Seminar Hilbert 10 - Homework 5

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In these exercises, $\mathcal{F}_0$ is the class of functions in real variables represented by expressions combining the variables with integers and the number π by addition, substraction, multiplication and the sin function.

Exercise 1 (4 points) We have shown that there is no algorithm for deciding for an arbitrary $\Phi(\chi_1, \dots, \chi_m) \in \mathcal{F}_0$ whether

$$\exists \chi_1, \dots, \chi_m \Phi(\chi_1, \dots, \chi_m) = 0$$

In this exercise, we'll improve this result by showing that it is undecidable to determine whether

$$\exists \chi_1, \dots, \chi_m \Phi(\chi_1, \dots, \chi_m) < 1$$

for arbitrary $\Phi \in \mathcal{F}_0$. To do this, we need to modify the function

$$D^2(\chi_1, \dots, \chi_m) + \sin^2(\pi\chi_1) + \dots + \sin^2(\pi\chi_m)$$

such that it takes values 0 precisely when the above does, and values > 1 otherwise. Of course, the function still needs to be in $\mathcal{F}_0$.

a) (2 points) For $\chi_1, \dots, \chi_m$ arbitrary real numbers, we denote by $(x_1, \dots, x_m)$ the point with integer coordinates closest to $(\chi_1, \dots, \chi_m)$. The distance between them is denoted by ϵ. Show that there is a polynomial B (computable in D!) such that for all $\chi_1, \dots, \chi_m$:

$$|D^2(\chi_1, \dots, \chi_m) - D^2(x_1, \dots, x_m)| < B(\chi_1, \dots, \chi_m)\epsilon.$$

Hint: Use Taylor's theorem.

Solution: We start by Taylor expanding

$$D^2(\chi_1, \dots, \chi_m) - D^2(x_1, \dots, x_m)$$

with variables $x_1, \dots, x_m$ in the point $(\chi_1, \dots, \chi_m)$:

$$D^2(\chi_1, \dots, \chi_m) - D^2(x_1, \dots, x_m) = 0 - \sum_{\alpha} \frac{\partial^\alpha D^2(x_1, \dots, x_m)}{\partial x^\alpha}(x_i - \chi_i)^{\alpha_i}$$

where the summation is over all multi-indices $\alpha = \alpha_1 \dots \alpha_m$ where $\alpha_i < n$, with n the degree of D^2. Since $|x_i - \chi_i| < \epsilon$, and $\frac{\epsilon}{m} < 1$ we can estimate for $\epsilon > 0$:

$$\begin{aligned} |D^2(\chi_1, \dots, \chi_m) - D^2(x_1, \dots, x_m)| &\le |\sum_{\alpha} \frac{\partial^\alpha D^2(x_1, \dots, x_m)}{\partial x^\alpha}(\chi_1, \dots, \chi_m)(x_i - \chi_i)^{\alpha_i}| \\ &\le |\sum_{\alpha} \frac{\partial^\alpha D^2(x_1, \dots, x_m)}{\partial x^\alpha}(\chi_1, \dots, \chi_m) m^{|\alpha|}|\epsilon \end{aligned}$$

Now we let B be the polynomial

$$B(\chi_1,\ldots,\chi_m) = 1 + \left(\sum_{\alpha} \frac{\partial^{\alpha} D^2(x_1,\ldots,x_m)}{\partial x^{\alpha}}(\chi_1,\ldots,\chi_m) m^{|\alpha|}\right)^2$$

Clearly B can be computed from D, and:

$$B(\chi_1,\ldots,\chi_m) > |\sum_{\alpha} \frac{\partial^{\alpha} D^2(x_1,\ldots,x_m)}{\partial x^{\alpha}}(\chi_1,\ldots,\chi_m) m^{|\alpha|}|$$

Then the above yields:

$$|D^2(\chi_1,\ldots,\chi_m) - D^2(x_1,\ldots,x_m)| < B(\chi_1,\ldots,\chi_m)\epsilon$$

as desired. □

Conclude that for

$$\epsilon < \frac{1}{2B(\chi_1,\ldots,\chi_m)} \tag{1}$$

we have

$$D^2(\chi_1,\ldots,\chi_m) > \frac{1}{2}$$

if $D(x_1,\ldots,x_n) \neq 0$.

b) (1 point) Show that if (1) does not hold, we have:

$$\sin^2(\pi\chi_1) + \ldots + \sin^2(\pi\chi_m) \geq \frac{1}{B^2(\chi_1,\ldots,\chi_m)}.$$

Solution: Recall that:

$$|\sin(\pi\chi)| \geq 2|\chi - x|$$

for all $\chi \in [-\frac{1}{2}+x, \frac{1}{2}+x]$ where x is the nearest integer to χ. Since for all χ for which x is the nearest integer, we have $\chi \in [-\frac{1}{2}+x, \frac{1}{2}+x]$, it follows that:

$$\sin(\pi\chi_i)^2 \geq 4(\chi_i - x_i)^2$$

Therefore

$$\sin(\pi\chi_1)^2 + \ldots + \sin(\pi\chi_m)^2 \geq 4((\chi_1 - x_1)^2 + \ldots + (\chi_m - x_m)^2) = 4\epsilon^2.$$

So if (1) does not hold:

$$\sin(\pi\chi_1)^2 + \ldots + \sin(\pi\chi_m)^2 \geq 4\left(\frac{1}{2B(\chi_1,\ldots,\chi_m)}\right)^2 = \frac{1}{B^2(\chi_1,\ldots,\chi_m)}$$

□

c) (1 point) Conclude that there is no algorithm for deciding for an arbitrary function $\Phi \in \mathcal{F}_0$ whether the inequality

$$\Phi(\chi_1, \dots, \chi_m) < 1.$$

has a solution in real $\chi_1, \dots, \chi_m$.

Solution: Suppose we can. Then we can decide for arbitrary D whether there is a real solution $\chi_1, \dots, \chi_m$ to the inequality:

$$2D^2(\chi_1, \dots, \chi_m) + B^2(\chi_1, \dots, \chi_m)(\sin^2(\pi\chi_1) + \dots + \sin^2(\pi\chi_m)) < 1$$

since B can be computed from D. But by the above estimates, this inequality holds if and only if $D(x_1, \dots, x_m) = 0$ where $(x_1, \dots, x_m) \in \mathbb{Z}^m$ is the closest point to $(\chi_1, \dots, \chi_m)$ with integer coordinates (or one of the closest points). This contradicts the undecidability of Hilberts tenth problem. □

Exercise 2 (6 points) In this exercise, we'll improve the undecidability result about functions in $\mathcal{F}_0$ to the same result about functions in $\mathcal{F}_0$ with only one real variable. The key is to prove that the image of the map

$$\chi \mapsto (\chi \sin(\chi), \chi \sin(\chi^3), \dots, \chi \sin(\chi^{2m-1})$$

lies dense in $\mathbb{R}^m$. For every m, we denote this map by f_m.

We will first prove the case where $m = 2$.

a) (1 point) Let $y_1, y_2, \delta \in \mathbb{R}$ be arbitary real numbers, with $\delta > 0$. Show that there are reals χ_1, χ_2 such that:

(i) $\chi_2 > \chi_1 > |y_2|$

(ii) $\chi_2 \sin(\chi_2) = y_1$

(iii) $\chi_2^3 - \chi_1^3 > 2\pi$

(iv) $(\chi_2 - \chi_1)(\chi_2 + 1) < \delta$.

Hint: Choose an appropriate χ_1 and define $\chi_2 = (\chi_1^2 + \delta/2)^{1/2}$.

Solution: First assume $\chi_1 > |y_2|+1$. Let $\chi_2 = (\chi_1^2 + \frac{\delta}{2})^{\frac{1}{2}}$. Then $\chi_2^3 - \chi_1^3 = \chi_2(\chi_1^2 + \delta/2)) - \chi_2^{2m+1} > \delta\chi_2/2$. Then $(\chi_2 - \chi_1)(\chi_2 + 1) < (\chi_2 - \chi_1)(\chi_2 + \chi_1) = \chi_2^2 - \chi_1^2 = \delta/2 < \delta$.

Now for $\chi_1 > 4\pi/\delta$, χ_1, χ_2 satisfy (i),(iii),(iv). Therefore pick $\chi_1 \in]\max\{4\pi/\delta, |y_m| + 1\}, \infty[$ such that (ii) holds, which can be done by the intermediate value theorem. □

b) (1 point) Let $y_1, y_2, \delta \in \mathbb{R}$ be arbitary real numbers, with $\delta > 0$. Show that there is a χ such that

$|f_1(\chi) - y_1| < \delta$ and ~~$f_2(\chi) = y_2$~~ $\chi \sin(\chi^3) = y_2$.

Conclude that the image of f_2 lies dense in $\mathbb{R}^2$. *Hint:* Choose an appropriate χ between χ_1 and χ_2 in (a) and use the mean value theorem on $f_1(\chi) - y_1 = f_1(\chi) - f_1(\chi_2)$ to make an estimate.

Solution: Pick χ_1, χ_2 such that they satisfy (i)-(iv) above for this y_1, y_2, δ. Since $\chi_2^3 - \chi_1^3 > 2\pi$, there is $\chi_1 < \chi < \chi_2$ such that $\chi \sin(\chi^3) = y_2$ (by the mean value theorem and the fact that $\chi_2 > \chi_1 > |y_2|$). Now I'll show that for this χ, we also have $|\chi \sin(\chi) - y_1| < \delta$. We use that $y_1 = \chi_2 \sin(\chi_2)$, so that:

$$\begin{aligned}
|\chi \sin(\chi) - y_1| &= |\chi \sin(\chi) - \chi_2 \sin(\chi_2)| \\
&\leq (\chi_2 - \chi) \cdot |\frac{d}{dt} t \sin t|_{t=x}| \text{ for some } \chi \leq x \leq \chi_2 \text{, by the Mean Value Theorem} \\
&\leq (\chi_2 - \chi_1) \cdot |x \cos(x) + \sin x| \text{ for some } \chi \leq x \leq \chi_2 \\
&\leq (\chi_2 - \chi_1)(\chi_2 + 1) < \delta
\end{aligned}$$

This finishes the proof. □

c) (1 point) Let $y_1, \ldots, y_m, y_{m+1}\delta \in \mathbb{R}$ be arbitary real numbers, with $\delta > 0$. Suppose f is a function such that $f(]\chi, \infty[)$ lies dense in $\mathbb{R}^m$ for any χ. Show that there are reals χ_1, χ_2 such that:

(i) $\chi_2 > \chi_1 > |y_{m+1}|$

(ii) $|f(\chi_2) - (y_1, \ldots, y_m)| < \delta$

(iii) $\chi_2^{2m+1} - \chi_1^{2m+1} > 2\pi$

(iv) $(\chi_2 - \chi_1)((2m-1)\chi_2^{2m-1} + 1) < \delta$.

Hint: Modify the proof in (a).

Solution: First assume $\chi_1 > |y_m| + 1$. Let $\chi_2 = (\chi_1^{2m} + \frac{\delta}{2(2m-1)})^{\frac{1}{2m}}$. Then $\chi_2^{2m+1} - \chi_1^{2m+1} = \chi_2(\chi_1^{2m} + \delta/(2(2m-1))) - \chi_2^{2m+1} > \delta\chi_2/(2(2m-1))$. Then $(\chi_2 - \chi_1)((2m-1)\chi_2^{2m-1} + 1) < (2m-1)(\chi_2 - \chi_1)(\chi_2^{2m-1} + \chi_1^{2m-1}) \leq 2(2m-1)(\chi_2^{2m} - \chi_1^{2m}) = \delta$.

Now for $\chi_1 > 2(2m-1)\pi/\delta$, χ_1, χ_2 satisfy (i),(iii),(iv). Therefore pick $\chi_1 \in]\max\{2(2m-1)\pi/\delta, |y_m| + 1\}, \infty[$ such that (ii) holds, this can be done by the assumption about f. □

d) (2 points) Prove that the image of f_m lies dense in $\mathbb{R}^m$, for every $m \geq 1$.

Solution: The case for $m = 1$,$m = 2$ follows from the above. We take the inductive hypothesis that $f_m(]\chi, \infty[)$ lies dense in $\mathbb{R}^m$ for any χ. Let $(y_1, \ldots, y_{m+1}) \in \mathbb{R}^{m+1}$ and $\epsilon > 0$ be arbitary. Pick χ_1, χ_2 as in (c) for $f = f_m$, and $\delta = \frac{\epsilon}{2(m-1)}$.

Since $\chi_2^{2m+1} - \chi_1^{2m+1} > 2\pi$, there is a $\chi_1 \leq \chi \leq \chi_2$ such that $\chi \sin(\chi^{2m+1}) = y_m$. Now for any $i \leq m-1$, we have by the mean value theorem:

$$\begin{aligned}
|(f_m(\chi))_i - y_i| &\leq |(f_m(\chi))_i - (f_m(\chi_2))_i| + |f_m(\chi_2)_i - y_i| \\
&\leq (\chi_2 - \chi)((f_m)_i'(x)) + \delta \text{ for some } \chi \leq x \leq \chi_2 \\
&= (\chi_2 - \chi)((2i-1)x^{2i-1} \cos x^{2i-1} + \sin(x^{2i-1})) + \delta \text{ for some } \chi \leq x \leq \chi_2 \\
&\leq (\chi_2 - \chi_1)((2i-1)\chi_2^{2i-1} + 1) \\
&\leq (\chi_2 - \chi_1)((2m-1)\chi_2^{2m-1} + 1) + \delta \\
&< \delta + \delta = 2\delta
\end{aligned}$$

Hence: $|(f_m(\chi) - (y_1, \ldots, y_m)| < 2\delta(m-1) = \epsilon$. Since $(f_{m+1}(\chi))_{m+1} = y_m$, it follows that $|f_{m+1}(\chi) - (y_1, \ldots, y_{m+1})| < \epsilon$.

Since we can pick χ_1, χ_2 arbitrarily large (by the proof of (c)), this proves the induction step. □

e) (1 point) Show, using exercise 1, that there is no algorithm for deciding for an arbitary function $\Psi \in \mathcal{F}_0$ in one real variable whether the equation

$$\Psi(\chi) < 1$$

has a real solution χ.

Solution: Assume we can. Then for an arbitrary $\Psi \in \mathcal{F}_0$ in m variables, we can determine whether

$$\Psi \circ f_m(\chi) < 1 \tag{2}$$

has a real solution χ, since $\Psi \circ f_m \in \mathcal{F}_0$. But $\Psi \circ f_m(\chi) < 1$ has a real solution χ if and only if

$$\Psi(\chi_1, \ldots, \chi_m) < 1$$

has a real solution $\chi_1, \ldots, \chi_m$: Suppose we have such a solution $\chi_1, \ldots, \chi_m$, then there is $\epsilon > 0$ such that:

$$\Psi(\chi_1, \ldots, \chi_m) < 1 - \epsilon.$$

Since Ψ is continuous, there is a $\delta > 0$ such that for all $(\nu_1, \ldots, \nu_m)$ such that

$$|(\nu_1, \ldots, \nu_m) - (\chi_1, \ldots, \chi_m)| < \delta$$

we have $|\Psi(\chi_1, \ldots, \chi_m) - \Psi(\nu_1, \ldots, \nu_m)| < \epsilon/2$. Since the image of f_m is dense, there is χ such that

$$|f_m(\chi) - (\chi_1, \ldots, \chi_m)| < \delta.$$

So for such χ, it follows that $\Psi \circ f_m(\chi) < 1 - \epsilon/2$.

So if we can decide (2), we contradict 1a above. □