Exercise 21 Let $K : \mathbb{N} \to \mathbb{N}$, $G : \mathbb{N}^{k+1} \to \mathbb{N}$ and $H : \mathbb{N}^{k+3} \to \mathbb{N}$ be functions. Define $F$ by:

$$F(0, \vec{y}, x) = G(\vec{y}, x)$$

$$F(z + 1, \vec{y}, x) = H(z, F(z, \vec{y}, K(x)), \vec{y}, x)$$

Suppose that $G$, $H$ and $K$ are primitive recursive.

a) Prove directly, using the pairing function $j$ and suitably adapting the proof of proposition 2.1.9: if $\forall x(K(x) \leq x)$, then $F$ is primitive recursive.

b) Define a new function $F'$ by:

$$F'(0, m, \vec{y}, x) = G(\vec{y}, K^m(x))$$

$$F'(n + 1, m, \vec{y}, x) = H(n, F'(n, m, \vec{y}, x), \vec{y}, K^m(n + 1))$$

Recall that $K^m(n + 1)$ means: the function $K$ applied $m$ times.

Prove: if $n \leq m$ then $\forall k[F'(n, m + k, \vec{y}, x) = F'(n, m, \vec{y}, K^k(x))]$

c) Prove by induction: $F(z, \vec{y}, x) = F'(z, \vec{y}, x)$ and conclude that $F$ is primitive recursive, also without the assumption that $K(x) \leq x$.

Solution: There is more than one way to solve a), which was the most challenging part of the exercise. Define the function $\tilde{F}$ by:

$$\tilde{F}(z, \vec{y}, x) = \langle F(z, \vec{y}, 0), \ldots, F(z, \vec{y}, x) \rangle$$

Then $F(z, \vec{y}, x) = (\tilde{F}(z, \vec{y}, x))_x$, so if we can show that $\tilde{F}$ is primitive recursive, then so is $F$, being defined from $\tilde{F}$ by composition with primitive recursive functions.

Define an auxiliary function $L$ by

$$L(z, u, \vec{y}, x) = \langle H(z, (u)_{K(x)}, \vec{y}, 0), \ldots, H(z, (u)_{K(x)}, \vec{y}, x) \rangle$$

Then

$$L(z, u, \vec{y}, 0) = \langle H(z, (u)_{K(0)}, 0) \rangle$$

$$L(z, u, \vec{y}, x + 1) = L(z, u, \vec{y}, x) * \langle H(z, (u)_{K(x+1)}, \vec{y}, x + 1) \rangle$$

so $L$ is defined by primitive recursion from primitive recursive functions, hence primitive recursive. Now for $\tilde{F}$ we have:

$$\tilde{F}(0, \vec{y}, x) = \langle G(\vec{y}, 0), \ldots, G(\vec{y}, x) \rangle$$

$$\tilde{F}(z + 1, \vec{y}, x) = L(z, \tilde{F}(z, \vec{y}, x), \vec{y}, x)$$

(this takes a few lines of checking!) where in the first line we have a function defined by course-of-values recursion from $G$ (so primitive recursive); and $\tilde{F}$ is defined by primitive recursion; so it is primitive recursive.
b) The only point here is to get the induction right. If one wishes to show \( \forall n \leq m \, P(m) \) then it suffices to show: \( P(0) \) and for all \( n < m \), if \( P(n) \) then \( P(n+1) \).

For \( n = 0 \) we have \( F'(n, m + k, \bar{y}, x) = F'(0, m + k, \bar{y}, x) = G(\bar{y}, K^{m+k}(x)) \) and also

\[
F'(n, m, \bar{y}, K^k(x)) = F'(0, m, \bar{y}, K^k(x)) = G(\bar{y}, K^n(K^k(x))) = G(\bar{y}, K^{m+k}(x))
\]

so the statement holds for \( n = 0 \). Suppose \( n < m \) and the statement holds for \( n \). Since \( n < m \) hence \( n + 1 \leq m \), we have \( m + k \cdot (n + 1) = (m - (n + 1)) + k \) (this is the point where the assumption \( n < m \) is used! This does not hold in general!), so using the induction hypothesis we have: \( F'(n + 1, m + k, \bar{y}, x) = H(n, F'(n, m + k, \bar{y}, x), \bar{y}, K^{m+k-n}(x)) = H(n, F'(n, m, \bar{y}, K^k(x)), \bar{y}, K^{m+n}(K^k(x))) = F'(n + 1, m, \bar{y}, K^k(x)) \). This completes the induction step.

c) We have \( F(0, \bar{y}, x) = G(\bar{y}, x) \) and \( F'(0, \bar{y}, x) = G(\bar{y}, K^0(x)) = G(\bar{y}, x) \), so for \( z = 0 \) the statement holds.

Suppose the statement holds for \( z \). Since \( z + 1 \cdot (z + 1) = 0 \) we have: \( F'(z + 1, z + 1, \bar{y}, x) = H(z, F'(z, z + 1, \bar{y}, x), \bar{y}, x) = H(z, F(z, z + 1, \bar{y}, x), \bar{y}, x) = F(z + 1, \bar{y}, x) \), which completes the induction step.

We see that the function \( F \) is defined by composition of \( F' \) (and projection functions); hence \( F \) is primitive recursive. Since we have never used that \( K(x) \leq x \) in this proof, \( F \) is primitive recursive without this assumption.

**Exercise 35.** Prove Smullyan’s Simultaneous Recursion Theorem: given two binary partial recursive functions \( F \) and \( G \), for every \( k \) there exist indices \( a \) and \( b \) satisfying for all \( x_1, \ldots, x_k \):

\[
a \cdot (x_1, \ldots, x_k) \simeq F(a, b) \cdot (x_1, \ldots, x_k)
\]

and

\[
b \cdot (x_1, \ldots, x_k) \simeq G(a, b) \cdot (x_1, \ldots, x_k)
\]

**Solution:** First, use the Recursion Theorem to find an index \( \alpha \) such that for all \( y, x_1, \ldots, x_k \):

\[
\alpha \cdot (y, x_1, \ldots, x_k) \simeq F(S^k_1(\alpha, y), y) \cdot (x_1, \ldots, x_k)
\]

Then, again applying the Recursion Theorem, find index \( \beta \) such that for all \( x_1, \ldots, x_k \):

\[
\beta \cdot (x_1, \ldots, x_k) \simeq G(S^k_1(\alpha, \beta), \beta) \cdot (x_1, \ldots, x_k)
\]

Let \( b = \beta \) and \( a = S^k_1(\alpha, \beta) \). Then:

\[
a \cdot (\bar{x}) \simeq S^k_1(\alpha, \beta) \cdot (\bar{x}) \\
\simeq \alpha \cdot (\bar{x}) \\
\simeq F(S^k_1(\alpha, \beta), \beta) \cdot (\bar{x}) \\
\simeq F(a, b) \cdot (\bar{x})
\]

and

\[
b \cdot (\bar{x}) \simeq \beta \cdot (\bar{x}) \\
\simeq G(S^k_1(\alpha, \beta), \beta) \cdot (\bar{x}) \\
\simeq G(a, b) \cdot (\bar{x})
\]

**Exercise 55:** Conclude from Theorem 3.3.3 that there cannot exist a total recursive function \( F \) which is such that for all \( e: \phi_e \) is constant on its domain if and only if \( F(e) \in K \).

**Solution:** Suppose there were such \( F \). Then we have that

\[
X = \{ e \mid \phi_e \text{ is constant on its domain} \}
\]

is reducible to \( K \) via \( F \), so \( X \) would be r.e. by Exercise 43.

It is also clear from the definition that \( X \) is extensional for indices of partial recursive functions.
Therefore, by Myhill-Shepherdson (3.3.3. part 1), the set $F = \{ \phi_e | e \in X \}$ is open in $PR$.
However, this would mean (by the remarks following Exercise 53) that $F$ is upwards closed. Since $F$ contains the empty function, therefore $F$ would be the set of all partial recursive functions; so every partial recursive function would be constant on its domain. This is clearly false.

**Exercise 72:** Find for each of the following relations an $n$, as small as you can, such that they are in $\Sigma_n$, $\Pi_n$ or $\Delta_n$:

i) $\{ e | W_e \text{ is finite} \}$  
ii) $\{ e | rge(\phi_e) \text{ is infinite} \}$  
iii) $\{ e | \phi_e \text{ is constant (possibly partial)} \} = \{ e | \phi_e \text{ has at most one value} \}$  
iv) $\{ j(e, f) | W_e \leq_m W_f \}$  
v) $\{ e | W_e \text{ is } m\text{-complete in } \Sigma_1 \}$

Then, classify the first three of these completely, by showing that they are $m$-complete in the class you found.

**Solution:** we do i) and ii) simultaneously. Let $DomFin$ be the set $\{ e | W_e \text{ is finite} \}$ and let $RgeInf$ be the set $\{ e | rge(\phi_e) \text{ is infinite} \}$. We have:

$$e \in DomFin \iff \exists y \forall k (T(1, e, y, k) \rightarrow y \leq x)$$
$$e \in RgeInf \iff \forall y \exists z \exists k (T(1, e, y, k) \land U(k) > x)$$

From this we see that $DomFin$ is in $\Sigma_2$ and $RgeInf$ is in $\Pi_2$.

From the Kleene Normal Form Theorem we know that the set $Tot = \{ e | \forall x \exists y T(1, e, x, y) \}$ is $m$-complete in $\Pi_2$ and its complement $NTot = N - Tot$ is therefore $m$-complete in $\Sigma_2$. Let $g$ be an index such that

$$g^*(e, x) \simeq \begin{cases} 
  x & \text{if } \exists z \forall i < x T(1, e, i, (z)_i) \\
  \text{undefined} & \text{otherwise}
\end{cases}$$

Let $G(e) = S^1_L(g, e)$. We have: $rg(\phi_{G(e)})$ is infinite if and only if $W_{G(e)}$ is infinite, if and only if $e \in Tot$; so $G$ reduces $Tot$ to $RgeInf$ and $NTot$ to $DomFin$. Therefore, $RgeInf$ is $m$-complete in $\Pi_2$ and $DomFin$ is $m$-complete in $\Sigma_2$.

iii): let $Const$ be the set from iii). We have

$$e \in Const \iff \forall u y k l (T(1, e, u, k) \land T(1, e, y, l) \rightarrow U(k) = U(l))$$

which establishes that $Const$ is in $\Pi_1$.

Let $g$ be an index satisfying:

$$g^*(e, x) \simeq \begin{cases} 
  0 & \text{if } \forall y \leq x T(1, e, e, y) \\
  z + 1 & \text{if } z \leq x \text{ is minimal with } T(1, e, e, z)
\end{cases}$$

Let $G(e) = S^1_L(g, e)$. We see that $G(e) \in Const$ precisely when $e \in N - K$. Since $K$ is $m$-complete in $\Sigma_1$ hence $N - K$ is $m$-complete in $\Pi_1$, we see that $Const$ is $m$-complete in $\Pi_1$.

iv): $W_e \leq_m W_f$ if and only if there is a total recursive function $\phi_u$ such that $W_e = \phi^{-1}_u(W_f)$. Therefore $W_e \leq_m W_f$ holds, if and only if the following condition is satisfied:

$$\exists u \ [ \forall x \exists y T(1, u, x, y) \land \forall z w v \exists a (T(1, e, z, v) \land T(1, u, z, w) \rightarrow T(1, f, U(w), a)) \land \forall b c d e g (T(1, u, b, c) \land T(1, f, U(c), d) \rightarrow T(1, e, b, g)) ]$$
We have an existential quantifier before an intersection of $\Pi_2$-sets. Since $\Pi_2$ is closed under intersections (proposition 4.2.4), the set \( \{j(e, f) \mid W_e \leq_m W_f\} \) is in $\Sigma_3$.

v): $W_e$ is $m$-complete in $\Sigma_1$ if and only if $K \leq_m W_e$. So the set of v) is in $\Sigma_3$ by the result of iv).

**Exercise 77.** Prove that for a set $X \subseteq \mathbb{N}$ the following assertions are equivalent:

i) $X$ is creative

ii) $X$ is 1-complete in $\Sigma_1$;

iii) $X$ is $m$-complete in $\Sigma_1$;

iv) There is a total recursive bijective function $h$ such that $h[X] = K$

Hint: use Exercises 75-76, proposition 4.3.5 and Theorem 4.3.3.

**Solution:** it is necessary to prove first that $K$ is 1-complete in $\Sigma_1$. In fact the usual proof of $m$-completeness of $K$ works, because $Smm$-functions can be assumed to be injective.

i) $\Rightarrow$ ii): Suppose $X$ is creative. Then by 4.3.5, $K \leq_1 X$. Since $K$ is 1-complete, $X$ is.

ii) $\Rightarrow$ iii): trivial.

iii) $\Rightarrow$ iv): Suppose $X$ is $m$-complete in $\Sigma_1$. Then $K \leq_m X$. Since $K$ is creative by Exercise 75, $X$ is creative by Exercise 76 iii); so $K \leq_1 X$. Because $K$ is 1-complete we also have $X \leq_1 K$. Statement iv) now follows from Theorem 4.3.3.

iv) $\Rightarrow$ i): Suppose $h: \mathbb{N} \to \mathbb{N}$ is a total recursive bijection with $h[X] = K$. Let $G$ be primitive recursive such that $W_{G(e)} = h[W_e]$ for all $e$. By 4.3.4, we may assume that $K$ is creative via a total recursive, injective function $H$. Let $F(e) = h^{-1}(H(G(e)))$. We claim that $X$ is creative via $F$. Indeed, suppose $W_e \cap X = \emptyset$. Then $W_{G(e)} \cap K = \emptyset$. So $H(G(e)) \notin W_{G(e)} \cup K$. Then $F(e) = h^{-1}(H(G(e))) \notin W_e \cup X$.

**Exercise 87.** Given sets $A$ and $B$, prove that the following assertions are equivalent:

i) $B \leq_T A$

ii) There exist total recursive functions $F$ and $G$ such that the following holds:

\[
\begin{align*}
x \in B & \text{ if and only if } \exists \sigma(\sigma \in W_{F(x)} \land \forall i < \lh(\sigma)(\sigma)_i = \chi_A(i)) \\
x \notin B & \text{ if and only if } \exists \sigma(\sigma \in W_{G(x)} \land \forall i < \lh(\sigma)(\sigma)_i = \chi_A(i))
\end{align*}
\]

(Hint: use proposition 5.1.8.)

**Solution:** i) $\Rightarrow$ ii): suppose i) holds. By proposition 5.1.8 we know that there is a number $e$ such that for all $x$:

\[
\begin{align*}
x \in B & \text{ if and only if } \exists \sigma(\sigma \leq \chi_A \land \exists w(T^\sigma(1, e, x, w) \land U(w) = 0)) \\
x \notin B & \text{ if and only if } \exists \sigma(\sigma \leq \chi_A \land \exists w(T^\sigma(1, e, x, w) \land U(w) = 1))
\end{align*}
\]

where we use $\sigma \leq \chi_A$ as short for: $\forall i < \lh(\sigma)(\sigma)_i = \chi_A(i)$. Let $f$ and $g$ be indices such that

\[
f(x, y) \simeq \begin{cases} 0 & \text{if } \exists w(T^\sigma(1, e, x, w) \land U(w) = 0) \\
\text{undefined} & \text{otherwise}
\end{cases}
\]

\[
g(x, y) \simeq \begin{cases} 0 & \text{if } \exists w(T^\sigma(1, e, x, w) \land U(w) = 1) \\
\text{undefined} & \text{otherwise}
\end{cases}
\]

and put $F(x) = S_1^1(g, x)$, $G(x) = S_1^1(g, x)$. Then

\[
\begin{align*}
W_{F(x)} &= \{\sigma \mid \exists w(T^\sigma(1, e, x, w) \land U(w) = 0)\} \\
W_{G(x)} &= \{\sigma \mid \exists w(T^\sigma(1, e, x, w) \land U(w) = 1)\}
\end{align*}
\]
Then ii) holds: suppose $x \in B$. Then by the choice of $e$ we have $\exists \sigma (\sigma \preceq \chi_A \land \exists w (T^{\sigma}(1, e, x, w) \land U(w) = 0))$ so $\exists \sigma (\sigma \preceq \chi_A \land \sigma \in W_{F(x)})$. The converse is immediate; and a similar equivalence holds for $x \notin B$.

ii) $\Rightarrow$ i): suppose ii) holds. In order to determine $\chi_B(x)$, find the least pair $\langle \sigma, w \rangle$ satisfying $\sigma \preceq \chi_A$ and $w$ testifies that $\sigma \in W_{F(x)}$ or $\sigma \in W_{G(x)}$. Note that only one of the two can happen. Output 0 if $\sigma \in W_{F(x)}$ and 1 if $\sigma \in W_{G(x)}$. This is recursive in $A$, so $B \leq_T A$. 
