Homework set 6 solutions

Hilbert's tenth problem seminar, Fall 2013, due November 4th

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1a: A possible way to define 0 is as $c = 0 \leftrightarrow \forall x : (x \cdot c = c)$. If $c = 0$, the right hand side is trivial. If the right hand side is true, we have for $x = 0$, $0 = 0 \cdot c = x \cdot c = c$.

1b: Take 1a as induction basis, hence 0 is arithmetically definable in terms of S, + and $\cdot$. Assume that we have proven that $n \epsilon \mathbf{Z}$ is arithmetically definable in terms of S and $\cdot$. Then $P(c) = c = S(n)$ and $M(c) = S(c) = n$ give arithmetical definitions of respectively $(n+1)$ and $(n-1)$ in terms of S and n (where n is defined in terms of S and $\cdot$). So by induction, all $n \epsilon \mathbf{Z}$ are arithmetically definable in S and $\cdot$.

1c: Notice that $c(a+b) = c^2 \Leftrightarrow S(a \cdot c) \cdot S(b \cdot c) = S[(c \cdot c) \cdot S(a \cdot b)]$. As long as c is not equal to zero, we have that $c(a+b) = c^2 \Leftrightarrow a + b = c$. If $c = 0$ then $a + b = c$ iff $a + b = 0$ which is if and only if $a \cdot a = b \cdot b$ and either a is not b or $a = 0$. In 1b, we have proven that zero is arithmetically definable in terms of S and $\cdot$. So we can construct a statement defining $a + b = c$ by $A(a, b, c) = [c = 0 \wedge a \cdot a = b \cdot b \wedge (a = 0 \vee \neg(a = b))] \vee [\neg(c = 0) \wedge S(a \cdot c) \cdot S(b \cdot c) = S[(c \cdot c) \cdot S(a \cdot b)]$.

2a: First note that the concept of positive integers (Pos) is arithmetically definable in terms of Int, + and $\cdot$ over rationals: $Pos(a) \leftrightarrow \neg(\forall x : (a \cdot x = a)) \wedge (\exists x, y, z, w : [Int(x) \wedge Int(y) \wedge Int(z) \wedge Int(w) \wedge a = x \cdot x + y \cdot y + z \cdot z + w \cdot w])$.

Then, for any positive denominator a of a rational b, we have that $Pos(a)$ and $Int(a \cdot b)$. Hence, the smallest denominator can be written as:
$a = den(b) \leftrightarrow Pos(a) \wedge Int(a \cdot b) \wedge \forall c : (Pos(c) \wedge Int(c \cdot b) \wedge \neg(c = a)) \rightarrow \exists d : (Pos(d) \wedge a + d = c)$

2b: Note that $a > b$ iff $a \cdot den(a) \cdot den(b) > b \cdot den(a) \cdot den(b)$. Since both sides of the second inequality are an integer we thus have that: $a > b \leftrightarrow \exists c : (Pos(c) \wedge (a \cdot den(a) \cdot den(b) = b \cdot den(a) \cdot den(b) + c))$

2c: $a = \lfloor b \rfloor$ is true if and only if a is an integer and both $a < b \vee a = b$, and $a + 1 > b$ are true. To avoid using 1, we can replace the second statement by $\forall c : Pos(c) \rightarrow a + c > b$. Hence $a = \lfloor b \rfloor \leftrightarrow Int(a) \wedge (a < b \vee a = b) \wedge \forall c : (Pos(c) \rightarrow b < (a + c))$

3: Note, it was given in the exercise that $\mathcal{U}(c) \leftrightarrow c = 1$, hence $\mathcal{U}(c)$ is true for a unique value of c. Denote 1 as the constant defined by $\mathcal{U}$. Secondly, it is needed that the relation Pos is closed under addition, which I forget to demand. People who explained that they needed this second fact and used good arguments to show that without them they couldn't solve the exercises, received all points.

B7: Take $\Phi(c)$ as the statement $\forall a, b : [Pos(a) \wedge Pos(b) \wedge (a + c = b + c)] \rightarrow a = b$. We will use induction to prove that this formula is true. Axiom B2 gives that $\Phi(1)$ is true, which is the induction basis. Assume now that for c we have $\Phi(c)$ is true. Take a and b such that $Pos(a)$ and

$Pos(b)$ are true and such that $a+(c+1) = b+(c+1)$. By B3 we have $a+(c+1) = (a+c)+1$ and $b+(c+1) = (b+c)+1$. So $(a+c)+1 = (b+c)+1$. So by B2 $a+c = b+c$. By induction hypothesis, we have $a = b$. So we can conclude that $a + (c + 1) = b + (c + 1)$ implies $a = b$, so $\Phi(c + 1)$ is true. So by B6 we have that for all c with $Pos(c)$, $\Phi(c)$ is true, which is precisely the statement B7.

B9: Let $\Phi(c)$ be the statement: $\forall a, b : [Pos(a) \wedge Pos(b)] \rightarrow a + (b + c) = (a + b) + c$. B3 tells us $\Phi(1)$ is true. Assume for c where $Pos(c)$ is true, that $\Phi(c)$ is true. Then for a and b with $Pos(a)$ and $Pos(b)$, we have $a + (b + (c + 1)) = a + ((b + c) + 1) = (a + (b + c)) + 1 = ((a + b) + c) + 1 = (a + b) + (c + 1)$ by B3,B3,Induction Hypothesis and B3 respectively. So $\Phi(c + 1)$ is true, hence $\Phi(c)$ implies $\Phi(c + 1)$. So by B6 we have that for all c with $Pos(c)$, $\Phi(c)$ is true.

B8: Let $\Phi(c)$ be the statement: $Pos(c) \rightarrow c + 1 = 1 + c$. We will use induction to prove that this formula is true. Trivially, $\Phi(1)$ is true $(1 + 1 = 1 + 1)$. Now assume that for c with $Pos(c)$ true, that $\Phi(c)$ is true, so $c + 1 = 1 + c$. Hence $(c + 1) + 1 = (1 + c) + 1$ which is by B3 equal to $1 + (c + 1)$. So $\Phi(c + 1)$ is true. By B6 we have that for all c with $Pos(c)$, $\Phi(c)$.
Let $\Psi(a)$ be the statement $\forall c : Pos(c) \rightarrow c + a = a + c$. We have just proven that $\Psi(1)$ is true. Now assume that for c, $Pos(c)$, $\Psi(c)$ is true. So $c + (a + 1) = (c + a) + 1 = (a + c) + 1 = a + (c + 1) = a + (1 + c) = (a + 1) + c$ by B3, induction hypothesis, B3, $\Phi(1)$ and B9 respectively. Hence $\Psi(a + 1)$ is true. So, by B6 we have that for all a with $Pos(a)$, $\Psi(a)$ is true, which is precisely B8.

Points:
1a: 1 point.
1b: 1 point (0.5 for positive integers and 0.5 for negative integers)
1c: 1 point (0.5 for statement and 0.5 for explanation)
2a: 2 points (0.5 for proving that positive integers are arithmetically definable (or ordering of integers is arithmetically definable). 1 for the statement and 0.5 for the explanation)
2b: 1 point (0.5 for statement and 0.5 for explanation)
2c: 1 point (0.5 for statement and 0.5 for explanation)
3a: 1 point
3b: 1 point (0.5 for each induction proof)
3c: 1 point