Category Theory and Topos Theory,
Spring 2014
Hand-In Exercises

Jaap van Oosten

March–June 2014

Exercise 1 (To be handed in February 17) Recall that a topological space is *normal* if every one-point subset is closed and for every pair A, B of disjoint closed subsets, there exist disjoint open subsets U, V with $A \subset U$, $B \subset V$. We denote by $\mathcal{N}$ the full subcategory of Top on the normal topological spaces.

a) Characterise the epimorphisms in $\mathcal{N}$. Hint: you may find it useful to invoke Urysohn's Lemma.

b) Show that for two morphisms $f, g : A \to B$ in $\mathcal{N}$ we have: $f = g$ if and only if for every morphism $h : B \to \mathbb{R}$, $hf = hg$ holds (this property of $\mathbb{R}$ in $\mathcal{N}$ is sometimes called a *coseparator*)

Exercise 2 (To be handed in March 10) Let $\mathcal{C}$ be a regular category.

a) Suppose that

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "g"'] & B \arrow[d, "f"] \\ C \arrow[r, "e"'] & D \end{tikzcd}$$

is a pullback diagram in $\mathcal{C}$ with e regular epi. Prove: if g is mono, then so is f.

b) Prove that the composition of two regular epis in $\mathcal{C}$ is again regular epi in $\mathcal{C}$.

Exercise 3 (To be handed in March 24) We are given an adjunction $\mathcal{E} \underset{I}{\overset{R}{\leftrightarrows}} \mathcal{S}$ with $R \dashv I$, unit η and counit ε.

a) Prove: I is faithful if and only if every component of ε is epi; and I is full if and only if every component of ε is split mono. Hint: you may use the fact that for an arrow $A \xrightarrow{f} B$ in $\mathcal{E}$, the composite arrow $RIA \xrightarrow{\varepsilon_A} A \xrightarrow{f} B$ transposes under the adjunction to the arrow $I(f) : I(A) \to I(B)$.

b) Now suppose I is full and faithful. Prove: if $F : \mathcal{A} \to \mathcal{E}$ is a diagram and IF has a limit in $\mathcal{S}$, then F has a limit in $\mathcal{E}$.

Exercise 4 (To be handed in April 7) Let Ω be a frame, as in Definition 4.13 of the Category Theory lecture notes. We consider the category $\mathcal{C}_\Omega$ defined there, and also the presheaf category $\mathrm{Set}^{\Omega^{\mathrm{op}}}$.

We have the Yoneda embedding $y : \Omega \to \mathrm{Set}^{\Omega^{\mathrm{op}}}$ and we have a functor $H : \Omega \to \mathcal{C}_\Omega$, which sends $p \in \Omega$ to the object (X, E_X) where $X = \{*\}$ and $E_X(*) = p$.

a) Show that there is an essentially unique functor $F : \mathcal{C}_\Omega \to \mathrm{Set}^{\Omega^{\mathrm{op}}}$ which preserves all small coproducts and moreover makes the diagram

$$\begin{tikzcd} \Omega \arrow[d, "H"'] \arrow[dr, "y"] & \\ \mathcal{C}_\Omega \arrow[r, "F"'] & \mathrm{Set}^{\Omega^{\mathrm{op}}} \end{tikzcd}$$

commute. Give a concrete description of $F(X, E_X)$ as a presheaf on Ω.

b) Suppose Ω has a (**nonempty!**-correction added later) subset B with the property that $\bigvee B \notin B$. Show that the functor F does not preserve regular epis.

c) Show that the functor F has a left adjoint L.

d) Show that the functor L from part c) does not preserve equalizers.

Exercise 5 (To be handed in April 28) We consider the category $\mathcal{C}$ whose objects are subsets of $\mathbb{N}$, and arrows $A \to B$ are *finite-to-one* functions, i.e. functions f satisfying the requirement that for every $b \in B$, the set $\{a \in A \mid f(a) = b\}$ is finite.

a) Show that $\mathcal{C}$ has pullbacks.

b) Define for every object A of $\mathcal{C}$ a set $\mathrm{Cov}(A)$ of sieves on A as follows: $R \in \mathrm{Cov}(A)$ if and only if R contains a finite family $\{f_1, \ldots, f_n\}$ of functions into A, which is *jointly almost surjective*, that is: the set

$$A - \bigcup_{i=1}^{n} \mathrm{Im}(f_i)$$

is finite.

Show that Cov is a Grothendieck topology.

c) Show that if $R \in \mathrm{Cov}(A)$, then R contains a family $\{f_1, \ldots, f_n\}$ which is jointly almost surjective and moreover, every f_i is injective.

d) Given a (**nonempty!**–correction added later) set X and an object A of $\mathcal{C}$, we define $F_X(A)$ as the set of equivalence classes of functions $\xi : A \to X$, where $\xi \sim \eta$ if $\xi(n) = \eta(n)$ for all but finitely many $n \in A$.

   Show that this definition can be extended to the definition of a presheaf F_X on $\mathcal{C}$.

e) Show that F_X is a sheaf for Cov.

Exercise 6 (To be handed in May 12) This exercise is about interpreting Logic in the category of sheaves on a site. There is a 'forcing' definition similar to the one for presheaves; it is explained on p. 32 of the lecture notes, with one regrettable inaccuracy. The definition of $C \Vdash_J \neg\varphi(a_1, \ldots, a_n)$ should be:

- $C \Vdash_J \neg\varphi(a_1, \ldots, a_n)$ if and only if for every arrow $g : D \to C$, if $D \Vdash_J \varphi(a_1 g, \ldots, a_n g)$ then $\emptyset \in \mathrm{Cov}(D)$

Now the exercise. We assume that we have a site $(\mathcal{C}, \mathrm{Cov})$ and an object I of $\mathcal{C}$ which satisfy the following conditions:

i) $\emptyset \notin \mathrm{Cov}(I)$

ii) If there is no arrow $I \to A$ then $\emptyset \in \mathrm{Cov}(A)$

iii) If there is an arrow $I \to A$ then every arrow $A \to I$ is split epi

We call a sheaf F in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ $\neg\neg$-*separated* if for every object A of $\mathcal{C}$ and all $x, y \in F(A)$,

$$A \Vdash_J \neg\neg(x = y) \to x = y$$

Prove that the following two assertions are equivalent, for a sheaf F:

a) F is $\neg\neg$-separated

b) For every object A of $\mathcal{C}$ and all $x, y \in F(A)$ the following holds: if for every arrow $\phi : I \to A$ we have $x\phi = y\phi$ in $F(I)$, then $x = y$

Solution to Exercise 1.

a) An arrow $f : X \to Y$ in $\mathcal{N}$ is epi if and only if the image of f is dense in Y. The 'if' part is easy since normal spaces are Hausdorff and a continuous map between Hausdorff spaces is completely determined by its restriction to a dense subset of its domain. For the 'only if' part, suppose f does not have dense image. Pick $y_0 \notin \overline{f(X)}$. By Urysohn's Lemma there is a continuous function $g : Y \to \mathbb{R}$ satisfying: $g(y) = 0$ for every $y \in \overline{f(X)}$, and $g(y_0) = 1$. Let $h : Y \to \mathbb{R}$ be the function constant 0. Then g and h agree on $f(X)$ yet $g \neq h$, so f is not epi.

b) Clearly, 'only if' is trivial. For the 'if' part, suppose $f \neq g$. Pick $a \in A$ with $f(a) \neq g(a)$. Again by Urysohn, there is a continuous $h : B \to \mathbb{R}$ with $h(f(a)) = 0$, $h(g(a)) = 1$. So $hf \neq hg$.

Solution to Exercise 2.

a) Suppose $E \overset{p_0}{\underset{p_1}{\rightrightarrows}} B$ is a parallel pair for which $fp_0 = fp_1$. Let

$$\begin{tikzcd} F \arrow[r, "h'"] \arrow[d, "g'"'] & E \arrow[d, "fp_0"] \\ C \arrow[r, "e"'] & D \end{tikzcd}$$

be a pullback. Then by the pullback property of the original diagram there are arrows $q_0, q_1 : F \to A$ such that $gq_0 = g', hq_0 = p_0h'$ and $gq_1 = g', hq_1 = p_1h'$:

$$\begin{tikzcd} F \arrow[rr, "h'"] \arrow[dr, shift left, "q_1"] \arrow[dr, shift right, "q_0"'] \arrow[ddr, "g'"'] & & E \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] \\ & A \arrow[r, "h"] \arrow[d, "g"] & B \arrow[d, "f"] \\ & C \arrow[r, "e"'] & D \end{tikzcd}$$

From $gq_0 = g' = gq_1$ and the assumption that g is mono, we get $q_0 = q_1$. Therefore $p_0h' = hq_0 = hq_1 = p_1h'$. Since h', being a pullback of the regular epi e, is regular epi (hence epi), we find $p_0 = p_1$. We conclude that f is mono.

b) Suppose in $A \xrightarrow{e_1} B \xrightarrow{e_2} C$ the arrows e_1, e_2 are both regular epi. In order to show that the composite e_2e_1 is regular epi, we factor this composite as me with m mono and e regular epi:

$$\begin{tikzcd} A \arrow[r, "e_1"] \arrow[dr, "e"'] & B \arrow[r, "e_2"] & C \\ & D \arrow[ur, "m"'] & \end{tikzcd}$$

If $E \overset{p_0}{\underset{p_1}{\rightrightarrows}} A$ is the kernel pair of e_1 then $mep_0 = e_2e_1p_0 = e_2e_1p_1 = mep_1$ so since m in mono, $ep_0 = ep_1$. Therefore, since e_1 is the coequalizer of p_0, p_1 we have a unique map $n : B \to D$ satisfying $ne_1 = e$. Then we also have: $mne_1 = me = e_2e_1$, so since e_1 is epi, $mn = e_2$ and the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "e_1"] \arrow[dr, "e"'] & B \arrow[r, "e_2"] \arrow[d, "n"] & C \\ & D \arrow[ur, "m"'] & \end{tikzcd}$$

Repeating the argument for the kernel pair q_0, q_1 of e_2, we get that $nq_0 = nq_1$; so since e_2 is the coequalizer of its kernel pair, we get a unique arrow $k : C \to D$ such that $ke_2 = n$.

Then $mke_2 = mn = e_2$ so since e_2 is epi, $mk = \mathrm{id}_C$; and $kme = ke_2e_1 = ne_1 = e$, so since e is epi, $km = \mathrm{id}_D$. We find that k is a two-sided inverse for m, which is therefore an isomorphism. We conclude that e_2e_1 is regular epi.

Solution to Exercise 3.

a) By the hint we have for every parallel pair $f, g : A \to B$, that $I(f) = I(g)$ if and only if $f\varepsilon_A = g\varepsilon_A$. From this it follows easily that I is faithful if and only if ε is epi.

Suppose I is full. Take $\alpha : A \to RIA$ such that $I(\alpha) = \eta_{IA} : IA \to IRIA$. Then both id_{RIA} and $\alpha\varepsilon_A$ are transposes of η_{IA}, so $\alpha\varepsilon_A = \mathrm{id}_{RIA}$ and ε is split monic.

Conversely, suppose ε_A is split monic, with retraction α. Any map $h : IA \to IB$ transposes to

$$RIA \overset{R(h)}{\to} RIB \overset{\varepsilon_B}{\to} B$$

which is equal to the composite

$$RIA \overset{\varepsilon_A}{\to} A \overset{\alpha}{\to} RIA \overset{R(h)}{\to} RIB \overset{\varepsilon_B}{\to} B$$

which is the transpose of $I(\varepsilon_B R(h)\alpha)$. Therefore $h = I(\varepsilon_B R(h)\alpha)$, and I is full.

b) Let $\mathcal{I}$ be an index category and $M : \mathcal{I} \to \mathcal{E}$ be a diagram. Suppose $\nu : \Delta_L \Rightarrow IM$ is a limiting cone for IM in $\mathcal{S}$, with vertex L. Then we have a cone $\Delta_{RL} \overset{\varepsilon\circ(R(\nu))}{\Rightarrow} M$ in $\mathcal{E}$, and therefore a cone $I(\varepsilon\circ(R(\nu))) : \Delta_{IRL} \Rightarrow IM$ in $\mathcal{S}$. Since ν is limiting we have a unique map of cones $d : IRL \to L$.

Moreover, for each object i of $\mathcal{I}$ we have, by naturality of η and the triangle identities, a commutative diagram

$$\begin{tikzcd}
IRL \arrow[r, "IR(\nu_i)"] & IRIM(i) \arrow[r, "I(\varepsilon)"] & IM(i) \\
L \arrow[u, "\eta"] \arrow[r, "\nu_i"'] & IM(i) \arrow[u, "\eta"] \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

which means that η is a map of cones from ν to $I(\varepsilon \circ (R(\nu)))$. Since ν is limiting, we have $d\eta = \mathrm{id}_L$.

Now consider $\eta d : IRL \to IRL$. Since I is full, this composition is of the form $I(e)$ for some $e : RL \to RL$. Let $\tilde{e} : L \to IRL$ be the transpose of

e. Then $\tilde{e} = I(e)\eta = \eta d\eta = \eta$, which is the transpose of id_{RL}. Therefore $e = \mathrm{id}_{RL}$ and η_L is an isomorphism with inverse d.

We also see that the cone ν is isomorphic to the cone $I(\varepsilon \circ R(\nu)) : \Delta_{IRL} \Rightarrow IM$, which is therefore limiting. It now follows readily from the full and faithfulness of I that the cone $\varepsilon \circ R(\nu) : RL \to M$ is limiting in $\mathcal{E}$.

c) Another proof of part b) is: prove that I is monadic and invoke the theorem (exercise 114) in the lecture notes that a monadic functor creates limits. So, let $h : IRX \to X$ be an IR-algebra. Then $h\eta_X = \mathrm{id}_X$ and just as in the last part of the proof given above, one proves that h is an isomorphism with inverse η.

Moreover, any object of the form IX has the structure of an IR-algebra: $IRIX \overset{I(\varepsilon)}{\to} IX$.

We see that the category IR-Alg is equivalent to the full subcategory of $\mathcal{S}$ on objects in the image of I. Since I is full and faithful, this subcategory is equivalent to $\mathcal{E}$ via I. So I is indeed monadic.

Solution to Exercise 4.

a) The first thing to recognize is that in $\mathcal{C}_\Omega$, every object (X, E_X) is the coproduct of the family $\{H(E_X(x)) \,|\, x \in X\}$. Therefore, if the functor F is to preserve coproducts and make the given diagram commute, there is no choice but to put

$$F(X, E_X) = \coprod \{y(E_X(x)) \,|\, x \in X\}$$

As a presheaf, $F(X, E_X)$ can be described like this: it is the P-indexed collection of sets $(A_p)_{p \in P}$ where

$$A_p \;=\; \{(x, p) \,|\, p \leq E_X(x)\}$$

and for $q \leq p$ the transition map $A_{qp} : A_p \to A_q$ sends (x, p) to (x, q). For a morphism $f : (X, E_X) \to (Y, E_Y)$ we have $E_X(x) \leq E_Y(f(x))$ so if the presheaf $F(Y, E_Y)$ is $(B_p)_{p \in \Omega}$, then $(x, p) \in A_p$ implies $(f(x), p) \in B_p$, so we have an arrow $F(f) : F(X, E_X) \to F(Y, E_Y)$ and this makes F a functor.

b) Here, we must know what regular epis look like in $\mathcal{C}_\Omega$. We have: $f : (X, E_X) \to (Y, E_Y)$ is regular epi if and only if f is a surjective function and moreover, for each $y \in Y$, $E_Y(y) = \bigvee \{E_X(x) \mid f(x) = y\}$.

Now suppose $B \subset \Omega$ and $\bigvee B \notin B$, so for all $b \in B$, $b < \bigvee B$. We consider the objects (B, id) and $H(\bigvee B)$ of $\mathcal{C}_\Omega$. The unique map $\pi : B \to \{*\}$ is a morphism from (B, id) to $H(\bigvee B)$ and it is regular epi (**for this, it has to be assumed that B is nonempty! This was a slight inaccuracy in the formulation of the exercise**).

However, the morphism $F(\pi)$ is not epi in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$, since $FH(\bigvee B) = y(\bigvee B)$ has an element at level $\bigvee B$, whereas $F(B, \mathrm{id})$ has no such element. Hence the component of $F(\pi)$ at $\bigvee B$ is not surjective.

c) Let $(A_p)_{p\in\Omega}$ be a presheaf on Ω, with maps $A_{qp} : A_p \to A_q$ for $q \le p$. Let $\perp$ denote the bottom element of Ω. Consider a morphism $f : (A_p)_{p\in\Omega} \to F(X, E_X)$. Suppose $\xi \in A_p$ and $\eta \in A_q$. By naturality of f, if $A_{\perp p}(\xi) = A_{\perp q}(\eta)$ and $f_p(\xi) = (x, p), f_q(\eta) = (y, q)$, then $x = y$. We see therefore, that f determines a function $\tilde{f} : A_\perp \to X$ with the property that for every element $\xi \in A_p$,

$$f_p(\xi) = (\tilde{f}(A_{\perp p}(\xi)), p)$$

Moreover, we must have for $\xi \in A_p$ that $p \le E_X(\tilde{f}(A_{\perp p}(\xi)))$. This gives us the idea to define L: define $L((A_p)_{p\in P})$ as $(A_\perp, E)$ where

$$E(\xi) \;=\; \bigvee\{p \in P \,|\, \text{for some } x \in A_p,\ A_{\perp p}(x) = \xi\}$$

We now see that the map $\tilde{f} : A_\perp \to X$ is a morphism $L((A_p)_{p\in P}) \to (X, E_X)$ in $\mathcal{C}_\Omega$. Coversely, given a map $g : L((A_p)_{p\in P}) \to (X, E_X)$ we have a map $\bar{g} : (A_p)_{p\in P} \to F(X, E_X)$ by putting

$$\bar{g}_p(\xi) \;=\; (g(A_{\perp p}(\xi)), p)$$

You can check yourself that $\bar{g}$ is well-defined and that the operations $\tilde{(\cdot)}$ and $\bar{(\cdot)}$ are each other's inverse. So, L is left adjoint to F.

d) For a concrete example we have to fix Ω. So let $\Omega = \{0 < 1\}$. Consider the presheaves A and B on Ω, where $A_1 = A_0 = \{*\}$, $B_0 = \{*\}$, $B_1 = \{a, b\}$ with $a \ne b$. We have two arrows, f_a and f_b, from A to B and their equalizer is the inclusion $E \subset A$ where $E_0 = \{*\}$, $E_1 = \emptyset$. Applying the functor L, we see that $L(A) = L(B) = H(1)$, and that $L(f_a) = L(f_b)$ is the identity map. So the equalizer of $L(f_a)$ and $L(f_b)$ is an isomorphism. However, $L(E) = H(0)$ and $L(E) \to L(A)$ is not an isomorphism. So L does not preserve equalizers.

Solution to Exercise 5.

a) Given $B \xrightarrow{f} A$, $C \xrightarrow{g} A$ in $\mathcal{C}$, let

$$\begin{tikzcd} X \arrow[r, "g'"] \arrow[d, "f'"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd}$$

a pullback diagram in Set. Then X is countable, so we may as well assume that $X \subseteq \mathbb{N}$. Because f, g are finite-to-one, so are f', g' and the diagram lives in $\mathcal{C}$; and it is a pullback in $\mathcal{C}$ because whenever we have arrows $Y \xrightarrow{a} A$, $Y \xrightarrow{b} B$ in $\mathcal{C}$ with $fa = gb$, then the unique factorization $Y \to X$ must be finite-to-one, and therefore in $\mathcal{C}$.

b) Certainly the maximal sieve is in $\mathrm{Cov}(A)$ since it contains the one-element family consisting of the identity on A.

For stability, suppose $R \in \mathrm{Cov}(A)$ and $g : B \to A$ is an arrow in $\mathcal{C}$. We have to prove that $g^*(R) \in \mathrm{Cov}(B)$. Let $\{f_1, \ldots, f_n\}$ a finite subfamily of R which is jointly almost surjective. It is enough to show that the sieve on B generated by $\{f_1', \ldots, f_n'\}$ is in $\mathrm{Cov}(B)$, where each f_i' is such that

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "f_i'"'] & \cdot \arrow[d, "f_i"] \\ B \arrow[r, "g"'] & A \end{tikzcd}$$

is a pullback. This is because this sieve is a subsieve of $f^*(R)$. Now the set

$$A - \bigcup_{i=1}^{n} \mathrm{Im}(f_i)$$

is a finite set, call it E. Since g is an arrow in $\mathcal{C}$, hence a finite-to-one function, its preimage under g, $g^{-1}(E)$, is finite. Hence we have that

$$B - \bigcup_{i=1}^{n} \mathrm{Im}(f_i')$$

is also finite, which shows that the sieve generated by $\{f_1', \ldots, f_n'\}$ is in $\mathrm{Cov}(B)$, as desired.

For local character, suppose R, S are sieves on A, $R \in \mathrm{Cov}(A)$ and for every $f : D \to A$ in R we have $f^*(S) \in \mathrm{Cov}(D)$. We have to prove that $S \in \mathrm{Cov}(A)$. Now if R contains the jointly almost surjective family $\{f_1, \ldots, f_n\}$ and for every i the sieve $f_i^*(S)$ contains the jointly almost surjective family $\{g_1^i, \ldots, g_{k_i}^i\}$, then the family

$$\{f_i g_j^i \,|\, 1 \le i \le n, 1 \le j \le k_i\}$$

is a jointly almost surjective family of arrows into A, and this family is contained in S. So $S \in \mathrm{Cov}(A)$, as desired.

c) Suppose $\{f_1, \ldots, f_n\} \subset R$ is jointly almost surjective. For each i let $e_i : \mathrm{Im}(f_i) \to \mathrm{dom}(f_i)$ be a section of f_i. Then R contains the family $\{f_1 e_1, \ldots, f_n e_n\}$ since R is a sieve. Moreover, every composition $f_i e_i$ is injective; and the joint image of the maps $f_i e_i$ is the same as the joint image of the maps f_i.

d) Again, we need the set X to be nonempty. For, if $A \subset \mathbb{N}$ is finite and nonempty, then $\emptyset \in \mathrm{Cov}(A)$ because the empty family is jointly almost surjective. However, if $X = \emptyset$ then there are no equivalence classes of functions $A \to X$.

Provided X is nonempty we define $F_X(A)$ as given. For an arrow $f : B \to A$ and $[\xi] \in F_X(A)$ we put: $[\xi]f = [\xi \circ f]$. This is well-defined, for if $\xi \sim \eta$ in $F_X(A)$ then $\xi \circ f \sim \eta \circ f$ in $F_X(B)$. Clearly, we have a prasheaf structure on F_X.

e) Suppose $\xi, \eta : A \to X$ are two functions such that for all $f : B \to A$ in some $R \in \mathrm{Cov}(A)$ we have $[\xi]f = [\eta]f$ in $F_X(B)$. Then in particular this holds for a finite, jointly almost surjective subfamily $\{f_1, \ldots, f_n\}$ of R. So for each i, the compositions $\xi \circ f_i$ and $\eta \circ f_i$ agree on all but finitely elements of their domain. Since the family is finite, ξ and η agree on all but finitely elements of A. So F_X is separated.

Now suppose we have a compatible family

$$\{[\xi_f] \in F_X(\mathrm{dom}(f)) \mid f \in R\}$$

indexed by some $R \in \mathrm{Cov}(A)$. We must produce an amalgamation. Now R contains a finite, jointly almost surjective subfamily $\{f_1, \ldots, f_n\}$ consisting of injective functions. Let A_i be the image of f_i. Clearly we have a unique function $\eta_i : A_i \to X$ such that $\eta_i \circ f_i = \xi_{f_i}$. For different indices i and j, there can be at most finitely many elements $x \in A_i \cap A_j$ for which $\eta_i(x) \neq \eta_j(x)$, by the compatibility of the family. So in the whole of A there are at most finitely many x such that either $x \notin \bigcup_{i=1}^n \mathrm{Im}(f_i)$, or for some $i \neq j$, $x \in A_i \cap A_j$ and $\eta_i(x) \neq \eta_j(x)$. Let the finite set of such x's be E. Then define $\eta : A \to X$ by: $\eta(x) = \eta_i(x)$, if $x \notin E$ and $x \in A_i$ (it doesn't matter which i we choose), and let $\eta(x)$ be an arbitrary element of X if $x \in E$. Then $[\eta]$ is an amalgamation for the family $\{[\xi_{f_i}] \mid 1 \leq i \leq n\}$ and hence, by compatibility, for the original family we started with.

Solution to Exercise 6.
a)⇒b): suppose A an object of $\mathcal{C}$, $x, y \in F(A)$ such that for all $\phi : I \to A$ we have $x\phi = y\phi$. We have to prove that $x = y$, but by assumption a) it is sufficient to prove that $A \Vdash_J \neg\neg(x = y)$, which, after some elementary logical operations, is equivalent to:

(*) For every arrow $B \xrightarrow{f} A$, if $\emptyset \notin \mathrm{Cov}(B)$ then there is an arrow $C \xrightarrow{g} B$ such that $\emptyset \notin \mathrm{Cov}(C)$ and $xfg = yfg$.

But given such $f : B \to A$ with $\emptyset \notin \mathrm{Cov}(B)$, we have some $g : I \to B$ by our assumptions on the site $(\mathcal{C}, \mathrm{Cov})$. By hypothesis on A and x, y, we have $xfg = yfg$. So we have proved (*).
b)⇒a): Suppose $A \Vdash_J \neg\neg(x = y)$ (which is equivalent to (*) above, as we saw), and let $f : I \to A$ be an arrow. By (*) there is an arrow $C \xrightarrow{g} I$ such that $xfg = yfg$ and $\emptyset \notin \mathrm{Cov}(C)$. This last fact gives us some map $I \to C$, so we know that $g : C \to I$ is split epi; let $h : I \to C$ be a retraction. Then $xfg = yfg$, hence

$$xf = xfgh = yfgh = yf$$

The map $f : I \to A$ was arbitrary, so we conclude that the hypothesis of part b) is satisfied. Hence $x = y$. Because also A wa sarbitrary, we conclude that F is $\neg\neg$-separated, as was to be shown.