Exercise 1 (Exercise 38 of the lecture notes). Let $K : \mathbb{N} \to \mathbb{N}$, $G : \mathbb{N}^{k+1} \to \mathbb{N}$ and $H : \mathbb{N}^{k+3} \to \mathbb{N}$ be functions. Define $F$ by:

\[
F(0, \vec{y}, x) = G(\vec{y}, x) \\
F(z + 1, \vec{y}, x) = H(z, F(z, \vec{y}, K(x)), \vec{y}, x)
\]

Suppose that $G$, $H$ and $K$ are primitive recursive.

a) (4 points) Prove directly, using the pairing function $j$ and suitably adapting the proof of proposition 3.9: if $\forall x (K(x) \leq x)$, then $F$ is primitive recursive.

b) (3 points) Define a new function $F'$ by:

\[
F'(0, m, \vec{y}, x) = G(\vec{y}, K^m(x)) \\
F'(n + 1, m, \vec{y}, x) = H(n, F'(n, m, \vec{y}, x), \vec{y}, K^m(n+1)(x))
\]

Recall that $K^{m=\langle n+1 \rangle}$ means: the function $K$ applied $m=\langle n+1 \rangle$ times.

Prove: if $n \leq m$ then $\forall k[F'(n, m + k, \vec{y}, x) = F'(n, m, \vec{y}, K^{k}(x))]$

c) (3 points) Prove by induction: $F(z, \vec{y}, x) = F'(z, z, \vec{y}, x)$ and conclude that $F$ is primitive recursive, also without the assumption that $K(x) \leq x$.

Solution: There is more than one way to solve a), which was the most challenging part of the exercise. Define the function $\tilde{F}$ by:

\[
\tilde{F}(z, \vec{y}, x) = \langle F(z, \vec{y}, 0), \ldots, F(z, \vec{y}, x) \rangle
\]
Then \( F(z, \bar{y}, x) = (\bar{F}(z, \bar{y}, x))_x \), so if we can show that \( \bar{F} \) is primitive recursive, then so is \( F \), being defined from \( \bar{F} \) by composition with primitive recursive functions.

Define an auxiliary function \( L \) by

\[
L(z, u, \bar{y}, x) = \langle H(z, (u)_{K(0)}, \bar{y}, 0), \ldots, H(z, (u)_{K(x)}, \bar{y}, x) \rangle
\]

Then

\[
L(z, u, \bar{y}, 0) = \langle H(z, (u)_{K(0)}, \bar{y}, 0) \rangle \\
L(z, u, \bar{y}, x + 1) = L(z, u, \bar{y}, x) \ast (H(z, (u)_{K(x+1)}, \bar{y}, x + 1))
\]

so \( L \) is defined by primitive recursion from primitive recursive functions, hence primitive recursive.

Now for \( \bar{F} \) we have:

\[
\bar{F}(0, \bar{y}, x) = \langle G(\bar{y}, 0), \ldots, G(\bar{y}, x) \rangle \\
\bar{F}(z + 1, \bar{y}, x) = L(z, \bar{F}(z, \bar{y}, x), \bar{y}, x)
\]

(this takes a few lines of checking!) where in the first line we have a function defined by course-of-values recursion from \( G \) (so primitive recursive); and \( \bar{F} \) is defined by primitive recursion; so it is primitive recursive.

b) The only point here is to get the induction right. If one wishes to show \( \forall n \leq m P(m) \) then it suffices to show: \( P(0) \) and for all \( n < m \), if \( P(n) \) then \( P(n + 1) \).

For \( n = 0 \) we have \( F'(n, m+k, \bar{y}, x) = F'(0, m+k, \bar{y}, x) = G(\bar{y}, K^{m+k}(x)) \)

and also

\[
F'(n, m, \bar{y}, K^k(x)) = F'(0, m, \bar{y}, K^k(x)) = G(\bar{y}, K^m(K^k(x))) = G(\bar{y}, K^{m+k}(x))
\]

so the statement holds for \( n = 0 \). Suppose \( n < m \) and the statement holds for \( n \). Since \( n < m \) hence \( n+1 \leq m \), we have \( m+k-(n+1) = (m-k-(n+1))+k \) (this is the point where the assumption \( n < m \) is used! This does not hold in general!), so using the induction hypothesis we have:

\[
F'(n+1, m+k, \bar{y}, x) = \\
H(n, F'(n, m+k, \bar{y}, x), \bar{y}, K^{m+k-(n+1)}(x)) = \\
H(n, F'(n, m, \bar{y}, K^k(x)), \bar{y}, K^{m-(n+1)}(K^k(x))) = \\
F'(n+1, m, \bar{y}, K^k(x))
\]

This completes the induction step.

c) We have \( F(0, \bar{y}, x) = G(\bar{y}, x) \) and \( F'(0, 0, \bar{y}, x) = G(\bar{y}, K^0(x)) = G(\bar{y}, x) \), so for \( z = 0 \) the statement holds.
Suppose the statement holds for $z$. Since $z + 1 ≡ (z + 1) = 0$ we have:

$$F'(z+1, z+1, g, x) = H(z, F'(z, z+1, g, x), g, x) = H(z, F'(z, g, K(x)), g, x) = F(z+1, g, x),$$

which completes the induction step.

We see that the function $F$ is defined by composition from $F'$ (and projection functions); hence $F$ is primitive recursive. Since we have never used that $K(x) ≤ x$ in this proof, $F$ is primitive recursive without this assumption.

**Exercise 2** Given a natural number $x > 0$ and a prime number $p$, by ord\(_p\)(\(x\)) (the order of $p$ at $x$) we mean the highest number $n$ such that $p^n$ divides $x$.

a) (2 points) Give a formula $ψ(v, x)$ in $L_{PA}$ (but you can use the abbreviations pr, pp and $x\upharpoonright v$ from the notes) which expresses that $x > 0$, $v$ is prime and ord\(_v\)(\(x\)) is even.

b) (2 points) Give also such a formula $χ(v, x)$, expressing: $x > 0$, $v$ is prime and ord\(_v\)(\(x\)) $≡ 1$ (modulo 3).

c) (3 points) For the formula $ψ(v, x)$ from a), prove:

$$PA ⊢ ∀x[∀v \leq x(pr(v) → ψ(v, x)) → ∃y(y · y = x)]$$

d) (3 points) Prove in $PA$ that “the root of a non-square is irrational”, that is:

$$PA ⊢ ∀xyz(x > 0 ∧ x · x = y · z · z → ∃v(y = v · v))$$

**Solution:** a) $x > 0 ∧ pr(v) ∧ ∃y(y · y = x \upharpoonright v)$
b) $x > 0 ∧ pr(v) ∧ ∃y(y · y · y · v = x \upharpoonright v)$
c) You will not be punished for assuming without proof that for $x, y > 0$ and $pr(v)$, $(x y) \upharpoonright v = (x \upharpoonright v)(y \upharpoonright v)$ but let’s do this first: since $(x \upharpoonright v) | x$ and $(y \upharpoonright v) | y$, $(x \upharpoonright v)(y \upharpoonright v) | xy$ and hence, since $(x \upharpoonright v)(y \upharpoonright v)$ is a $v$-power by Exercise 56a) and by the definition of $(\cdot)\upharpoonright v$, $(x \upharpoonright v)(y \upharpoonright v) | (xy) \upharpoonright v$. Conversely, if $x = (x \upharpoonright v) · w$ and $y = (y \upharpoonright v) · z$, then $v \upharpoonright wz$ and $x y = (x \upharpoonright v)(y \upharpoonright v) wz$, so $(xy) \upharpoonright v(x \upharpoonright v)(y \upharpoonright v)$.

To prove c) we employ well-founded induction. Let $χ(x)$ be the formula

$$∀v ≤ x(pr(v) → ψ(v, x)) → ∃y(y y = x)$$

and assume

1. $∀x' < xχ(x')$
2. $∀v ≤ x(pr(v) → ψ(v, x))$
We have to prove that \( x \) is a square. This is trivial if \( x \leq 1 \) so let \( x > 1 \). Then \( x \) has a prime divisor \( v \) by Proposition 4.5. By assumption (2), let \( y \) satisfy \( x|v = yy \). Then \( v|y \) so \( vv|x \); let \( z \) satisfy \( x = vvz \). We now have:

\[
\begin{align*}
(3) & \quad x|v = vv(z|v) \\
(4) & \quad \text{for } pr(w), w \neq v, x|w = z|w
\end{align*}
\]

From (3) and assumption (2) we get that \( z|v \) is a square, and (4) says that if \( pr(w) \) and \( w \neq v \) then \( z|w = x|w \), hence also a square by assumption (2).

By well-founded induction, we are done.

d) Suppose \( x > 0 \) and \( xx = yzz \). We have to prove that \( y \) is a square, which (again) is trivial if \( y \leq 1 \). So, let \( y > 1 \) and \( v \) a prime divisor of \( y \). We see that \( (z|v)(z|v) \leq (x|v)(x|v) \) so \( z|v \leq x|v \), so \( (z|v)|(x|v) \). Let \( x = x'(z|v) \), \( z = z'(z|v) \). Then \( xx' = yy'z'z' \) so

\[
y|v = (yz'z')|v = (x'|v)(x'|v)
\]

so \( y|v \) is a square. The number \( v \) was an arbitrary prime divisor of \( y \), therefore by c) we can conclude that \( y \) is a square.

Remark: the induction in c) is necessary: without the induction axioms, it is possible that there is a (nonstandard) model in which “\( \sqrt{2} \) is rational”: there are nonstandard elements \( p, q \) for which \( p^2 = 2q^2 \).

Exercise 3 This combines exercises 65 and 71 from the notes: give a full proof of Theorem 4.13 but now, with “\( \Sigma_1 \)-formula” replaced by “\( \Delta_1 \)-formula” (in definition 4.12).

Solution. For a primitive recursive function \( F \), let us write \( \varphi_F \) for the representing \( \Sigma_1 \)-formula constructed in the proof of 4.13. To be explicit:

If \( F \) is \( \lambda x.0 \) then \( \phi_F \) is \( z = 0 \)
If \( F \) is \( \lambda x.x + 1 \) then \( \phi_F \) is \( z = x + 1 \)
If \( F \) is \( \lambda x_1 \cdots x_k.x_i \) then \( \varphi_F \) is \( z = x_i \)
If \( F(\vec{x}) = G(H_1(\vec{x}), \ldots, H_m(\vec{x})) \) then \( \varphi_F \) is

\[
\exists w_1 \cdots w_m(\varphi_{H_1}(\vec{x}, w_1) \land \cdots \land \varphi_{H_m}(\vec{x}, w_m) \land \varphi_G(\vec{w}, z))
\]

If \( F(\vec{x}, 0) = G(\vec{x}) \) and \( F(\vec{x}, y + 1) = H(\vec{x}, F(\vec{x}, y), y) \) then \( \varphi_F \) is

\[
\exists a(\varphi_G(\vec{x}, (a, m)_0) \land \forall i < y \varphi_H(\vec{x}, (a, m)_i, i, (a, m)_{i+1}) \land (a, m)_y = z)
\]

We prove the following things, all by induction on the definition of \( F \) as a primitive recursive function:
1. For all \(n_1, \ldots, n_k \in \mathbb{N}\), PA \(\vdash \varphi_F(\overline{n_1}, \ldots, \overline{n_k}, F(n_1, \ldots, n_k))\)

2. PA \(\vdash \forall \overline{x} \exists ! z \varphi_F(\overline{x}, z)\)

3. The formula \(\varphi_F\) is, in PA, equivalent to a \(\Pi_1\)-formula.

For the basic functions, assertions 1–3 are immediate; note that \(\varphi_F\) is a \(\Delta_0\)-formula in these cases.

In the case of composition: \(F(\overline{x}) = G(H_1(\overline{x}), \ldots, H_m(\overline{x}))\) we assume 1–3 for \(\varphi_G, \varphi_{H_1}, \ldots, \varphi_{H_m}\).

1. Suppose \(\overline{x} = x_1, \ldots, x_k\). Given \(n_1, \ldots, n_k \in \mathbb{N}\) we have
PA \(\vdash \varphi_{H_i}(\overline{n_1}, \ldots, \overline{n_k}, H_i(n_1, \ldots, n_k))\) and PA \(\vdash \varphi_G(H_i(n_1, \ldots, n_k), F(n_1, \ldots, n_k))\)

so
\[
\text{PA} \vdash \exists \overline{w}(\varphi_{H_1}(\overline{n_1}, \ldots, \overline{n_k}, w_1) \land \cdots \land \varphi_{H_m}(\overline{n_1}, \ldots, \overline{n_k}, w_m) \land \varphi_G(\overline{w}, F(n_1, \ldots, n_k)))
\]
so PA \(\vdash \varphi_F(\overline{n_1}, \ldots, \overline{n_k}, F(n_1, \ldots, n_k))\).

2. Reason in PA: given \(\overline{x}\), we have \(w_1, \ldots, w_m\) with \(\varphi_{H_1}(\overline{x}, w_1), \ldots, \varphi_{H_m}(\overline{x}, w_m)\), by induction hypothesis on \(H_1, \ldots, H_m\). By induction hypothesis on \(G\) we get a \(z\) with \(\varphi_G(w_1, \ldots, w_m, z)\). So we have a \(z\) with \(\varphi_F(\overline{x}, z)\).

For uniqueness, suppose \(\varphi_F(\overline{x}, z) \land \varphi_F(\overline{x}, z')\). Then we have \(w_1, \ldots, w_m, w'_1, \ldots, w'_m\) with \(\varphi_{H_1}(\overline{x}, w_1), \ldots, \varphi_{H_m}(\overline{x}, w_m)\) and \(\varphi_{H_1}(\overline{x}, w'_1), \ldots, \varphi_{H_m}(\overline{x}, w'_m)\) and \(\varphi_G(\overline{w}, z)\), \(\varphi_G(\overline{w'}, z')\). The uniqueness in the induction hypothesis for \(H_1, \ldots, H_m\) gives \(w_1 = w'_1, \ldots, w_m = w'_m\); the uniqueness in the induction hypothesis for \(G\) now gives \(z = z'\).

3. Let \(\psi_G\) be a \(\Pi_1\)-formula such that PA \(\vdash \varphi_G(\overline{x}, z) \leftrightarrow \psi_G(\overline{x}, z)\). Define the formula \(\psi'(\overline{x}, z)\) by
\[
\forall \overline{w}(\varphi_{H_1}(\overline{x}, w_1) \land \cdots \land \varphi_{H_m}(\overline{x}, w_m) \rightarrow \psi_G(\overline{w}, z))
\]
Since the \(\varphi_{H_i}\) are \(\Sigma_1\), the formula \(\psi_G\) is \(\Pi_1\), the logical equivalence \(\forall x(\exists y A \rightarrow \forall w B) \leftrightarrow \forall xyw(A \rightarrow B)\) gives a \(\Pi_1\)-formula \(\psi_F\) equivalent to \(\psi'_F\).

We prove that \(\psi'_F\) is equivalent to \(\varphi_F\). Given \(\overline{x}, z\), assume \(\psi'_F(\overline{x}, z)\). By property 2 for \(H_1, \ldots, H_m\), there are \(w_1, \ldots, w_m\) with \(\varphi_{H_1}(\overline{x}, w_1), \ldots, \varphi_{H_m}(\overline{x}, w_m)\). Hence by \(\psi'_F(\overline{x}, z)\) we obtain \(\psi_G(\overline{w}, z)\) hence \(\varphi_G(\overline{x}, z)\). So we have \(\varphi_F(\overline{x}, z)\).

Conversely, suppose \(\varphi_F(\overline{x}, z)\) and assume \(\varphi_{H_1}(\overline{x}, w_1), \ldots, \varphi_{H_m}(\overline{x}, w_m)\).

By \(\varphi_F(\overline{x}, z)\) we find \(w'_1, \ldots, w'_m\) such that
\[
\varphi_{H_1}(\overline{x}, w'_1) \land \cdots \land \varphi_{H_m}(\overline{x}, w'_m) \land \varphi_G(\overline{w'_1}, \ldots, \overline{w'_m}, z)
\]
The uniqueness in the induction hypothesis for \(H_1, \ldots, H_m\) gives \(w_i = w'_i\) for \(i = 1, \ldots, n\). So we get \(\varphi_G(\overline{w}, z)\) and hence \(\psi_G(\overline{w}, z)\) using the induction hypothesis on \(G\). Hence \(\psi'_F(\overline{x}, z)\) follows.
In the case of primitive recursion:
1. Given $n_1, \ldots, n_k, l$ we prove that $PA \vdash \varphi_F(n_1, \ldots, n_k, l, F(n_1, \ldots, n_k))$ by induction on $l$.
   
   For $l = 0$ we must prove $PA \vdash \exists a, m (\varphi_G(n_1, \ldots, n_k, (a, m)_0) \land (a, m)_0 = F(n_1, \ldots, n_k))$ which follows from 4.9 i).
   
   Inductively, suppose $PA \vdash \varphi_F(n_1, \ldots, n_k, l, F(n_1, \ldots, n_k))$ so there is $a, m$ with
   
   $PA \vdash \varphi_G(n_1, \ldots, n_k, (a, m)_0) \land \forall i < l \varphi_H(n_1, \ldots, (a, m)_i, (a, m)_{i+1}) \land (a, m)_l = F(n_1, \ldots, n_k, l)$
   
   By 4.9 ii), find $b, n$ such that $\forall i < l (a, m)_i = (b, n)_i$ and $(b, n)_{l+1}$ is the unique $w$ such that $\varphi_H(n_1, \ldots, n_k, (a, m)_l, w)$. Then this $(b, n)$ testifies that $PA \vdash \varphi_F(n_1, \ldots, n_k, l + 1, F(n_1, \ldots, n_k, l + 1))$.

2. In PA, let $\vec{x}, y$ be given; to show $\exists z \varphi_F(\vec{x}, y, z)$. Induction on $y$. For $y = 0$ this is similar to case 1: use 4.9 i). For the induction step one uses 4.9 ii) again in a very similar way to the proof of 1.
   
   To get uniqueness of $z$: suppose $\varphi_F(\vec{x}, y, z) \land \varphi_F(\vec{x}, y, z')$. Then there are $a, m, b, n$ such that
   
   $\varphi_G(\vec{x}, 0, (a, m)_0) \land \forall i < y \varphi_H(\vec{x}, (a, m)_i, (a, m)_{i+1}) \land z = (a, m)_y$
   
   $\varphi_G(\vec{x}, 0, (b, n)_0) \land \forall i < y \varphi_H(\vec{x}, (b, n)_i, (b, n)_{i+1}) \land z' = (b, n)_y$
   
   One proves, using the uniqueness in the induction hypothesis for $G$ and $H$, that $\forall i < y (a, m)_i = (b, n)_i$, hence $z = z'$.

3. Let $\psi_F(\vec{x}, y, z)$ be the formula
   
   $\forall am(\varphi_G(\vec{x}, (a, m)_0) \land \forall i < y \varphi_H(\vec{x}, (a, m)_i, (a, m)_{i+1}) \rightarrow z = (a, m)_y$
   
   and $\psi_F$ the obvious $\Pi_1$-equivalent of $\psi_F$. Again, one employs induction on $y$ to prove the equivalence
   
   $\varphi_F(\vec{x}, y, z) \leftrightarrow \psi_F(\vec{x}, y, z)$
   
   In the proof, one uses the uniqueness property in the induction hypothesis, much in the way as property 3 was proved for composition.

**Exercise 4.** Recall that the following functions are primitive recursive.
The function assigning to $x$ the Gödel number $\neg \chi$ if $x$ is the Gödel number of some formula $\chi$. Otherwise, its value is 0.

The function assigning to $(x, y, i)$ the Gödel number $\chi[t/v_i]$ if $x$ is the Gödel number of some term $t$, $y$ is the Gödel number of some formula $\chi$ and $t$ is free for $v_i$ in $\chi$. Otherwise, its value is 0.

The function assigning to a number $x$ the Gödel number $\downarrow x$.

Let $\text{Neg}$, $\text{Sub}$ and $\text{Num}$ be formulas representing these functions in $\text{PA}$, in such a way that the recursions of the latter two are provable in $\text{PA}$.

We define the sequence of theories $(T_n)_{n \in \mathbb{N}}$ by recursion: $T_0 = \text{PA}$ and for $n \in \mathbb{N}$, $T_{n+1}$ is $\text{PA} + \text{Con}_{T_n}$.

a) Prove that $T_n$ is consistent for every $n \in \mathbb{N}$.

Thus, the given sequence is an ascending hierarchy of consistent theories, where each theory claims the consistency of the previous one. The goal of this exercise is to create a similar descending hierarchy, where each theory claims the consistency of the next one.

Now define the formula $\phi(v_0, v_1)$ as:

$$
\exists a \exists b \exists c \neg \Box(c) \land \neg \text{Neg}(b, c) \land \text{Sub}(\downarrow 0, v_0, 1, b) \land \text{Num}(v_1 + 1, a).
$$

b) Apply the Diagonalisation Lemma to $\phi$ to obtain a formula $\psi(v_1)$ and define $S_n := \text{PA} + \psi(n)$. Show that, in $\text{PA}$, the formula $\psi(n)$ naturally expresses the consistency of $S_{n+1}$.

It may look as though we have our desired sequence. However, the $S_n$ also need to be consistent.

c) Prove that $\text{PA} \vdash \Box(\forall x, \neg \psi(x)) \rightarrow \forall x, \neg \psi(x)$. (Please don’t explicitly formalize the argument in $\text{PA}$; just make it clear that the argument may be so formalized.)

d) Deduce that $S_n$ is inconsistent for all $n \in \mathbb{N}$. [Hint: use Löb’s Theorem]

This shows we have to be a bit more clever to solve our problem. Let $\phi'(v_0, v_1)$ be the formula

$$
\neg [\exists a \exists b \text{Proof}(v_1, b) \land \neg \text{Neg}(a, b) \land \text{Sub}(\downarrow 0, v_0, 1, a)] \rightarrow \phi(v_0, v_1).
$$

e) As before, apply Lemma 5.1 to $\phi'$ to obtain a formula $\psi'(v_1)$ and define $S'_n := \text{PA} + \psi'(n)$. Prove that $S'_n$ is consistent.
f) Show that, in PA, the formula $\psi'(\bar{n})$ naturally expresses the consistency of $S'_{n+1}$.

g) Prove that $S'_n$ is consistent for all $n \in \mathbb{N}$.

h) Can you explain why the argument that showed the $S_n$ to be inconsistent doesn’t work now?