Category Theory and Topos Theory, Spring 2016 Hand-In Exercises

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1 Exercises

Exercise 1 (To be handed in February 22) The category Rel of *relations* has as objects sets, and an arrow $X \to Y$ is a relation from X to Y, i.e. a subset of $X \times Y$. Composition of arrows is composition of relations: given $R: X \to Y$ and $S: Y \to Z$, the composite $SR: X \to Z$ is the set

 $SR = \{(x, z) \mid \text{for some } y \in Y, (x, y) \in R \text{ and } (y, z) \in S\}$

- a) Prove that Rel has both a terminal and an initial object, and give these explicitly.
- b) Characterize the monomorphisms in Rel. Show in particular that every mono $R: X \to Y$ in Rel is a *total relation*: for all $x \in X$ there is a $y \in Y$ with $(x, y) \in R$.
- c) Characterize the epimorphisms in Rel (Hint: you may find it useful to do part d) first).
- d) Give a concrete description (up to equivalence of categories) of Rel^{op}.

Exercise 2 (To be handed in March 7) The category Set_p has as objects sets and as arrows *partial functions*: an arrow $X \to Y$ is a function f from a subset D_f of X to Y. Composition in Set_p is given as follows: for $X \xrightarrow{f} Y \xrightarrow{g} Z$ we have

$$D_{qf} = \{x \in X \mid x \in D_f \text{ and } f(x) \in D_q\}$$

and gf(x) = g(f(x)) for $x \in D_{gf}$.

- a) Prove that Set_p has finite products.
- b) Prove that Set_p has equalizers.
- c) Prove that Set_p has coequalizers of kernel pairs.

- d) Prove that Set_p is regular.
- **Exercise 3 (To be handed in March 21)** a) A meet-semilattice is a poset A which has finite limits. Concretely, it has a top element \top and for $x, y \in A$, the meet (or glb) $x \wedge y$ exists. A morphism of meet-semilattices $A \rightarrow B$ is a function which preserves \top and commutes with \wedge . We have a category **MSL** of meet-semilattices and their morphisms. Show that the forgetful functor **MSL** \rightarrow Set has a left adjoint.
- b) A sup-lattice is a poset which has least upper bounds of arbitrary subsets; a suplattice homomorphism preserves least upper bounds. We have a category **SL** of sup-lattices; prove that the forgetful functor **SL** \rightarrow Set has a left adjoint.
- c) The category **Frm** of frames (see section 4.5 of the lecture notes) has as objects frames and as morphisms functions which preserve arbitrary joins, top element and binary meets. Show that the forgetful functor **Frm** \rightarrow Set has a left adjoint.

Exercise 4 (To be handed in April 4) Let C be a category, $S : C \to C$ a functor and (T, μ, η) a monad on C. A *distributive law* for T over S is a natural transformation $c : TS \Rightarrow ST$ such that the following diagrams commute for each object x of C:

$$\begin{array}{cccc} TSx & T^2Sx \xrightarrow{T(c_x)} TSTx \xrightarrow{c_{Tx}} ST^2x \\ \downarrow c_x & \mu_{Sx} \downarrow & \downarrow S(\mu_x) \\ Sx \xrightarrow{S(\eta_x)} STx & TSx \xrightarrow{c_x} STx \end{array}$$

a) Prove that a distributive law for T over S gives rise to a functor T-S: $T-Alg \rightarrow T-Alg$ for which the following diagram commutes:

$$\begin{array}{c} T-\operatorname{Alg} \xrightarrow{T-S} T-\operatorname{Alg} \\ \downarrow U^T \\ \mathcal{C} \xrightarrow{\qquad S} \mathcal{C} \end{array}$$

b) Prove that for the functor T-S constructed in a), the law c induces a natural transformation

$$F^T S \Rightarrow (T-S)F^T$$

c) Conclude that in the case that c is a natural isomorphism, the functor T-S restricts to an endofunctor on the Kleisli category of T.

Exercise 5 (To be handed in April 18) We compare the categories $\operatorname{Set}^{\mathcal{C}^{\operatorname{op}}}$ and $\operatorname{Set}^{\mathcal{C}_0}$: the latter is just the category with as objects \mathcal{C}_0 -indexed families of sets $(X_C)_{C \in \mathcal{C}_0}$ and as arrows \mathcal{C}_0 -indexed families of functions. There is an obvious forgetful functor $U: \operatorname{Set}^{\mathcal{C}^{\operatorname{op}}} \to \operatorname{Set}^{\mathcal{C}_0}$.

- a) Prove that the functor U is monadic (that is, U has a left adjoint L such that Set^{C^{op}} is equivalent to the category of UL-algebras).
- b) Conclude from a) that "limits are calculated point-wise" in $\operatorname{Set}^{\mathcal{C}^{\operatorname{op}}}$
- c) Prove that U also has a *right adjoint R*. For 1 extra bonus-point, you may try your hand at the question: is U also *comonadic*?

Exercise 6 (To be handed in/sent in May 11) Let X be a topological space with set of opens $\mathcal{O}(X)$. We define, for every open U of X, a set $\operatorname{Cov}(U)$ of "covering families" for U as follows: a (downwards closed) family \mathcal{W} of open subsets of U is in $\operatorname{Cov}(U)$ if and only if there is a *finite* subset $\{W_1, \ldots, W_n\}$ of \mathcal{W} such that $\bigcup_{i=1}^n W_i = U$. We allow n = 0: so $\emptyset \in \operatorname{Cov}(\emptyset)$.

- a) Prove that Cov is a Grothendieck topology on $\mathcal{O}(X)$. We write $\operatorname{Sh}(\mathcal{O}(X), \operatorname{Cov})$ for the category of sheaves for this topology.
- b) Let Shv(X) be the category of 'normal' sheaves on X, i.e. with respect to the topology where W covers U iff $\bigcup W = U$.

Show that $\operatorname{Shv}(X) \subset \operatorname{Sh}(\mathcal{O}(X), \operatorname{Cov})$ and that the inclusion is, in general, strict: give an example of a space X and a presheaf F on X which is a sheaf for Cov but not a normal sheaf on X.

c) Show that the inclusion $\text{Shv}(X) \to \text{Sh}(\mathcal{O}(X), \text{Cov})$ has a left adjoint which preserves finite limits.

2 Solutions

Solution to Exercise 1

a) (2 points) The empty set is initial in Rel since for any set X there is exactly one relation from \emptyset to X: the empty relation. By the same token, \emptyset is also terminal in Rel.

b) (3 points) Relations from X to Y correspond bijectively to union-preserving functions $\mathcal{P}(X) \to \mathcal{P}(Y)$ (where $\mathcal{P}(X)$ denotes the power set of X): $R \subseteq X \times Y$ corresponds to the function \hat{R} defined by

$$\hat{R}(A) = \{ y \in Y \mid \exists x \in A(x, y) \in R \}$$

Note that if $R \subseteq X \times Y$ corresponds to \hat{R} and $S \subseteq Y \times Z$ to \hat{S} , then SR corresponds to the composite $\hat{S}\hat{R}$. Therefore, R is mono if and only if for any set W and $S, T \subseteq W \times X$, $\hat{R}\hat{S} = \hat{R}\hat{T}$ implies S = T. If we take $Z = \{*\}$, we see that this means that \hat{R} is injective.

The following reformulation was worth 1 bonus point: R is mono if and only if there is a function $f: X \to Y$ with the property that for all $x \in X$ the following hold:

i) $(x, f(x)) \in R$

ii) For no $x' \neq x$ is $(x', f(x)) \in R$

From this last condition it is obvious that if R is mono, R is a total relation. Following the hint in the exercise, we first solve part d) before we do c).

d) (3 points) For every relation R from X to Y, we have the relation $R^{\circ} = \{(y,x) \mid (x,y) \in R\}$, which is a relation from Y to X, so a morphism $X \to Y$ in Rel^{op}. Moreover, for relations $R: X \to Y$ and $S: Y \to Z$, we have $(SR)^{\circ} = R^{\circ}S^{\circ}$ and of course, for the identity relation I on X we have $I^{\circ} = I$. This means that we have a functor $F: \text{Rel} \to \text{Rel}^{\text{op}}$. Then F^{op} is a functor $\text{Rel}^{\text{op}} \to \text{Rel}$, and it is trivial to see that the two functors are each othe's inverse. So Rel^{op} is isomorphic, and hence equivalent, to Rel.

c) (2 points) $R: X \to Y$ is epi in Rel if and only if R° is mono in Rel (by d)). In order to characterize this in terms of R, we use the condition given in the solution of part b): R is epi if and only if there is a function $f: Y \to X$ such that for every $y \in Y$ the following hold:

- i) $(f(y), y) \in R$
- ii) For no $y' \neq y$ is $(f(y), y') \in R$.

Solution to Exercise 2 There is a straightforward solution and a slick one; let us do the straightforward solution first.

a) (3 points) The empty set is terminal in Set_p since for any set X the unique function f with $D_f = \emptyset$, is the unique partial function $X \to \emptyset$.

For sets X and Y, consider the disjoint sum $X * Y = (X \times Y) + X + Y$ together with projections $p_X : X * Y \to X$ and $p_Y : X * Y \to Y$ with $D_{p_X} = (X \times Y) + X$, $D_{p_Y} = (X \times Y) + Y$. Given two partial functions $Z \xrightarrow{f} X, Z \xrightarrow{g} Y$ we have a partial function $\langle f, g \rangle : Z \to X * Y$ defined by

$$\langle f,g\rangle(z) = \begin{cases} (f(z),g(z)) \in X \times Y & \text{if } z \in D_f \cap D_g \\ f(z) \in X & \text{if } z \in D_f - D_g \\ g(z) \in Y & \text{if } z \in D_g - D_f \end{cases}$$

So $D_{\langle f,g \rangle} = D_f \cup D_g$. One can check that $\langle f,g \rangle$ is the unique partial function satisfying $p_X \langle f,g \rangle = f$ and $p_Y \langle f,g \rangle = g$. So X * Y, together with the maps p_X and p_Y is a product cone.

b) (2 points) Given a parallel pair of partial functions $X \xrightarrow{f} Y$, a partial function $h: Z \to X$ satisfies fh = gh if and only if for each $z \in D_h$ the following holds: either $h(z) \in D_f \cap D_g$ and f(h(z)) = g(h(z)), or $h(z) \notin D_f \cup D_g$. Hence, such h factors through the inclusion $E \subset X$ where

$$E = \{x \in D_f \cap D_g \,|\, f(x) = g(x)\} \cup (X - (D_f \cup D_g))$$

and therefore this inclusion is the equalizer of f and g.

c) (3 points) First we do general pullbacks, constructed out of products and equalizers. For a diagram



the vertex of the limiting cone is

$$P = \{(w,z) \in D_h \times D_g \,|\, h(w) = g(z)\} \cup (W - D_h) \times (Z - D_g) + (W - D_h) + (Z - D_g)$$

Hence, given $X \xrightarrow{f} Y$, the kernel pair of f has as vertex

$$K_f = \{(x, y) \in D_f \times D_f \mid f(x) = f(y)\} \cup (X - D_f) \times (X - D_f) + (X - D_f)$$

Now, if $g: X \to Z$ coequalizes the two projections $p_1, p_2: K_f \to X$ then for $(x, y) \in D_f \times D_f$, if f(x) = f(y) then g(x) = g(y). Moreover, if $x \notin D_f$ then g(x) is undefined (because there is $\bar{x} \in K_f$ such that $x = p_1(\bar{x})$ and $p_2(\bar{x})$ is undefined, or vice versa). We conclude that g factors through the image of f; uniquely since $f: D_f \to \text{Im}(f)$ is surjective.

d) (2 points) We see that a partial map $f: X \to Y$ is a regular epi in Set_p if and only if the function $f: D_f \to Y$ is surjective.

If



is a pullback diagram as in the solution of c), then it is easy to see that if h is surjective, so is p_Z : if $z \in D_g$, pick w such that (h(w) = g(z)); then $z = p_Z((w, z))$. If $z \notin D_g$, then $z \in (Z - D_g)$, hence z is also in the image of p_Z . We conclude that regular epis are stable under pullback. Hence Set_p is regular.

Slick solution: prove that Set_p is equivalent to the category Set_* of *pointed* sets: an object of Set_* is a pair (X, x) with X a set and $x \in X$; an arrow $(X, x) \to (Y, y)$ is a function $f: X \to Y$ satisfying f(x) = y.

Let $F : \operatorname{Set}_p \to \operatorname{Set}_*$ the functor which sends X to $(X \cup \{X\}, X)$ (mind you, set theory teaches us that $X \notin X$ always!), and a partial function $f : X \to Y$ to the function

$$F(f)(x) = \begin{cases} f(x) & \text{if } x \in D_f \\ Y & \text{if } x \in X - D_f \\ Y & \text{if } x = X \end{cases}$$

In the converse direction we have $G : \operatorname{Set}_* \to \operatorname{Set}_p$ which sends (X, x) to $X - \{x\}$; for $f : (X, x) \to (Y, y)$ we have $G(f) : (X - \{x\}) \to (Y - \{y\})$ with $D_{G(f)} = \{x' \in X - \{x\} \mid f(x) \neq y\}$. I leave the proof that this is an equivalence to you, as well as the (pretty easy) proof that Set_* is regular.

Solution to exercise 3.

a) (3 points) A meet-semilattice (A, \top, \wedge) can be seen as a monoid with the properties:

- i) the monoid operation \wedge is commutative
- ii) the operation is idempotent: $a \wedge a = a$

This means that given a set X and a function $f: X \to A$ where (A, \top, \wedge) is a meet-semilattice, its mate (under the adjunction $\operatorname{Mon} \underbrace{\stackrel{(\cdot)^*}{\longrightarrow}}_{U}$ Set) $\tilde{f}: X^* \to A$ will send two sequences of elements of X to the same element of A if these sequences enumerate the same finite subset of X. We see that the left adjoint to $U: \operatorname{\mathbf{MSL}} \to \operatorname{Set}$ sends a set X to the set $\mathcal{P}_{\operatorname{fin}}(X)$ of finite subsets of X, ordered by *reverse* inclusion. The meet operation is *union*, the top element is the empty set. For a function $f: X \to A, \ \tilde{f}: \mathcal{P}_{\operatorname{fin}}(X) \to A$ sends U to $f(x_1) \wedge \cdots \wedge f(x_n)$ if $U = \{x_1, \ldots, x_n\}; \ \tilde{f}(\emptyset) = \top$.

b) (3 points) The free sup-lattice on a set X is the powerset $\mathcal{P}(X)$ of X. Given a function $f: X \to U$ where U is the underlying set of a sup-lattice, its mate is the sup-lattice homomorphism $\tilde{f}: \mathcal{P}(X) \to U$ defined by

$$\tilde{f}(A) = \bigvee \{ f(a) \, | \, a \in A \}$$

c) (4 points) Every frame is a meet-semilattice (and every frame homomorphism is a meet-semilattice morphism), so there is an inclusion functor $\mathbf{Frm} \to \mathbf{MSL}$ and the forgetful functor $\mathbf{Frm} \to \mathbf{Set}$ factors through this inclusion. Since adjunctions compose, it is sufficient to calculate the left adjoint to the inclusion $\mathbf{Frm} \to \mathbf{MSL}$.

For any meet-semilattice A we have a frame $\mathcal{D}A$ consisting of all downwards closed subsets of A:

$$\mathcal{D}A = \{ U \subseteq A \mid \forall x, y \in A (x \in U, y \le x \Rightarrow y \in U) \}$$

There is a meet-semilattice morphism $A \xrightarrow{\eta} \mathcal{D}A$ sending $a \in A$ to $\downarrow a = \{x \in A \mid x \leq a\}$. By definition of meets, we have $\downarrow (a \land b) = \downarrow a \cap \downarrow b$ (and meets in $\mathcal{D}A$ are given by intersection). And $\downarrow \top = A$, which is the top element of $\mathcal{D}A$. Moreover, for any **MSL**-morphism $f : A \to X$ where X is a frame, there is a unique factorization



by the frame homomorphism $\tilde{f} : \mathcal{D}A \to X$ which sends a downwards closed subset U of A to $\bigvee \{f(a) \mid a \in U\}$. So we have a functor $\mathcal{D} : \mathbf{MSL} \to \mathbf{Frm}$ which is left adjoint to the inclusion functor. Calculating the composition of the two left adjoints Set \rightarrow **MSL** \rightarrow **Frm**, we find the functor F : Set \rightarrow **Frm** which sends a set X to the collection of upwards closed sets of finite subsets of X, that is: $U \in F(X)$ if and only if $U \subseteq \mathcal{P}_{\text{fin}}(X)$ and for $A, B \in \mathcal{P}_{\text{fin}}(X)$, if $A \in U$ and $A \subseteq B$, then $B \in U$. And F(U) is ordered by inclusion.

(1 bonuspoint) The functor F can also be described as follows. Given a set X, the *Scott topology* on $\mathcal{P}(X)$ has as opens those subsets U of $\mathcal{P}(X)$ which satisfy:

- i) whenever $A \subseteq B \subseteq X$ and $A \in U$, also $B \in U$
- ii) whenever $A \in U$, there is a finite subset A' of A which also belongs to U.

Then F(X) is isomorphic to the frame of open subsets of $\mathcal{P}(X)$ for the Scott topology.

Solution to Exercise 4.

a) (5 points) Define (T-S) on objects by: $(T-S)(Tx \xrightarrow{h} x$ is the composition of

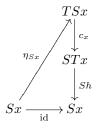
$$TSx \xrightarrow{c_x} STx \xrightarrow{S(h)} Sx$$

and on maps $f:(Tx \xrightarrow{h} x) \to (Ty \xrightarrow{k} y)$ by

$$\begin{array}{ccc} TSx \xrightarrow{TSf} TSy \\ c_x & \downarrow c_y \\ STx \xrightarrow{STf} STy \\ sh & \downarrow sk \\ Sx \xrightarrow{Sf} Sy \end{array}$$

It is clear that the diagram defining (T-S)(f) commutes, that the definition is functorial, and that the diagram given in the exercise commutes; so it only remains to check that $(T-S)(Tx \xrightarrow{h} x)$ is a T-algebra when h is.

Now the unit diagram



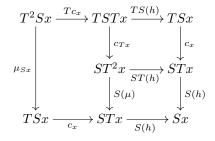
commutes since

$$S(h)c_x\eta_{Sx} = S(h)S(\eta_x)$$

= $S(h\eta_x)$
= $S(\mathrm{id}) = \mathrm{id}$

The first equality by naturality of c, and the second by the assumption that h is an algebra.

The associativity diagram

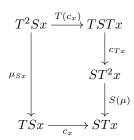


commutes: the left-hand square because of the second diagram defining a distributive law in a); the right-hand upper square by naturality of c, and the lower right-hand square because h is an algebra.

b) (3 points) We have that $(T-S)(F^T(x))$ is the composition

$$TSTx \xrightarrow{c_{Tx}} ST^2x \xrightarrow{S(\mu)} STx$$

and the second diagram defining a distributive law just says that there is an algebra map



from $F^T(Sx)$ to $(T-S)(F^Tx)$, which is obviously natural.

c) (2 points) The exercise was sloppily formulated. What one can conclude from b) is that, when we apply the functor T-S to a free algebra, we get something that is isomorphic to a free algebra. Therefore, the functor T-S restricts to the full subcategory of T – Alg on those algebras that are isomorphic to a free algebra. This category is *equivalent* to the Kleisli category for T, but not (necessarily) isomorphic to it. If you had spotted this glitch and got stuck, you received at least 1 point.

Solution to Exercise 5.

a) (4 points) We denote a typical object of $\operatorname{Set}^{\mathcal{C}_0}$ by $X = (X_C)_C$ ($C \in \mathcal{C}_0$), and typical presheaves on \mathcal{C} by F, G.

So, $U(F) = (F(C))_C$.

For $X = (X_C)_C$, let L(X) be defined by

$$L(X)(C) = \{ (f,\xi) \mid \text{dom}(f) = C, \xi \in X_{\text{cod}(f)} \}$$

Then L(X) is a presheaf: for $g: C' \to C$ and $(f,\xi) \in L(X)(C)$, we let

$$L(X)(g)(f,\xi) = (fg,\xi)$$

Moreover, the assignment $X \mapsto L(X)$ is functorial: given an arrow $\alpha = (\alpha_C : X_C \to Y_C)$ from X to Y in Set^{\mathcal{C}_0}, we have $L(\alpha) : L(X) \to L(Y)$ defined by

$$L(\alpha)_C(f,\xi) = (f, \alpha_{\operatorname{cod}(f)}(\xi))$$

This defines L as a functor.

There is a natural transformation $\varepsilon : LU \Rightarrow \text{id: given a presheaf } F$, the component $\varepsilon_C^F : LUF(C) \rightarrow F(C)$ sends (f,ξ) to $F(f)(\xi)$. This is a natural transformation: for $\alpha : F \rightarrow G$ we have

$$\begin{aligned} \varepsilon_C^G(LU(\alpha)_C(f,\xi)) &= & \varepsilon_C^G(f,\alpha_{\operatorname{cod}(f)}(\xi)) &= \\ G(f)(\alpha_{\operatorname{cod}(f)}(\xi)) &= & \alpha_C(F(f)(\xi)) &= \\ & \alpha_C(\varepsilon_C^F(f,\xi)) \end{aligned}$$

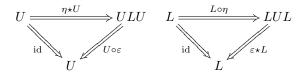
so the diagram

commutes.

There is a natural transformation η : id \Rightarrow UL: given X, we have $\eta_C : X_C \rightarrow UL(X)_C$ defined by

$$\eta_C(\xi) = (\mathrm{id}_C, \xi) \in LU(X)_C$$

The naturality of η is obvious. We check the triangle equalities:



For a presheaf F and $\xi \in (UF)_C = F(C)$, we have $(\eta \star U)_C(\xi) = (\mathrm{id}_C, \xi)$ and $(U \circ \varepsilon)_C(\mathrm{id}_C, \xi) = F(\mathrm{id}_C)(\xi) = \xi$, so the left-hand triangle commutes.

For X in Set^{\mathcal{C}_0} and $(f,\xi) \in L(X)(C)$, $(L \star \eta)(f,\xi) = (\mathrm{id}_C, (f,\xi))$ and

$$(\varepsilon \star L)_C(\mathrm{id}_C, (f, \xi)) = L(X)(\mathrm{id}_C)(f, \xi) = (f\mathrm{id}_C, \xi) = (f, \xi)$$

so also the right-hand triangle commutes. We conclude that indeed, L is left adjoint to U.

Consider the monad T = UL on $\operatorname{Set}^{\mathcal{C}^{\operatorname{op}}}$: $T(X)_C = \{(f,\xi) \mid \operatorname{dom}(f) = C, \xi \in X_{\operatorname{cod}(f)}\}$, so

$$T^{2}(X)_{C} = \{(g, (f, \xi)) | \operatorname{dom}(g) = C, \operatorname{cod}(g) = \operatorname{dom}(f), \xi \in X_{\operatorname{cod}(f)}\}$$

The multiplication $\mu: T^2 \Rightarrow T$ is given by $(g, (f, \xi)) = (fg, \xi)$, and the unit η is the unit of the adjunction $L \dashv U$.

Now suppose X is a T-algebra: we have $h: T(X) \to X$ such that the diagrams

$$\begin{array}{cccc} T^{2}X \xrightarrow{\mu} TX & TX \\ Th & & & \\ TX \xrightarrow{h} X & X & \\ \end{array} \xrightarrow{\eta} & & \\ TX \xrightarrow{h} X & X \xrightarrow{\eta} & \\ \end{array}$$

commute.

These diagrams mean that $h_C(f,\xi) \in X_C$ for all C, and for $C \xrightarrow{g} C' \xrightarrow{f} D$ and $\xi \in X_D$ we have the equalities

$$h_C(g, h_{C'}(f, \xi)) = h_C(fg, \xi)h_C(\mathrm{id}_C, \xi) = \xi$$

These equations just mean that X has a presheaf structure if, for $\xi \in X_D$ and $f: C \to D$ we put $X(f)(\xi) = h_C(f,\xi)$. Conversely, every structure of a presheaf on X defines a T-algebra structure on X. We see that $\operatorname{Set}^{\mathcal{C}^{\operatorname{op}}}$ is equivalent to the category T-Alg, and that U is monadic.

b) (2 points) In $\operatorname{Set}^{\mathcal{C}_0}$, limits are pointwise (a diagram in $\operatorname{Set}^{\mathcal{C}_0}$ is just a \mathcal{C}_0 -indexed family of diagrams in Set).

Since U is monadic, it creates limits: for any diagram $P: I \to \operatorname{Set}^{\mathcal{C}_0}$, i.e. family of diagrams $(P_C: I \to \operatorname{Set})_{C \in \mathcal{C}_0}$ with limits Λ_C , there is a unique presheaf structure on Λ_C , and this is the vertex of a limiting cone for P. So, limits are calculated pointwise.

c) (4 points) Given an object X of $\text{Set}^{\mathcal{C}_0}$, define R(X) as follows: R(X)(C) consists of all families

$$(\xi_f)_{\operatorname{cod}(f)=C}$$

for which $\xi_f \in X_{\operatorname{dom}(f)}$, for all f with codomain C.

Then R(X) is a presheaf if we define, for $g: C' \to C$,

$$R(X)(g)((\xi_f)_{\operatorname{cod}(f)=C}) = (\xi_{gh})_{\operatorname{cod}(h)=C'}$$

For $\alpha: X \to Y$ in Set^{\mathcal{C}_0} we have $R(\alpha): R(X) \to R(Y)$ given by

$$R(\alpha)_C((\xi_f)_{\operatorname{cod}(f)=C}) = (\alpha_{\operatorname{dom}(f)}(\xi_f))_{\operatorname{cod}(f)=C}$$

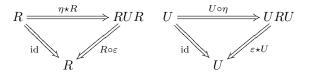
This defines R as a functor: $\operatorname{Set}^{\mathcal{C}_0} \to \operatorname{Set}^{\mathcal{C}^{\operatorname{op}}}$.

We have a natural transformation $\eta : id \Rightarrow RU$: for a presheaf F, define η_C on F(C) by

$$\eta_C(\xi) = (F(f)(\xi))_{\operatorname{cod}(f)=C}$$

Also, we have a natural transformation $\varepsilon : UR \Rightarrow \text{id}$: for an object X of $\text{Set}^{\mathcal{C}_0}$, ε_C sends the family $(\xi_f)_{\text{cod}(f)=C}$ to $\xi_{\text{id}_C} \in X_C$.

Again, we check the triangle equalities



For the left-hand triangle, if $\xi = (\xi_f)_{\operatorname{cod}(f)=C}$ is an element of R(X)(C) then $(\eta \star R)_C(\xi)$ is the family of families $(A_f)_{\operatorname{cod}(f)=C}$, where, for $f: C' \to C$,

$$A_f = (\xi_{fg})_{\operatorname{cod}(g) = C'}$$

And

$$(R \circ \varepsilon)_C ((A_f)_{\operatorname{cod}(f)=C}) = (\varepsilon_{C'}(A_f))_{f:C' \to C} = (\xi_f)_{f:C' \to C} = \xi$$

so the left-hand triangle commutes. Virtually the same calculation shows that the right-hand triangle commutes; we conclude that R is right adjoint to U.

For the last statement (1 bonus point), we look at the comonad $\bot = UR$ on $\operatorname{Set}^{\mathcal{C}_0}$. We have the *counit* $\varepsilon : \bot \Rightarrow \operatorname{id}$, which is the counit of the adjunction $U \dashv R$, and a *comultiplication* $\delta : \bot \Rightarrow \bot^2$, which is $U \circ (\eta \star R)$.

A \bot -coalgebra in Set^{C_0} is an object X together with a morphism $h: X \to \bot X$, making the diagrams

$$\begin{array}{cccc} X & \stackrel{h}{\longrightarrow} & \mathbb{L}X & X \\ h & & \downarrow \mathbb{L}h & h \\ \mathbb{L}X & \stackrel{id}{\longrightarrow} & \mathbb{L}^{2}X & \mathbb{L}X & \stackrel{id}{\longrightarrow} X \end{array}$$

commute.

So we have a map h such that in particular, for $\xi \in X_C$, $h_C(\xi)$ is a family $(\zeta_f)_{f:C'\to C}$ such that (by the right-hand triangle) $\zeta_{id} = \xi$.

For $f : C' \to C$ let us denote $\zeta_f = (h_C(\xi))_f$ by $X(f)(\xi)$. By the above paragraph, $X(\mathrm{id}_C)(\xi) = \xi$. So $h_C(\xi) = (X(f)(\xi))_{\mathrm{cod}(f)=C}$ whence

$$(\bot h)_C(h_C(\xi)) = ((X(g)(X(f)(\xi)))_{g:C'' \to C'})_{f:C' \to C}$$

and

$$\delta_C(h_C(\xi)) = (X(fg)(\xi)_{g:C'' \to C'})_{f:C' \to C}$$

From this we see that the requirement that the left-hand diagram commute, just means that

$$X(fg)(\xi) \quad X(g)(X(f)(\xi))$$

always; that is, X is a presheaf. We conclude that U is in fact comonadic (and, hence, creates colimits; which are, thus, calculated pointwise in $\operatorname{Set}^{\mathcal{C}^{\operatorname{op}}}$).