Hilbert 10 Seminar: Homework 15 Solutions

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1. First we verify $B_n(1) = n$ for all n. We proceed by induction on n.

$n = 0$: By definition $B_0(1) = 0$

$n = 1$: Again by definition we have $B_1(1) = 1$

$n + 2 > 1$: $B_{n+2}(1) = 2 \cdot B_{n+1}(1) - B_n(1)$ Applying the inductive hypothesis we obtain $B_{n+2}(1) = 2 \cdot (n+1) - n = n + 2$ as required.

Next the fact that $deg(B_{n+1}(T)) = n$ for all n. We proceed by induction on n.

$n = 0$: $deg(B_1(T)) = deg(1) = 0$

$n = 1$: $deg(B_2(T)) = deg(2T) = 1$

$n + 2 > 1$: $deg(B_{n+2}(T)) = deg(2T \cdot B_{n+1}(T) - B_n(T))$. Applying the inductive hypothesis we have $deg(B_{n+1}(T)) = n$ and $deg(B_n(T)) = max\{0, n-1\} < n + 1$. Hence $deg(2T \cdot B_{n+1}(T)) = n + 1$ and so $deg(B_{n+2}(T)) = n + 1$ as required.

2. We can prove this directly by a tedious induction, but there is a simpler method using the results from the lecture. First recall

Lemma 1.

(a) $\forall x, y \in \mathbb{N} : x^2 - (a^2 - 1)y^2 = 1 \longleftrightarrow \exists n(x + \sqrt{a^2-1}y = (a + \sqrt{a^2-1})^n$

(b) $a_n + \sqrt{a^2-1}a'_n = (a + \sqrt{a^2-1})^n$

Hence for $a \in \mathbb{N}^+$ we have $X_n(a) = a_n$ and $Y_n(a) = a'_n$. Similarly we used the result

Lemma 2. $\forall a \in \mathbb{N}^+ : A_n(a) = a_n \wedge B_n(a) = a'_n$.

Thus it follows that the polynomials $X_n(T) - A_n(T)$ and $Y_n - B_n(T)$ have infinitely many roots and are therefore both the constant zero polynomial. That is, $X_n(T) = A_n(T)$ and $Y_n(T) = B_n(T)$ as required.

3. a) As in the lemma proved in the lecture, as we take the squares of $U(T)$ and $V(T)$ we can assume without loss of generality that the lead coefficients of both are positive. Hence there exists $N \in \mathbb{N}$ such that for $x > N$ we have $U(x)$ and $V(x)$ positive and strictly increasing. Now let $a > N$ be a natural number. Since U, V solve the polynomial Pell equation and $U(a), V(a) \in \mathbb{N}$ we have an instance of the integer Pell equation $U(a)^2 - (a^2-1)V(a)^2 = 1$. Hence it follows that $U(a) = a_n$ and $V(a) = a'_n$ for some $n \in \mathbb{N}$. Let $f : \mathbb{N} \setminus N \to \mathbb{N}$ be the function taking a to the index $f(a)$ such that $U(a) = a_{f(a)}$ and $V(a) = a'_{f(a)}$.

b) By lemma 1 we have that, for $a > N$:

$$U(a) + \sqrt{a^2-1}(V(a)) = a_{f(a)} + \sqrt{a^2-1}(a'_{f(a)}) = (a + \sqrt{a^2-1})^{f(a)}$$

Hence as $a \to \infty$:

$$f(a) = \frac{\log(U(a) + V(a)\sqrt{a^2-1})}{\log(a + \sqrt{a^2-1})} = \frac{\mathcal{O}(\log a)}{\log a + \mathcal{O}(1)} = \mathcal{O}(1)$$

Hence for some $K \in \mathbb{N}$ for all $a > N$ we have $f(a) < K$.

c) From b) we have that $f : \mathbb{N} \setminus N \to K$. Thus by the pigeonhole principle there exists n with $0 \le n < K$ such that $f(a) = n$ for infinitely many values of a. We know by lemma 2 that $A_n(a) = a_n$ and $B_n(a) = a'_n$. Thus it follows that the polynomials $A_n(T) - U(T)$ and $B_n(T) - V(T)$ have infinitely many roots a_n and a'_n respectively, where $a \in \{a | f(a) = n\}$. Hence $A_n(T) = U(T)$ and $B_n(T) = V(T)$ identically. We may conclude that the sequences $A_n(T)$ and $B_n(T)$ exhaust all solutions to the polynomial Pell equation.

Mark Scheme

1.
   - 1 mark for a correct inductive proof of $B_n(1) = n$
   - 1 mark for a correct inductive proof of $deg(B_{n+1}(T)) = n$
2.
   - 1 mark for showing $X_n(a) = a_n$ and $Y_n(a) = a'_n$
   - 1 mark for concluding from this the identities.
   - Alternatively, 2 marks for a correct inductive proof of the identities.
   - Alternatively, 2 marks for showing Jetze's definition satisfies the recursive definition.
3. a)
   - 1 mark for showing we can assume the lead coefficients positive, hence finding N
   - 1 mark for identifying the particular cases with the integer Pell equation and arguing the existence of f.

   b)
   - 1 mark for finding the identity $U(a) + \sqrt{a^2-1}(V(a)) = (a + \sqrt{a^2-1})^{f(a)}$
   - 1 mark for taking logs and correctly deducing the existence of K from the behaviour $a \to \infty$

   c)
   - 1 mark for correctly arguing (via Pigeonhole principle or otherwise) that for some n $f(a) = n$ for infinitely many values of a.
   - 1 mark for concluding the identity of the polynomials from this fact.