In this homework we prove some properties of the Chebyshev polynomials:

\[ A_0(T) = 1 \quad B_0(T) = 0 \]
\[ A_1(T) = T \quad B_1(T) = 1 \]
\[ A_{n+2}(T) = 2T \cdot A_{n+1}(T) - A_n(T) \quad B_{n+2}(T) = 2T \cdot B_{n+1}(T) - B_n(T) \]

1. Show \( B_n(1) = m \) and \( \text{deg}(B_{n+1}) = n \) for all \( n \).

2. Recall the definition of the Chebyshev polynomials given in Jetze’s lecture:
   The \( n \)-th Chebyshev polynomials are given by
   \[
   (T + \sqrt{T^2 - 1})^n = X_n(T) + Y_n(T)\sqrt{T^2 - 1}
   \]
   Show the two definitions coincide.

3. In the lecture we showed the Chebyshev polynomials give solutions to the \( \mathbb{Z}[T] \) Pell equation \( X^2 - (T^2 - 1)Y^2 = 1 \). In this exercise we prove the sequences exhaust all solutions to the equation.

Let \( U(T), V(T) \in \mathbb{Z}[T] \) be such that \( [U(T)]^2 - (T^2 - 1)[V(T)]^2 = 1 \)

a) Show that we may assume there exists \( N \in \mathbb{Z}^+ \) such that for all \( a \in \mathbb{Z}^+ \) if \( a > N \) then \( U(a) = a f(a) \) and \( V(a) = a f(a) \) for some function \( f \).

b) Show there exists \( K \in \mathbb{N} \) such that for all \( \mathbb{Z}^+ \ni a > N, f(a) < K \)[Hint: Consider the characterisation of solutions of the integer Pell equation and use this to determine the behaviour of \( f(a) \) as \( a \to \infty \)]

c) Conclude that for some \( n \in \mathbb{N} \): \( U(T) = A_n(T), \ V(T) = B_n(T) \)