eta_i(x)$, if $x \notin E$ and $x \in A_i$ (it doesn’t matter which $i$ we choose), and let $\eta(x)$ be an arbitrary element of $X$ if $x \in E$. Then $[\eta]$ is an amalgamation for the family $\{[\xi_f] \mid 1 \leq i \leq n\}$ and hence, by compatibility, for the original family we started with.

**Solution to Exercise 6.**

a) $\Rightarrow$ b): suppose $A$ an object of $\mathcal{C}$, $x, y \in F(A)$ such that for all $\phi : I \to A$ we have $x \phi = y \phi$. We have to prove that $x = y$, but by assumption a) it is sufficient to prove that $A \models J \neg \neg (x = y)$, which, after some elementary logical operations, is equivalent to:

$$(* \quad \text{For every arrow } B \xrightarrow{f} A, \text{ if } \emptyset \notin \text{Cov}(B) \text{ then there is an arrow } C \xrightarrow{g} B \text{ such that } \emptyset \notin \text{Cov}(C) \text{ and } xfg = yfg.$$ 

But given such $f : B \to A$ with $\emptyset \notin \text{Cov}(B)$, we have some $g : I \to B$ by our assumptions on the site $(\mathcal{C}, \text{Cov})$. By hypothesis on $A$ and $x, y$, we have $xfg = yfg$. So we have proved $(*)$.

b) $\Rightarrow$ a): Suppose $A \models J \neg \neg (x = y)$ (which is equivalent to $(*)$ above, as we saw), and let $f : I \to A$ be an arrow. By $(*)$ there is an arrow $C \xrightarrow{g} I$ such that $xfg = yfg$ and $\emptyset \notin \text{Cov}(C)$. This last fact gives us some map $I \to C$, so we know that $g : C \to I$ is split epi; let $h : I \to C$ be a retraction. Then $xfg = yfg$, hence

$$xf = xfg h = yfg h = yf$$
The map $f : I \to A$ was arbitrary, so we conclude that the hypothesis of part b) is satisfied. Hence $x = y$. Because also $A$ was arbitrary, we conclude that $F$ is $\neg\neg$-separated, as was to be shown.