1. First we verify $B_n(1) = n$ for all $n$. We proceed by induction on $n$.

$n = 0$: By definition $B_0(1) = 0$

$n = 1$: Again by definition we have $B_1(1) = 1$

$n + 2 > 1$: $B_{n+2}(1) = 2 \cdot B_{n+1}(1) - B_n(1)$ Applying the inductive hypothesis we obtain $B_{n+2}(1) = 2 \cdot (n + 1) - n = n + 2$ as required.

Next the fact that $\deg(B_{n+1}(T)) = n$ for all $n$. We proceed by induction on $n$.

$n = 0$: $\deg(B_1(T)) = \deg(1) = 0$

$n = 1$: $\deg(B_2(T)) = \deg(2T) = 1$

$n + 2 > 1$: $\deg(B_{n+2}(T)) = \deg(2T \cdot B_{n+1}(T) - B_n(T))$. Applying the inductive hypothesis we have $\deg(B_{n+1}(T)) = n$ and $\deg(B_n(T)) = \max\{0, n - 1\} < n + 1$. Hence $\deg(2T \cdot B_{n+1}(T)) = n + 1$ and so $\deg(B_{n+2}(T)) = n + 1$ as required.

2. We can prove this directly by a tedious induction, but there is a simpler method using the results from the lecture. First recall

**Lemma 1.**

(a) $\forall x, y \in \mathbb{N} : x^2 - (a^2 - 1) y^2 = 1 \iff \exists n(x + \sqrt{a^2 - 1} y = (a + \sqrt{a^2 - 1})^n$

(b) $a_n + \sqrt{a^2 - 1} a'_n = (a + \sqrt{a^2 - 1})^n$

Hence for $a \in \mathbb{N}^+$ we have $X_n(a) = a_n$ and $Y_n(a) = a'_n$. Similarly we used the result

**Lemma 2.** $\forall a \in \mathbb{N}^+ : A_n(a) = a_n \land B_n(a) = a'_n$.

Thus it follows that the polynomials $X_n(T) - A_n(T)$ and $Y_n - B_n(T)$ have infinitely many roots and are therefore both the constant zero polynomial. That is, $X_n(T) = A_n(T)$ and $Y_n(T) = B_n(T)$ as required.

3. a) As in the lemma proved in the lecture, as we take the squares of $U(T)$ and $V(T)$ we can assume without loss of generality that the lead coefficients of both are positive. Hence there exists $N \in \mathbb{N}$ such that for $x > N$ we have $U(x)$ and $V(x)$ positive and strictly increasing. Now let $a > N$ be a natural number. Since $U, V$ solve the polynomial Pell equation and $U(a), V(a) \in \mathbb{N}$ we have an instance of the integer Pell equation $U(a)^2 - (a^2 - 1)V(a)^2 = 1$. Hence it follows that $U(a) = a_n$ and $V(a) = a'_n$ for some $n \in \mathbb{N}$. Let $f : \mathbb{N} \setminus N \to \mathbb{N}$ be the function taking $a$ to the index $f(a)$ such that $U(a) = a_{f(a)}$ and $V(a) = a'_{f(a)}$. 


b) By lemma 1 we have that, for $a > N$:

$$U(a) + \sqrt{a^2 - 1}V(a) = af(a) + \sqrt{a^2 - 1}(a'_f(a)) = (a + \sqrt{a^2 - 1})f(a)$$

Hence as $a \to \infty$:

$$f(a) = \frac{\log(U(a) + V(a)\sqrt{a^2 - 1})}{\log(a + \sqrt{a^2 - 1})} = \Theta(\log a) \Theta(1)$$

Hence for some $K \in \mathbb{N}$ for all $a > N$ we have $f(a) < K$.

c) From b) we have that $f : \mathbb{N} \setminus N \to K$. Thus by the pigeonhole principle there exists $n$ with $0 \leq n < K$ such that $f(a) = n$ for infinitely many values of $a$. We know by lemma 2 that $A_n(a) = a_n$ and $B_n(a) = a'_n$. Thus it follows that the polynomials $A_n(T) - U(T)$ and $B_n(T) - V(T)$ have infinitely many roots $a_n$ and $a'_n$ respectively, where $a \in \{a | f(a) = n\}$. Hence $A_n(T) = U(T)$ and $B_n(T) = V(T)$ identically. We may conclude that the sequences $A_n(T)$ and $B_n(T)$ exhaust all solutions to the polynomial Pell equation.

Mark Scheme

1. • 1 mark for a correct inductive proof of $B_n(1) = n$
   • 1 mark for a correct inductive proof of $\deg(B_{n+1}(T)) = n$

2. • 1 mark for showing $X_n(a) = a_n$ and $Y_n(a) = a'_n$
   • 1 mark for concluding from this the identities.
   • Alternatively, 2 marks for a correct inductive proof of the identities.
   • Alternatively, 2 marks for showing Jetze’s definition satisfies the recursive definition.

3. a) • 1 mark for showing we can assume the lead coefficients positive, hence finding $N$
   • 1 mark for identifying the particular cases with the integer Pell equation and arguing the existence of $f$.

b) • 1 mark for finding the identity $U(a) + \sqrt{a^2 - 1}(V(a)) = (a + \sqrt{a^2 - 1})f(a)$
   • 1 mark for taking logs and correctly deducing the existence of $K$ from the behaviour $a \to \infty$

c) • 1 mark for correctly arguing (via Pigeonhole principle or otherwise) that for some $n$ $f(a) = n$ for infinitely many values of $a$.
   • 1 mark for concluding the identity of the polynomials from this fact.