Topos Theory, Spring 2024
Hand-In Exercises

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February–May 2024

1 Exercises

Exercise 1 (Deadline: February 29) We consider a small category $\mathcal{C}$ and a monoid M. An *M-presheaf on $\mathcal{C}$* is a presheaf F on $\mathcal{C}$ endowed with, for every object $C \in \mathcal{C}$, a right M-action $F(C) \times M \to F(C)$ (written: $(x, m) \mapsto xm$) which, besides the usual axioms for an M-action, also satisfies: $f^*(xm) = f^*(x)m$ for $f : D \to C$, $x \in F(C)$ and $m \in M$. A morphism of M-presheaves $F \to G$ is a natural transformation $\mu : F \Rightarrow G$ such that $\mu_C(xm) = \mu_C(x)m$ for all $C \in \mathcal{C}$, $x \in F(C)$. Clearly, we have a category $M\text{-}\widehat{\mathcal{C}}$ of M-presheaves and morphisms.

a) Let $\Delta : \widehat{\mathcal{C}} \to M\text{-}\widehat{\mathcal{C}}$ be the functor which endows each presheaf F with the trivial (identity) M-action. Show that Δ has a right adjoint, and describe it explicitly.

b) Show that $M\text{-}\widehat{\mathcal{C}}$ is a topos.

Exercise 2 (Deadline: March 14) Recall the definition (before Proposition 3.14) of the map $\exists_f : \Omega^X \to \Omega^Y$ for any monomorphism $f : X \to Y$.

a) (4 pts) Show that $\exists f$ induces a function $\sum_f : \mathrm{Sub}(X) \to \mathrm{Sub}(Y)$, and describe this function explicitly.

b) (6 pts) Show that for any subobject A of X, the inequality $A \leq f^*(\sum_f(A))$ holds.

Exercise 3 (Deadline: March 28) Let $T : \mathcal{E} \to \mathcal{F}$ be a logical functor between toposes.

a) (4 pts) Let X be an object of $\mathcal{E}$. Show that the functor $T/X : \mathcal{E}/X \to \mathcal{F}/TX$ which sends $(Y \stackrel{f}{\to} X)$ to $(TY \stackrel{Tf}{\to} TX$ (with the straightforward action on morphisms) is logical.

b) (3 pts) Suppose the functor T has a left adjoint F. Show that T/X has a left adjoint.

c) (3 pts) Under the assumption in b), show that T/X has a right adjoint. Can you describe it explicitly?

Exercise 4 (Deadline: April 11) We consider a universal closure operation c on a topos $\mathcal{E}$.

a) (2 pts) Let

$$\begin{array}{ccc} A' & \xrightarrow{f'} & B' \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\ A & \xrightarrow[f]{} & B \end{array}$$

be a commutative square with m a dense mono and n a closed mono. Prove that there is a unique "filler" $g : A \to B'$ (i.e., a map such that $gm = f'$ and $ng = f$).

b) (2 pts) For a subobject A' of A, show that $c_A(A')$ is the unique subobject A'' of A with the property that $A' \to A''$ is dense and $A'' \to A$ is closed.

c) (3 pts) Show that the composition of two dense monos is dense; and the same for closed monos.

d) (3 pts) Show that for $A', A'' \in \mathrm{Sub}(A)$ we have: $c_A(A' \cap A'') = c_A(A') \cap c_A(A'')$. [Hint: one inclusion is clear since c is order-preserving. For the other, show that it suffices to prove that $(A' \cap A'') \to (c_A(A') \cap c_A(A''))$ is dense and $(c_A(A') \cap c_A(A'')) \to A$ is closed.]

Exercise 5 (Deadline: April 25) This exercise is about the Heyting algebra structure of subobject lattices. $\mathrm{Sub}(X)$ denotes the lattice of subobjects of X.

a) Let $i : X \to Y$ be a monomorphism. Prove: if $\mathrm{Sub}(Y)$ is a Boolean algebra, then so is $\mathrm{Sub}(X)$.

b) Let $p : X \to Y$ be an epimorphism. Prove: if $\mathrm{Sub}(X)$ is a Boolean algebra, then so is $\mathrm{Sub}(Y)$.

Exercise 6 (Deadline: May 16) Let L be a one-sorted first-order language and suppose R is an L-structure in a topos $\mathcal{E}$.

a) Construct, for every object X of $\mathcal{E}$, an L-structure R_X on the *set* $\mathcal{E}(X, R)$.

b) Let ϕ be an L-formula with n free variables $x_1, \ldots, x_n$, and suppose the only logical symbols occurring in ϕ are $\wedge$ and $\to$. Prove: for any n-tuple $\alpha_1, \ldots, \alpha_n : X \to R$ the following are equivalent:

   i) $X \Vdash \phi[\alpha_1, \ldots, \alpha_n]$

   ii) For all arrows $f : Y \to X$, $R_Y \models \phi(\alpha_1 f, \ldots, \alpha_n f)$

2 Solutions

Exercise 1 a) Clearly, if $\mu : \Delta(F) \to G$ is any morphism in M-$\widehat{\mathcal{C}}$ then by the definition of such morphisms, for all $C \in \mathcal{C}$, $x \in F(C)$ and $m \in M$ we have $(\mu_C(x))m = \mu_C(x)$, so μ lands in the part of G which is invariant under the M-action. We have a functor from M-$\widehat{\mathcal{C}}$ to $\widehat{\mathcal{C}}$ which sends each M-presheaf to its invariant part. This is right adjoint to Δ, the verification of which is left to you.
b) This is most easily done by observing that M-$\widehat{\mathcal{C}}$ is equivalent to a presheaf category: it is equivalent to the category of presheaves on the product category $\mathcal{C} \times M$.

Exercise 2 a) Elements of $\mathrm{Sub}(X)$ are in 1-1 correspondence with maps $1 \to \Omega^X$: take the exponential transpose of the classifying map.

Define $\sum_f$ as follows: for $A \in \mathrm{Sub}(X)$, corresponding to the map $a : 1 \to \Omega^X$, define $\sum_f(A) \in \mathrm{Sub}(Y)$ as the subobject corresponding to the composition $\exists_f \circ a : 1 \to \Omega^Y$.

One can prove that this is simply the subobject $A \to X \xrightarrow{f} Y$, although it is hard to argue that this operation is *induced by* $\sum_f$.
b) Let $\widetilde{\exists_f}$ be the exponential transpose of $\exists_f$. Then $\sum_f(A)$ is classified by the composition

$$Y \xrightarrow{\langle a, \mathrm{id}_Y \rangle} \Omega^X \times Y \xrightarrow{\widetilde{\exists_f}} \Omega.$$

Then $f^*(\sum_f(A))$ is classified by

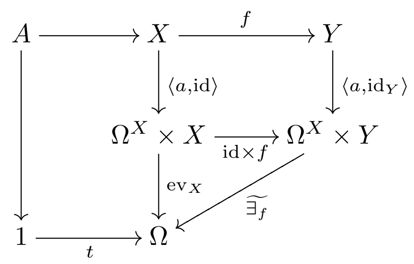
$$X \xrightarrow{f} Y \xrightarrow{\langle a, \mathrm{id}_Y \rangle} \Omega^X \times Y \xrightarrow{\widetilde{\exists_f}} \Omega.$$

and the inequality $A \leq f^*(\sum_f(A))$ holds if and only if the composition

$$A \to X \xrightarrow{f} Y \xrightarrow{\langle a, \mathrm{id}_Y \rangle} \Omega^X \times Y \xrightarrow{\widetilde{\exists_f}} \Omega$$

factors through the subobject classifier $1 \xrightarrow{t} \Omega$.

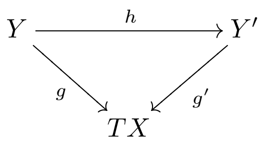
This, however, follows from the commutative diagram:



The lower right-hand triangle commutes by Proposition 3.14, and the left hand square commutes because the composite $\mathrm{ev}_X \circ \langle a, \mathrm{id} \rangle$ equals the transpose of a, that is: the map which classifies A as subobject of X.

Exercise 3 a) Assume $T : \mathcal{E} \to \mathcal{F}$ is logical; this means that T preserves finite limits, subobject classifiers and exponentials. So $T(1 \xrightarrow{t} \Omega)$ is a subobject classifier in $\mathcal{F}$ and moreover, if $\chi_A : X \to \Omega$ classifies the subobject A of X in $\mathcal{E}$, then $T(\chi_A)$ classifies $T(A) \in \mathrm{Sub}(TX)$ in $\mathcal{F}$. It follows that the map $\Delta : X \times X \to \Omega$, which classifies the diagonal $\delta : X \to X \times X$, is preserved by T. Also, T commutes with taking exponents and also with exponential transposes. So, for example, the singleton map $\{\cdot\} : X \to \Omega^X$ is preserved by T. We see that partial map classifiers are preserved by T. It is now a matter of inspection to see that the whole topos structure of $\mathcal{E}/X$ is preserved by T/X. We conclude that T/X is logical.

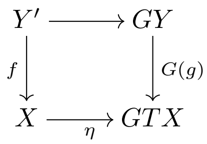
b) Let $F \dashv T$. Define $F^X : \mathcal{F}/TX \to \mathcal{E}/X$ as follows: for an object $(Y \xrightarrow{g} TX)$ of $\mathcal{F}/TX$ let $F^X(g)$ be the map $FY \xrightarrow{\tilde{g}} X$, the transpose of g along the adjunction $F \dashv T$. On morphisms



the image $F^X(h)$ is the map $F(h) : \tilde{g} \to \tilde{g'}$ obtained by transposing. The adjunction is straightforward.

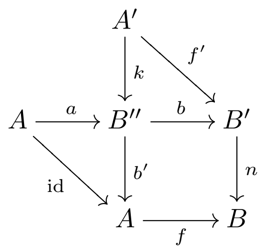
c) The existence of a right adjoint is an immediate application of Corollary 3.20: T/X is logical and has a left adjoint, so it has a right adjoint by 3.20.

In order to exhibit the right adjoint, we use the assumption in b) once more, and conclude that T has a right adjoint. Let $G : \mathcal{F} \to \mathcal{E}$ be right adjoint to T. Define $G_X : \mathcal{F}/TX \to \mathcal{E}/X$ as follows: $G_X(Y \xrightarrow{g} TX)$ is the map $Y' \xrightarrow{f} X$, from the pullback diagram



where $\eta : X \to GTX$ is the unit of the adjunction $T \dashv G$. Again, the adjunction $T/X \dashv G_X$ is left to you.

Exercise 4 a) Commutativity of the square gives that $m \leq f^*(n)$ in $\mathrm{Sub}(A)$, so by the order-preservingness of the closure operation, we have $c_A(m) \leq c_A(f^*(n))$. Now $c_A(m) = \mathrm{id}_A$ since m is dense, and $c_A(f^*(n)) = f^*(c_X(n)) = f^*(n)$ by stability of closure and the assumption that n is closed. The resulting inequality $\mathrm{id}_A \leq f^*(n)$ in $\mathrm{Sub}(A)$ gives us a commutative diagram

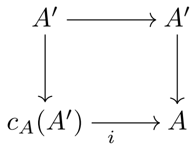


where the square is a pullback, and $m : A' \to A$ is the composite $b'k$. Note that $b'ab' = b'$ hence $ab' = \mathrm{id}$ since b' is mono. Therefore the map $g = ba : A \to B'$ satisfies the stated equalities.

b) We have inclusions $A' \to c_A(A') \to A$. Clearly, the second one is closed. To see that the first one is dense we must prove the equality

$$c_{c_A(A')}(A') = c_A(A').$$

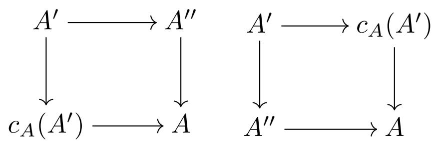
Consider the pullback



The desired equality is now clear from:

$$c_{c_A(A')}(A') = c_{c_A(A')}(i^*A') = i^*(c_A(A')) = c_A(A')$$

Now, we need to see that $c_A(A')$ is unique with the stated property. So assume A'' is such that $A' \to A''$ is dense and $A'' \to A$ is closed. We have commutative diagrams



which in turn, by applying part a), yield $c_A(A') \leq A''$ and $A'' \leq c_A(A')$. We conclude that $A'' = c_A(A')$.

c) Let $N \to M \to X$ be subobjects. First assume both inclusions are dense; we show that $N \to X$ is dense. Let $i : M \to X$ the inclusion. We have:

$$c_X(N) \cap M = i^*(c_X(N)) = c_M(i^*N) = c_M(M \cap N) = c_M(N)$$

Now since $N \to M$ is dense we have

$$M = c_M(N) = c_X(N) \cap M,$$

so $M \subseteq c_X(N)$. Since c is order preserving and idempotent we have $X = c_X(M) \subseteq c_X(c_X(N)) = c_X(N)$, giving that $N \to X$ is dense as required.

Now assume both inclusions are closed. We have (as used before) $c_X(N) \cap M = c_M(N) = N$, so we have a pullback:

$$\begin{array}{ccc} N & \longrightarrow & M \\ \big\downarrow & & \big\downarrow \\ c_X(N) & \longrightarrow & X \end{array}$$

Since $M \to X$ is closed, $N \to c_X(N)$ is closed. But $N \to c_X(N)$ is also dense. We conclude $N = c_X(N)$.

d) First a little remark: if $A \stackrel{i}{\to} B \stackrel{j}{\to} X$ is a composition of monos and $c_X(A) = B$, then i is dense. Indeed,

$$c_B(A) = c_X(A) \cap B = j^*(c_X(A)) = B.$$

For the proof that $c_A(A' \cap A'') = c_A(A') \cap c_A(A'')$, we prove:

d1) the map $A' \cap A'' \to c_A(A') \cap c_A(A'')$ is dense;

d2) the map $c_A(A') \cap c_A(A'') \to A$ is closed.

Let us first see that this is enough. We first show that statements d1) and d2) also hold for $c_A(A' \cap A'')$ in the place of $c_A(A') \cap c_A(A'')$, so that by the uniqueness of part b) we will be done after proving d1) and d2).

We have that $c_A(A' \cap A'') \to A$ is clearly closed, and $c_A(A' \cap A'') \to c_A(A') \cap c_A(A'')$ is dense by the little remark, since

$$\begin{array}{rclcl} c_{c_A(A'')}(c_A(A') \cap c_A(A'')) & = & c_{c_A(A'')}((j')^*(c_A(A'))) & = & \\ (j')^* c_A(c_A(A')) & = & (j')^* c_A(A') & = & c_A(A') \cap c_A(A'') \end{array}$$

(where j' is the mono $c_A(A'') \to A$).

Now for the proof of d1) and d2).

d1): this arrow is a composite of $A' \cap A'' \to c_A(A') \cap A'' \to c_A(A') \cap c_A(A'')$. Let j be the mono $A'' \to A$. Then

$$c_{A''}(A' \cap A'') = c_{A''}(j^*(A')) = j^*(c_A(A')) = c_A(A') \cap A''$$

so the first arrow in the composite is dense; the second one is dense because it is a pullback (intersection with $c_A(A')$) of the dense map $A'' \to c_A(A'')$. By part c) we conclude that d1) has been proved.

For d2), we split this as $c_A(A') \cap c_A(A'') \to c_A(A'') \to A$. For the first of these arrows, we have (again, let j' be the arrow $c_A(A'') \to A$):

$$c_A(A') \cap c_A(A'') = (j')^*(c_A(A')) = c_{c_A(A'')}((j')^*(A'))$$

We see that $c_A(A') \cap c_A(A'')$ is closed in $c_A(A'')$. The second arrow of the composite is trivially closed, so (invoking once again part c)) we are done.