Seminar on Hilbert's Tenth Problem
Homework week 12, due December 16

Exercise 1. (5 points)
Let R be a commutative ring with 1, let $k \geq 1$ and let $S, T \subset R^k$ be Diophantine. Prove:
(a) If R is an integral domain, then $S \cup T$ is also a Diophantine subset of R^k.
(b) If the fraction field of R is not algebraically closed, then $S \cap T$ is also a Diophantine subset of R^k.

Exercise 2. (10 points)
In this exercise, we will prove Proposition 2. Recall that p is assumed to be odd. The first two exercises are preliminary.
(a) Prove that, for every $a \in \mathbb{N}$, we have $\deg X_a = a$ and $X_{-a} = X_a$.
(b) Prove that, for every $a \in \mathbb{Z}$ and $b \in \mathbb{N}$, we have $X_{ap^b} = (X_a)^{p^b}$.

Now we prove Proposition 2: let $A, B \in \mathbb{F}_q[Z]$ with B non-constant. Then

$$\exists k \in \mathbb{N} \left(A = B^{p^k}\right) \Leftrightarrow \exists m \in \mathbb{Z}\ (A = X_m(B)) \ \wedge\ \exists n \in \mathbb{Z}\ (A + 1 = X_n(B+1)).$$

(c) Suppose that $A = B^{p^k}$ for some $k \in \mathbb{N}$. Prove that the right-hand side holds with $m = n = p^k$.
(d) Now suppose the right-hand side holds. Explain why we can assume that $m, n \in \mathbb{N}$. Show that $X_n(B+1) = X_n(B) + 1$.
(e) Since $X_0 = 1$, we cannot have $n = 0$. Now write $n = cp^k$ with $c, k \in \mathbb{N}$ and $p \nmid c$. Prove that $X_c(B+1) = X_c(B) + 1$.
(f) Show that the assumption that $c \geq 2$ leads to a contradiction. Conclude that proposition 2 holds. *Hint: write* $X_c(Z) = \alpha Z^c + \beta Z^{c-1} +$ *terms of lower degree.*

Exercise 3. (5 points)
Let n be a positive integer. Prove that

$$Z^n - 1 = \prod_{d|n} \Phi_d(Z).$$