Topos Theory, Spring 2023
Hand-In Exercises

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February–May 2023

1 Exercises

Exercise 1 (To be handed in March 2, 2023) We consider the poset $\mathbb{N}$ of natural numbers $0 < 1 < 2 < \cdots$, and the category $\widehat{\mathbb{N}}$ of presheaves on $\mathbb{N}$.

a) Show that in $\widehat{\mathbb{N}}$, the terminal object is not projective.

b) Show that in $\widehat{\mathbb{N}}$, if an object F is projective then every restriction map $F(n+1) \to F(n)$ is injective.

c) Show: an object of $\widehat{\mathbb{N}}$ is projective if and only if it is a coproduct of representables.

Exercise 2 (To be handed in March 23, 2023) Given a monomorphism $g : X \to Y$ in $\mathcal{E}$, consider the map $g^* : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ on subobjects, given by pullback along g.

a) Show that g^* has a left adjoint L_g given as follows: if $B \in \mathrm{Sub}(X)$ is represented by a mono $n : B \to X$ then $L_g(B) \in \mathrm{Sub}(Y)$ is represented by the mono $gn : B \to Y$.

b) Show that the map L_g can also be constructed as the composite

$$\mathrm{Sub}(X) \simeq \mathcal{E}(1, \Omega^X) \overset{\mathcal{E}(1,\exists g)}{\longrightarrow} \mathcal{E}(1, \Omega^Y) \simeq \mathrm{Sub}(Y)$$

where the map $\exists g : \Omega^X \to \Omega^Y$ is as defined in the lecture notes, just before Proposition 1.12.

c) In the notation of the pullback diagram in Lemma 1.13, show that the Lemma implies: for any subobject A of Y we have $L_g(f^*A) = k^*(L_h(A))$.

Exercise 3 (To be handed in April 6, 2023) As usual, $\mathcal{E}$ is a topos.

a) Deduce from Proposition 1.28 that every arrow $0 \to X$ in $\mathcal{E}$ is monic.

b) Use propositions 1.28 and 1.29 to show that if $\mathcal{E}^{\mathrm{op}}$ is also a topos, then $\mathcal{E}$ is trivial (i.e., equivalent to the one-arrow category).

c) Let $B \in \mathcal{E}$. The *coslice* $B/\mathcal{E}$ has as objects arrows $B \xrightarrow{f} X$ in $\mathcal{E}$, and as arrows

$$(B \xrightarrow{f} X) \to (B \xrightarrow{g} Y)$$

morphisms $\alpha : X \to Y$ in $\mathcal{E}$ such that $\alpha f = g$.

Prove that such a map α is monic in $B/\mathcal{E}$ if and only if α is monic in $\mathcal{E}$.

d) Use parts a) and c) to conclude that if the unique map $B \to 1$ is not monic, then the coslice $B/\mathcal{E}$ is not a topos.

Exercise 4 (To be handed in April 16, 2023) Let $\mathcal{E}$ be a topos with Lawvere-Tierney topology j. Let $X \in \mathcal{E}$.

a) Show that the diagram

$$\begin{tikzcd} \Omega \times X \arrow[rr, "j \times \mathrm{id}_X"] \arrow[dr, "p_1"'] & & \Omega \times X \arrow[dl, "p_1"] \\ & X & \end{tikzcd}$$

is a Lawvere-Tierney topology in $\mathcal{E}/X$. Call this topology j_X.

b) Let

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "m"'] & & A \arrow[dl, "a"] \\ & X & \end{tikzcd}$$

a mono in $\mathcal{E}/X$. Show that this mono is closed for j_X if and only if the map ϕ factors through Ω_j, where Ω_j is as defined in the proof of Lemma 1.48, and $\phi : A \to \Omega$ is such that the square

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "m"'] & A \arrow[d, "\langle \phi, a \rangle"] \\ X \arrow[r, "\langle t, \mathrm{id} \rangle"'] & \Omega \times X \end{tikzcd}$$

is a pullback.

c) Let $\alpha : X \to Y$ be an arrow in $\mathcal{E}$. Show that if $B \xrightarrow{g} X$ is a sheaf for j_X, then $\prod_\alpha(g)$ is a sheaf for j_Y.

Exercise 5 (To be handed in May 4, 2023) Call a functor $F : \mathcal{A} \to \mathcal{B}$ *flat* if for each object B of $\mathcal{B}$ the functor $\mathcal{B}(B, F(-)) : \mathcal{A} \to \mathrm{Set}$ is flat in the sense of the lecture notes.

a) Prove: if $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ are flat, then the composition $GF : \mathcal{A} \to \mathcal{C}$ is flat as well.

b) For every category $\mathcal{A}$ and object $A \in \mathcal{A}$, the representable functor $\mathcal{A}(A, -) : \mathcal{A} \to \mathrm{Set}$ is flat.

c) Suppose $G : \mathcal{A} \to \mathcal{B}$ has a left adjoint. Then G is flat.

Exercise 6 (To be handed in June1, 2023) In this exercise we discuss Grothendieck topologies on a small category $\mathcal{C}$; if Cov is such, we denote by $L : \widehat{\mathcal{C}} \to \mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ the sheafification functor induced by Cov. We order Grothendieck topologies by saying that $\mathrm{Cov} \leq \mathrm{Cov}'$ if $\mathrm{Cov}(C) \subseteq \mathrm{Cov}'(C)$ for all $C \in \mathcal{C}$.

a) Let R be a subpresheaf of a presheaf F. Show that there is a least Grothendieck topology Cov on $\mathcal{C}$ with the property that the L-image of the inclusion $R \to F$ is an isomorphism.

b) Deduce from part a): for every arrow $f : X \to Y$ there is a least Grothendieck topology Cov on $\mathcal{C}$ with the property that $L(f)$ is an epimorphism.

c) Similar as b), but now with monomorphism instead of epimorphism.

d) Similar as b), but now with isomorphism instead of epimorphism.

2 Solutions

Exercise 1. Let us agree on some notation: for a presheaf X on $\mathbb{N}$ and $m \leq n$ in $\mathbb{N}$, write X_{mn} for the restriction map $X(n) \to X(m)$. Note that since coproducts are calculated pointwise, we have $(X + Y)(0) = X(0) + Y(0)$ (disjoint sum in Set), and so, X is indecomposable if and only if $X(0)$ is a one-element set.
a): Suppose 1 is projective. Just as any presheaf, 1 is covered by a sum of representables: we have an epi $\sum_i X_i \to 1$, where each X_i is representable. This map must have a section, and since 1 is indecomposable, the section must map 1 into one representable X_i. But if X_i is the representable y_n then $X_i(n+1)$ is the empty set whereas $1(n + 1)$ is a singleton. We obtain a contradiction (a function from a nonempty set to the empty set), so 1 is not projective.
b): Suppose we have distinct elements $a, b \in X(n + 1)$ such that $X_{n(n+1)}(a) = X_{n(n+1)}(b)$. Split $X(n + 1)$ into two disjoint sets A, B such that $a \in A$ and $b \in B$. Define subpresheaves X_A and X_B of X: if $n + 1 \leq m$, then $X_A(m)$ consists of all elements of $X(m)$ such that $X_{(n+1)m}(x) \in A$; if $m < n + 1$, then $X_A(m) = X_{m(n+1)}[A]$. The presheaf X_B is defined similarly, with B in the role of A.

Now the two inclusions of X_A and X_B into X give a map from $X_A + X_B$ to X, which is readily seen to be epimorphic. Now for a section s we must have $s_{n+1}(a) \in X_A(n+1)$ and $s_{n+1}(b) \in X_B(n+1)$, but for their common restriction in $X(n)$ we can not find a consistent mapping.

c) A coproduct of projective objects is projective, as is easily seen. Moreover, a sum of objects is projective iff each of these objects is. Therefore, the projective objects are sums of indecomposable projectives; that is (by Proposition 0.7 of the notes) sums of retracts of representable presheaves. I leave it to you to establish that in our case, every retract of a representable is itself representable. So, projective = sum of representables, as claimed.

Exercise 2. a) We have to show, for monos $m : A \to Y$ and $n : B \to X$, that $B \leq g^*(A)$ if and only if $L_g(B) \leq A$, where L_g is as described in the exercise. Recall that if A is classified by $\phi_A : Y \to \Omega$, then $g^*(A)$ is classified by $\phi_A \circ g : X \to \Omega$. We therefore have equivalences:

$$\begin{array}{rcl} B \leq g^*(A) & \Leftrightarrow & \phi_A \circ g \circ n \text{ factors through } t \\ & \Leftrightarrow & g \circ n \text{ factors through } A \\ & \Leftrightarrow & L_g(B) \leq A \end{array}$$

which prove the adjunction.
b) The operation $L_g : \mathrm{Sub}(X) \to \mathrm{Sub}(Y)$ sends a subobject $n : B \to X$ to te composition $B \xrightarrow{n} X \xrightarrow{g} Y$.

We also have the operation

$$\mathrm{Sub}(X) \simeq \mathcal{E}(1, \Omega^X) \xrightarrow{\mathcal{E}(1, \exists g)} \mathcal{E}(1, \Omega^Y) \simeq \mathrm{Sub}(Y)$$

and we wish to prove that these two operations coincide.

The first operation sends $(B \xrightarrow{n} X)$ to a the subobject of Y corresponding to a map $(1 \xrightarrow{\psi} \Omega^Y)$ where the transpose of ψ classifies the composition $B \xrightarrow{n} X \xrightarrow{g} Y$.

The second operation sends $(B \xrightarrow{n} X)$ to the composition

$$1 \xrightarrow{\ulcorner \phi_n \urcorner} \Omega^X \xrightarrow{\exists g} \Omega^Y$$

(where $\phi_n : X \to \Omega$ classifies B) which transposes to

$$Y \xrightarrow{\ulcorner \phi_n \urcorner \times \mathrm{id}} \Omega^X \times Y \xrightarrow{\widetilde{\exists g}} \Omega$$

and it is easily seen that this composition classifies the composition

$$B \xrightarrow{n} X \xrightarrow{g} Y.$$

So indeed the two operations are the same.
c) The lemma asserts that the following diagram commutes:

$$\begin{tikzcd} \Omega^Y \arrow[r, "Pf"] \arrow[d, "\exists h"'] & \Omega^X \arrow[d, "\exists g"] \\ \Omega^T \arrow[r, "Pk"'] & \Omega^Z \end{tikzcd}$$

So it will also commute if we apply the functor $\mathcal{E}(1,-)$ to it, and observe that $\mathcal{E}(1,\Omega^Y) \simeq \mathrm{Sub}(Y)$, $\mathcal{E}(1,Pf) \simeq f^*$ and $\mathcal{E}(1,\exists h) \simeq L_h$.

Exercise 3. a) Suppose we have a diagram $Y \underset{b}{\overset{a}{\rightrightarrows}} 0 \xrightarrow{i} X$ with $ia = ib$. By 1.28, a is an isomorphism. Therefore, Y is initial in $\mathcal{E}$; hence $a = b$. So i is monic.
b) Suppose $\mathcal{E}$ and $\mathcal{E}^{\mathrm{op}}$ are toposes. From 1.28 for $\mathcal{E}^{\mathrm{op}}$ we have that in $\mathcal{E}^{\mathrm{op}}$, every arrow $X \to 0$ is an isomorphism, where 0 is the initial object of $\mathcal{E}^{\mathrm{op}}$. This means that in $\mathcal{E}$, every arrow $1 \to X$ is an isomorphism. In particular, $t : 1 \to \Omega$ is an isomorphism, whence every mono is an isomorphism. Since, by a), every morphism $0 \to X$ is monic, we obtain that every object is initial in $\mathcal{E}$. So $\mathcal{E}$ is trivial.
c) If α is mono in $\mathcal{E}$ and

$$\begin{tikzcd}
 & B \arrow[dl, "h"'] \arrow[dr, "f"] & \\
Y \arrow[rr, shift left, "\beta"] \arrow[rr, shift right, "\gamma"'] & & X
\end{tikzcd}$$

is a parallel pair with $\alpha\beta = \alpha\gamma$ then clearly $\beta = \gamma$, so α is mono in $B/\mathcal{E}$.

Conversely, if α is not mono in $\mathcal{E}$, let $Y \underset{\gamma}{\overset{\beta}{\rightrightarrows}} X$ be a parallel pair with $\alpha\beta = \alpha\gamma$ and $\beta \neq \gamma$. We then have a parallel pair

$$\begin{tikzcd}
 & B \arrow[dl, "\mu"'] \arrow[dr, "f"] \arrow[drr, "g"] & & \\
B+Y \arrow[rr, shift left, "\beta'"] \arrow[rr, shift right, "\gamma'"'] & & X \arrow[r, "\alpha"'] & Y
\end{tikzcd}$$

in $B/\mathcal{E}$, showing that α is not mono in $B/\mathcal{E}$.
d) The initial object of $B/\mathcal{E}$ is the identity arrow on B. For $b : B \to 1$ in $\mathcal{E}$ we have the arrow

$$\begin{tikzcd}
 & B \arrow[dl, "\mathrm{id}"'] \arrow[dr, "b"] & \\
B \arrow[rr, "b"'] & & 1
\end{tikzcd}$$

in $B/\mathcal{E}$ which is, by assumption on B and part c), not mono in $\mathcal{E}$. So not every arrow from the initial object is mono in $B/\mathcal{E}$, which therefore is not a topos.

Exercise 4. a) Let us denote by $X \overset{t_X}{\to} \Omega_X$ the subobject classifier in $\mathcal{E}/X$. So Ω_X is the projection $\Omega \times X \overset{p_1}{\to} X$, which is $X^*(\Omega)$; and t_X is the map $\langle t, \mathrm{id}_X\rangle : X \to \Omega \times X$. Now X^* is a logical functor, and it follows that $X^*(t) = t_X$, that $X^*(\wedge)$ is the map $\wedge_X : \Omega_X \times \Omega_X \to \Omega_X$ in $\mathcal{E}/X$. All equalities are preserved by X^*, so j_X is a Lawvere-Tierney topology in $\mathcal{E}/X$.
b) Pullbacks in $\mathcal{E}/X$ are pullbacks in $\mathcal{E}$, so the given square means that the map $\langle \phi, a\rangle : A \to \Omega_X$ classifies the mono f in $\mathcal{E}/X$. Again since X^* is logical, we

have that $(\Omega_X)_{j_X}$, which is the image of j_X, classifies j_X-closed monos in $\mathcal{E}/X$. So f is closed if and only if $\langle \phi, a\rangle$ factors through $(\Omega_X)_{j_X}$.
c) Suppose $B \xrightarrow{g} X$ is a sheaf for j_X. Recall that we have an adjunction

$$\mathcal{E}/X \underset{\prod_\alpha}{\overset{\alpha^*}{\leftrightarrows}} \mathcal{E}/Y \ , \quad \alpha^* \dashv \prod_\alpha$$

Given a diagram

$$\begin{tikzcd} W \arrow[r, "u"] \arrow[d, "f"'] & Z \\ \prod_\alpha(g) & \end{tikzcd}$$

in $\mathcal{E}/Y$ with u a j_Y-dense mono, we consider its transpose along the adjunction, to get

$$\begin{tikzcd} \alpha^* W \arrow[r, "\alpha^* u"] \arrow[d, "\bar{f}"'] & \alpha^* Z \\ g & \end{tikzcd}$$

in $\mathcal{E}/X$. Since the functor α^* is logical, $\alpha^* u$ is j_X-dense. And because g is a j_X-sheaf, we have a unique filler. Transposing back, we find the unique filler for the original diagram, showing that $\prod_\alpha(g)$ is a j_Y-sheaf.

Exercise 5. a) We need to show that for each object C of $\mathcal{C}$, the functor $\mathcal{C}(C, GF(-))$ is filtered, that is: the category of elements of this functor is a filtered category. We check the conditions of Definition 2.6.

i) We have to show that for some $A \in \mathcal{A}$, $\mathcal{C}(C, GFA)$ is nonempty. Since G is flat, there is a $B \in \mathcal{B}$ and an arrow $f : C \to GB$. Since F is flat, there is an $A \in \mathcal{A}$ and an arrow $g : B \to FA$. Then $G(g)f : C \to GFA$ is an element of $\mathcal{C}(C, GFA)$.

ii) Suppose $(f, A), (f', A')$ are objects of $\mathrm{Elts}(\mathcal{C}(C, GF(-)))$, so $f : C \to GFA$ and $f' : C \to GFA'$. This gives that (f, FA) and (f', GFA') are objects of $\mathrm{Elts}(\mathcal{C}(C, G(-)))$. Since this category is filtered, we find (g, B) with $g : C \to GB$ and arrows $u : B \to FA, v : B \to FA'$ such that $(Gu)g : C \to GFA$ and $(Gv)g : C \to GFA'$ satisfy $(Gu)g = f$ and $(Gv)g = f'$.

   The u and v determine objects (u, A) and (v, A') of $\mathrm{Elts}(\mathcal{B}(B, F(-)))$. By filteredness of this category, we find (w, A'') and morphisms $x : (w, A'') \to (u, A)$ and $y : (w, A") \to (v, A')$ satisfying $(Fx)w = u$ and $(Fy)w = v$. Finally, the pair $((Gw)g, A'') \in \mathrm{Elts}(\mathcal{C}(C, GF(-)))$ satisfies:

$$(f, A) \xleftarrow{x} ((Gw)g, A'') \xrightarrow{y} (f', A')$$

   as desired.

iii) Consider a parallel pair $(x, y) : (f, A) \to (f', A')$ in $\mathrm{Elts}(\mathcal{C}(C, GF(-)))$, so x and y are arrows $A \to A'$ satisfying $(GFx)f = f' = (GFy)f$. This gives morphisms Fx, Fy in $\mathrm{Elts}(\mathcal{C}(C, G(-))$ which form a parallel pair of arrows $(f, FA) \to (f', FA')$. Since $\mathrm{Elts}(\mathcal{C}(C, G(-)))$ is filtered we get an arrow $u : (g, B) \to (f, FA)$ which equalizes Fx, Fy:

$$(g, B) \xrightarrow{u} (f, FA) \underset{Fy}{\overset{Fx}{\rightrightarrows}} (f', FA')$$

Also note that $(Gu)g = f$.

Let $v = (Fx)u = (Fy)u$; we have a parallel pair $x, y : (u, A) \to (v, A')$ in $\mathrm{Elts}(\mathcal{B}(B, F(-)))$. Finally, again exploiting the filteredness of this category we obtain an arrow $z : (w, A'') \to (v, A)$ such that $xz = yz$ and $(Fz)w = u$. Since $(GFz)(Gw)g = (Gu)g = f$, the morphism z is an arrow $((Gw)g, A'') \to (f, A)$ which equalizes the given parallel pair, as desired.

b) We have to show that for an object A of a category $\mathcal{A}$, the representable functor $\mathcal{A}(A, -) : \mathcal{A} \to \mathrm{Set}$ is flat; in other word that the category $\mathrm{Elts}(\mathcal{A}(A, -))$ is filtered. But this category has an initial object (id_A, A) and every category with an initial object is trivially filtered.
[An alternative proof notices that the left Kan extension: $\widehat{\mathcal{A}} \to \mathrm{Set}$ of the given representable is the functor which sends a presheaf X to $X(A)$; and this functor preserves finite limits]
c) By the adjunction, the functor $\mathcal{B}(B, G(-))$ is isomorphic to the functor $\mathcal{A}(FB, -)$ which is representable, hence flat by part b).

Exercise 6. We start by making two remarks which, easy in themselves, might have merited more explicit mention in the notes.
Remark 1. The pointwise intersection of any set of Grothendieck topologies is again a Grothendieck topology. This is immediate from the definition.
Remark 2. Let $m : R \to F$ be a monomorphism, and L the sheafification functor. Then m is dense if and only if $L(m)$ is an isomorphism.
Proof of remark 2: Let i be the inclusion of the category of sheaves, i.e. the right adjoint to L. We also write L for the composite iL.

From the proof of the factorization theorem (Theorem 2.20) we know that, if we write $c_F(R)$ for the closure of R as subobject of F, the following square is a pullback:

$$\begin{tikzcd} c_F(R) \arrow[r] \arrow[d] & L(R) \arrow[d] \\ F \arrow[r, "\eta"'] & L(F) \end{tikzcd}$$

where η is the unit of the adjunction $L \dashv i$. From this it follows that the L-image of the inclusion $c_F(R) \to F$ is (isomorphic to) $L(R) \to L(F)$ (use that $L(\eta)$ is an isomorphism, and that L preserves pullbacks). We now see: if R is dense in F then $c_F(R) \to F$ is an isomorphism, hence $L(R) \to L(F)$ is an isomorphism;

conversely, if $L(R) \to L(F)$ is iso then so is $c_F(R) \to F$, which is to say that $R \to F$ is dense.

Now for the exercise: a) For each $C \in \mathcal{C}$ and each $x \in F(C)$, we have the sieve

$$S_x^C \quad = \quad \{f : C' \to C \mid F(f)(x) \in R(C')\}$$

on C. Clearly, R is dense in F if and only if every S_x^C is in $\mathrm{Cov}(C)$.

So the least Grothendieck topology for which $R \to F$ is dense, is the intersection of all Grothendieck topologies which contain all sieves S_x^C for $C \in \mathcal{C}, x \in F(C)$ (Remark 1).

b) Given $f : X \to Y$ let $X \stackrel{e}{\to} E \stackrel{m}{\to} Y$ be the epi-mono factorization of f. We know that L preserves both monos and epis, hence also such factorizations; and $L(f)$ is epi if and only if $L(m)$ is an isomorphism. By Remark 2, this is the case if and only if $E \stackrel{m}{\to} Y$ is dense. So the required Grothendieck topology is the one we get from part a), applied to the mono $m : E \to Y$.

c) Given $f : X \to Y$, let $K_f \to X \times X$ be the kernel pair of f and $\Delta \to K_f$ the factorization of the diagonal $\delta : X \to X \times X$ through K_f.

Now f is mono if and only if $\Delta \to K_f$ is epi, so the least topology for which $L(f)$ is mono is the least one for which $L(\Delta \to K_f)$ is epi; apply part b).

d) $L(f)$ is an isomorphism if and only if $L(f)$ is both epi and mono. If Cov_e and Cov_m are, respectively, the least Grothendieck topologies for which $L(f)$ is epi (mono), then we want the least topology which contains $\mathrm{Cov}_e \cup \mathrm{Cov}_m$.