Topos Theory

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Preface

These lecture notes were written during a *Mastermath* (Dutch national programme for master-level courses in mathematics) course, taught in the fall of 2018.

The main sources I used are:

1) My course notes *Basic Category Theory and Topos Theory* ([8]), material for the lecture course to which the present course is a sequel. Referred to in the text as the *Basic Course*.

2) MacLane's *Categories for the Working Mathematician* ([5]). Referred to as "MacLane".

3) Peter Johnstone's *Topos Theory* ([3]). This is referred to in the text by **PTJ**.

4) MacLane and Moerdijk's *Sheaves in Geometry and Logic* ([6]). Referred to by **MM**.

5) Francis Borceux's *Handbook of Categorical Algebra* ([1]).

6) Peter Johnstone's *Sketches of an Elephant* ([4]). Referred to by **Elephant**.

7) Moerdijk's *Classifying Spaces and Classifying Topoi* ([7]).

8) Olivia Caramello's *Theories, Sites, Toposes* ([2]).

9) Jaap van Oosten's *Realizability: an Introduction to its Categorical Side* ([9]).

There is no original material in the text, except for a few exercises and some proofs.

0.1 The plural of the word "topos"

Everyone knows the quip at the end of the Introduction of [3], which asks *those toposophers who persist in talking about topoi whether, when they go out for a ramble on a cold day, they carry supplies of hot tea with them in thermoi*. Since then, everyone has to declare what, in his or her view, is the plural of "topos". The form "topoi", of course, is the plural of the ancient Greek word for "place". However, Topology is not the science of places, and the name Topology is what inspired Grothendieck to introduce the word Topos.

Someone (I forget who) proposed: the word “topos” is *French*, and its plural is “topos”. True, but English has adopted many French words, which are then treated as English words. The French plural of “bus” is “bus”, but in English it is “buses”.

I stick with “toposes”.

Contents

Introduction

After the definition of a topos, we discuss some standard terminology and fix notation. In the second section, for motivation we exhibit the elementary notions at work in the example of presheaves over a small category. Subsequent sections treat (succinctly) sheaves on a topological space and sheaves on a site. Section 0.6 reviews a few miscellaneous elements of category theory that we shall need and that are not always covered in a basic course.

0.1 Definition and notations

If $\mathcal{C}$ denotes a category, we write $\mathcal{C}_0$ and $\mathcal{C}_1$ for the collection of objects of $\mathcal{C}$ and the collection of arrows of $\mathcal{C}$, respectively. For objects X, Y of $\mathcal{C}$ we write $\mathcal{C}(X, Y)$ for the collection of arrows of $\mathcal{C}$ with domain X and codomain Y. Such an arrow f is indicated by $f : X \to Y$ or $X \xrightarrow{f} Y$. By id or id_X we denote the identity arrow on object X.

Definition 0.1 An *elementary topos*, or *topos* for short, is a category with finite limits which is cartesian closed and has a *subobject classifier*. A subobject classifier is a monic arrow $t : T \to \Omega$ such that every monomorphism is a pullback of t in a unique way: for every mono $m : X \to Y$ there is a unique arrow $\chi_m : Y \to \Omega$ (the *classifying map*, or *characteristic map* of m) such that there is a pullback diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "m"'] & T \arrow[d, "t"] \\ Y \arrow[r, "\chi_m"'] & \Omega \end{tikzcd}.$$

Since pullbacks are only defined up to isomorphism, the pullback of t along χ_m in the diagram of 0.1 can be any monic arrow into Y which represents the same subobject as m; and if n is any mono into Y, with characteristic map χ_n, then $\chi_m = \chi_n$ if and only if m and n represent the same subobject of Y.

In Set, any two element set $\{a, b\}$ together with a specific choice of one of them, say b (considered as arrow $1 \to \{a, b\}$) acts as a subobject classifier: for $A \subset B$ we have the unique characteristic function $\phi_A : B \to \{a, b\}$ defined by $\phi_A(x) = b$ if $x \in A$, and $\phi_A(x) = a$ otherwise.

It is no coincidence that in Set, the domain of $t : T \to \Omega$ is a terminal object: T is always terminal. Indeed, for any object A the arrow $\phi : A \to \Omega$

which classifies the identity on A factors as tn for some $n : A \to T$. So there always is a morphism $A \to T$. Moreover, if $k : A \to T$ is any arrow, then we have pullback diagrams

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, "\mathrm{id}_A"'] & T \arrow[r, "\mathrm{id}_T"] \arrow[d, "\mathrm{id}_T"] & T \arrow[d, "t"] \\ A \arrow[r, "k"'] & T \arrow[r, "t"'] & \Omega \end{tikzcd}$$

so tk classifies id_A. By uniqueness of the classifying map, $tn = tk$; since t is mono, $n = k$. So T is terminal. Henceforth we shall write $1 \xrightarrow{t} \Omega$ for the subobject classifier, or, by abuse of language, just Ω.

We see that Definition 0.1 consists of three requirements, and each of these has its own notations, so let us deal with that first.

Finite limits: if $A \leftarrow C \to B$ is a product cone we shall write $A \times B$ for C. Given arrows $f : D \to A$ and $g : D \to B$ we let $\langle f, g \rangle : D \to A \times B$ be the unique factorization through the product cone. The projections are written $p_0 : A \times B \to A$ and $p_1 : A \times B \to B$. We write 1 for the terminal object and we sometimes use $!_X$ to denote the unique map $X \to 1$. The *diagonal* $\langle \mathrm{id}, \mathrm{id} \rangle : X \to X \times X$ is often denoted δ or δ_X; I also use δ to refer to the subobject of $X \times X$ represented by this map.

If the square

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d] \\ A \arrow[r] & X \end{tikzcd}$$

is a pullback, we shall often indicate this by writing $A \times_X B$ for C, and p_0, p_1 for f and g, respectively. For any map $f : Y \to X$ and subobject A of X, we have a well-defined subobject $f^*(A)$ of Y, which is obtained by taking a pullback along f of any mono representing A.

For any arrow $f : Y \to X$ we have a mono $\langle \mathrm{id}, f \rangle : Y \to Y \times X$; the subobject of $Y \times X$ this represents, is called the *graph* of f, $\mathrm{graph}(f)$.

Exercise 1 If $f, g : Y \to X$ are arrows and $\mathrm{graph}(f) = \mathrm{graph}(g)$, then $f = g$.

Subobject classifier: for an object X, we let $\Delta : X \times X \to \Omega$ be the map which classifies the diagonal δ_X, and we write $\{\cdot\} : X \to \Omega^X$ for the exponential transpose of Δ. We call $\{\cdot\}$ the *singleton map*. Anticipating the treatment of logic in toposes, we think of Ω^X as the "object of subobjects of X".

Cartesian closure: for any object X, the natural map $\Omega^X \times X \to \Omega$ (the component at Ω of the counit of the exponential adjunction) is denoted ev_X; the subobject of $\Omega^X \times X$ it classifies, is denoted by $\in_X$; we think of it as (the converse of) the *element relation*.

A type of argument one frequently encounters is based on the uniqueness of classifying maps: if f and g are maps $X \to \Omega$ and their pullbacks along $1 \xrightarrow{t} \Omega$ give the same subobject of X, then $f = g$.

Exercise 2 Show, using Exercise 1, that the singleton map is always monic.

Exercise 3 Let $f : Y \to X$ be a map.

a) Show that the maps

$$X \times Y \xrightarrow{\{\cdot\}\times\mathrm{id}} \Omega^X \times Y \xrightarrow{\mathrm{id}\times f} \Omega^X \times X \xrightarrow{\mathrm{ev}_X} \Omega$$

and

$$X \times Y \xrightarrow{\mathrm{id}\times f} X \times X \xrightarrow{\Delta} \Omega$$

are equal.

b) Let $Pf : \Omega^X \to \Omega^Y$ be the exponential transpose of the map

$$\Omega^X \times Y \xrightarrow{\mathrm{id}\times f} \Omega^X \times X \xrightarrow{\mathrm{ev}_X} \Omega\ \cdot$$

Show that the exponential transpose of the map

$$X \xrightarrow{\{\cdot\}} \Omega^X \xrightarrow{Pf} \Omega^Y$$

is the map

$$Y \times X \xrightarrow{f\times\mathrm{id}} X \times X \xrightarrow{\Delta} \Omega\ \cdot$$

0.2 Presheaf categories

We review the category $\widehat{\mathcal{C}} = \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ of contravariant functors from $\mathcal{C}$ to Set. $\mathcal{C}$ is assumed to be a small category throughout. Objects of $\widehat{\mathcal{C}}$ are called *presheaves on* $\mathcal{C}$.

We have the Yoneda embedding $y : \mathcal{C} \to \widehat{\mathcal{C}}$; we write its effect on objects C and arrows f as y_C, y_f respectively. So for $f : C \to D$ we have $y_f : y_C \to y_D$. Recall: $y_C(C') = \mathcal{C}(C', C)$, the set of arrows $C' \to C$ in $\mathcal{C}$; for $\alpha : C'' \to C'$ we have $y_C(\alpha) : y_C(C') \to y_C(C'')$ which is defined by composition with α, so $y_C(\alpha)(g) = g\alpha$ for $g : C' \to C$. For $f : C \to D$ we have $y_f : y_C \to y_D$ which is a natural transformation with components

$$(y_f)_{C'} : y_C(C') \to y_D(C')$$

given by $(y_f)_{C'}(g) = fg$. Note, that the naturality of y_f is just the associativity of composition in $\mathcal{C}$.

Presheaves of the form y_C are called *representable*.

The *Yoneda Lemma* says that there is a 1-1 correspondence between elements of $X(C)$ and arrows in $\widehat{\mathcal{C}}$ from y_C to X, for presheaves X and objects C of $\mathcal{C}$, and this correspondence is natural in both X and C. To every element $x \in X(C)$ corresponds a natural transformation $\mu : y_C \to X$ such that $(\mu)_C(\mathrm{id}_C) = x$; and natural transformations from y_C are completely determined by their effect on id_C. An important consequence of the Yoneda lemma is that the Yoneda embedding is actually an embedding, that is: full and faithful, and injective on objects.

Examples of presheaf categories

1. A first example is the category of presheaves on a monoid (a one-object category) M. Such a presheaf is nothing but a set X together with a right M-*action*, that is: we have a map $X \times M \to X$, written $x, f \mapsto xf$, satisfying $xe = x$ (for the unit e of the monoid), and $(xf)g = x(fg)$. There is only one representable presheaf.

2. If the category C is a poset $(P, \leq)$, for $p \in P$ we have the representable y_p with $y_p(q) = \{*\}$ if $q \leq p$, and $\emptyset$ otherwise. So we can identify the representable y_p with the *downset* $\downarrow(p) = \{q \,|\, q \leq p\}$.

3. The category of directed graphs and graph morphisms is a presheaf category: it is the category of presheaves on the category with two objects e and v, and two non-identity arrows $\sigma, \tau : v \to e$. For a presheaf X on this category, $X(v)$ can be seen as the set of vertices, $X(e)$ the set of edges, and $X(\sigma), X(\tau) : X(e) \to X(v)$ as the source and target maps.

4. A *tree* is a partially ordered set T with a least element, such that for any $x \in T$, the set $\downarrow(x) = \{y \in T \,|\, y \leq x\}$ is a finite linearly ordered subset of T. A *morphism* of trees $f : T \to S$ is an order-preserving function wth the property that for any element $x \in T$, the restriction of f to $\downarrow(x)$ is a bijection from $\downarrow(x)$ to $\downarrow(f(x))$. A *forest* is a set of trees; a map of forests $X \to Y$ is a function $\phi : X \to Y$ together with

an X-indexed collection $(f_x \,|\, x \in X)$ of morphisms of trees such that $f_x : x \to \phi(x)$. The category of forests and their maps is just the category of presheaves on ω, the first infinite ordinal.

Recall the definition of the category $y{\downarrow}X$ (an example of a 'comma category' construction): objects are pairs (C, μ) with C an object of $\mathcal{C}$ and $\mu : y_C \to X$ an arrow in $\widehat{\mathcal{C}}$. A morphism $(C, \mu) \to (C', \nu)$ is an arrow $f : C \to C'$ in $\mathcal{C}$ such that the triangle

$$\begin{tikzcd} y_C \arrow[rr, "y_f"] \arrow[dr, "\mu"'] & & y_{C'} \arrow[dl, "\nu"] \\ & X & \end{tikzcd}$$

commutes.

Note that if this is the case and $\mu : y_C \to X$ corresponds to $\xi \in X(C)$ and $\nu : y_{C'} \to X$ corresponds to $\eta \in X(C')$, then $\xi = X(f)(\eta)$.

There is a functor $U_X : y{\downarrow}X \to \mathcal{C}$ (the forgetful functor) which sends (C, μ) to C and f to itself; by composition with y we get a diagram

$$y{\circ}U_X : y{\downarrow}X \to \widehat{\mathcal{C}}$$

Clearly, there is a natural transformation ρ from $y{\circ}U_X$ to the constant functor Δ_X from $y{\downarrow}X$ to $\widehat{\mathcal{C}}$ with value X: let $\rho_{(C,\mu)} = \mu : y_C \to X$. So there is a cocone in $\widehat{\mathcal{C}}$ for $y{\circ}U_X$ with vertex X.

Proposition 0.2 *The cocone $\rho : y{\circ}U_X \Rightarrow \Delta_X$ is colimiting.*

Proof. Suppose $\lambda : y{\circ}U_X \Rightarrow \Delta_Y$ is another cocone. Define $\nu : X \to Y$ by $\nu_C(\xi) = (\lambda_{(C,\mu)})_C(\mathrm{id}_C)$, where $\mu : y_C \to X$ corresponds to ξ in the Yoneda Lemma.

Then ν is natural: if $f : C' \to C$ in $\mathcal{C}$ and $\mu' : y_{C'} \to X$ corresponds to $X(f)(\xi)$, the diagram

$$\begin{tikzcd} y_{C'} \arrow[rr, "y_f"] \arrow[dr, "\mu'"'] & & y_C \arrow[dl, "\mu"] \\ & X & \end{tikzcd}$$

commutes, so f is an arrow $(C', \mu') \to (C, \mu)$ in $y{\downarrow}X$. Since λ is a cocone, we have that

$$\begin{tikzcd} y_{C'} \arrow[rr, "y_f"] \arrow[dr, "\lambda_{(C',\mu')}"'] & & y_C \arrow[dl, "\lambda_{(C,\mu)}"] \\ & Y & \end{tikzcd}$$

commutes; so

$$\begin{array}{lcrc} \nu_{C'}(X(f)(\xi)) & = & (\lambda_{(C',\mu')})_{C'}(\mathrm{id}_{C'}) & = \\ (\lambda_{(C,\mu)})_{C'}((y_f)_{C'}(\mathrm{id}_{C'})) & = & (\lambda_{(C,\mu)})_{C'}(f) & = \\ Y(f)((\lambda_{(C,\mu)})_C(\mathrm{id}_C)) & = & Y(f)(\nu_C(\xi)) & \end{array}$$

It is easy to see that $\lambda : y{\circ}U_X \Rightarrow \Delta_Y$ factors through ρ via ν, and that the factorization is unique. ∎

Proposition 0.2 is often referred to by saying that "every presheaf is a colimit of representables".

Let us note that the category $\widehat{\mathcal{C}}$ is complete and cocomplete, and that limits and colimits are calculated 'pointwise': if I is a small category and $F : I \to \widehat{\mathcal{C}}$ is a diagram, then for every object C of $\mathcal{C}$ we have a diagram $F_C : I \to \mathit{Set}$ by $F_C(i) = F(i)(C)$; if X_C is a colimit for this diagram in Set, there is a unique presheaf structure on the collection $(X_C \,|\, C \in \mathcal{C}_0)$ making it into the vertex of a colimit for F. The same holds for limits. Some immediate consequences of this are:

i) An arrow $\mu : X \to Y$ in $\widehat{\mathcal{C}}$ is mono (resp. epi) if and only if every component μ_C is an injective (resp. surjective) function of sets.

ii) The category $\widehat{\mathcal{C}}$ is regular, and every epimorphism is a regular epi.

iii) The initial object of $\widehat{\mathcal{C}}$ is the constant presheaf with value $\emptyset$.

iv) An object X is terminal in $\widehat{\mathcal{C}}$ if and only if every set $X(C)$ is a singleton.

v) for every presheaf X, the functor $(-) \times X : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$ preserves colimits.

Furthermore we note the following fact: the Yoneda embedding $\mathcal{C} \to \widehat{\mathcal{C}}$ is the 'free colimit completion' of $\mathcal{C}$. That is: for any functor $F : \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is a cocomplete category, there is, up to isomorphism, exactly one *colimit preserving* functor $\tilde{F} : \widehat{\mathcal{C}} \to \mathcal{D}$ such that the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "y"'] & & \mathcal{D} \\ & \widehat{\mathcal{C}} \arrow[ur, "\tilde{F}"'] & \end{tikzcd}$$

commutes. $\tilde{F}(X)$ is computed as the colimit in $\mathcal{D}$ of the diagram

$$y{\downarrow}X \stackrel{U_X}{\to} \mathcal{C} \stackrel{F}{\to} \mathcal{D}$$

The functor $\tilde{F}$ is also called the 'left Kan extension of F along y'.

We shall now calculate explicitly some structure of $\widehat{\mathcal{C}}$. Exponentials can be calculated using the Yoneda Lemma and proposition 0.2. For Y^X, we need a natural 1-1 correspondence

$$\widehat{\mathcal{C}}(Z, Y^X) \simeq \widehat{\mathcal{C}}(Z \times X, Y)$$

In particular this should hold for representable presheaves y_C; so, by the Yoneda Lemma, we should have a 1-1 correspondence

$$Y^X(C) \simeq \widehat{\mathcal{C}}(y_C \times X, Y)$$

which is natural in C. This leads us to *define* a presheaf Y^X by: $Y^X(C) = \widehat{\mathcal{C}}(y_C \times X, Y)$, and for $f : C' \to C$ we let $Y^X(f) : Y^X(C) \to Y^X(C')$ be defined by composition with $y_f \times \mathrm{id}_X : y_{C'} \times X \to y_C \times X$. Then certainly, Y^X is a well-defined presheaf and for representable presheaves we have the natural bijection $\widehat{\mathcal{C}}(y_C, Y^X) \simeq \widehat{\mathcal{C}}(y_C \times X, Y)$ we want. In order to show that it holds for arbitrary presheaves Z we use proposition 0.2. Given Z, we have the diagram $y \circ U_Z : y{\downarrow}Z \to \mathcal{C} \to \widehat{\mathcal{C}}$ of which Z is a colimit. Therefore arrows $Z \to Y^X$ correspond to cocones on $y \circ U_Z$ with vertex Y^X. Since we have our correspondence for representables y_C, such cocones correspond to cocones on the diagram

$$y{\downarrow}Z \stackrel{U_Z}{\to} \mathcal{C} \stackrel{y}{\to} \widehat{\mathcal{C}} \stackrel{(-)\times X}{\to} \widehat{\mathcal{C}}$$

with vertex Y. Because, as already noted, the functor $(-) \times X$ preserves colimits, these correspond to arrows $Z \times X \to Y$, as desired.

It is easy to see that the construction of Y^X gives a functor $(-)^X : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$ which is right adjoint to $(-) \times X$, thus establishing that $\widehat{\mathcal{C}}$ is cartesian closed. The *evaluation map* $\mathrm{ev}_{X,Y} : Y^X \times X \to Y$ is given by

$$(\phi, x) \ \mapsto \ \phi_C(\mathrm{id}_C, x)$$

Exercise 4 Show that the map $\mathrm{ev}_{X,Y}$, thus defined, is indeed a natural transformation.

Exercise 5 Prove that $y : \mathcal{C} \to \widehat{\mathcal{C}}$ preserves all limits which exist in $\mathcal{C}$. Prove also, that if $\mathcal{C}$ is cartesian closed, y preserves exponents.

First a remark about subobjects in $\widehat{\mathcal{C}}$. A subobject of X can be identified with a *subpresheaf* of X: that is, a presheaf Y such that $Y(C) \subseteq X(C)$ for each C, and $Y(f)$ is the restriction of $X(f)$ to $Y(\mathrm{cod}(f))$. This follows easily from epi-mono factorizations pointwise, and the corresponding fact in Set.

Again, we use the Yoneda Lemma to compute the subobject classifier in $\widehat{\mathcal{C}}$. We need a presheaf Ω such that at least for each representable presheaf y_C, $\Omega(C)$ is in 1-1 correspondence with the set of subobjects (in $\widehat{\mathcal{C}}$) of y_C. So we define Ω such that $\Omega(C)$ is the set of subpresheaves of y_C; for $f : C' \to C$ we have $\Omega(f)$ defined by the action of pulling back along y_f.

What do subpresheaves of y_C look like? If R is a subpresheaf of y_C then R can be seen as a set of arrows with codomain C such that if $f : C' \to C$ is in R and $g : C'' \to C'$ is arbitrary, then fg is in R (for, $fg = y_C(g)(f)$). Such a set of arrows is called a *sieve* on C.

Under the correspondence between subobjects of y_C and sieves on C, the operation of pulling back a subobject along a map y_f (for $f : C' \to C$) sends a sieve R on C to the sieve $f^*(R)$ on C' defined by

$$f^*(R) \ = \ \{g : D \to C' \mid fg \in R\}$$

So Ω can be defined as follows: $\Omega(C)$ is the set of sieves on C, and $\Omega(f)(R) = f^*(R)$. The map $t : 1 \to \Omega$ sends, for each C, the unique element of $1(C)$ to the maximal sieve on C (i.e., the unique sieve which contains id_C).

Exercise 6 Suppose $\mathcal{C}$ is a preorder $(P, \leq)$. For $p \in P$ we let $\downarrow(p) = \{q \in P \mid q \leq p\}$. Show that sieves on p can be identified with downwards closed subsets of $\downarrow(p)$. If we denote the unique arrow $q \to p$ by qp and U is a downwards closed subset of $\downarrow(p)$, what is $(qp)^*(U)$?

Let us now prove that $t : 1 \to \Omega$, thus defined, is a subobject classifier in $\widehat{\mathcal{C}}$. Let Y be a subpresheaf of X. Then for any C and any $x \in X(C)$, the set

$$\phi_C(x) \ = \ \{f : D \to C \mid X(f)(x) \in Y(D)\}$$

is a sieve on C, and defining $\phi : X \to \Omega$ in this way gives a natural transformation: for $f : C' \to C$ we have

$$\begin{array}{rcr}
\phi_{C'}(X(f)(x)) & = & \{g : D \to C' \mid X(g)(X(f)(x)) \in Y(D)\} \\
 & = & \{g : D \to C' \mid X(gf)(x) \in Y(D)\} \\
 & = & \{g : D \to C' \mid fg \in \phi_C(x)\} \\
 & = & f^*(\phi_C(x)) \\
 & = & \Omega(f)(\phi_C(x))
\end{array}$$

Moreover, if we take the pullback of t along ϕ, we get the subpresheaf of X consisting of (at each object C) of those elements x for which $\mathrm{id}_C \in \phi_C(x)$; that is, we get Y. So ϕ classifies the subpresheaf Y.

On the other hand, if $\phi : X \to \Omega$ is any natural transformation such that pulling back t along ϕ gives Y, then for every $x \in X(C)$ we have that $x \in Y(C)$ if and only if $\mathrm{id}_C \in \phi_C(x)$. But then by naturality we get for any $f : C' \to C$ that

$$X(f)(x) \in Y(C') \;\Leftrightarrow\; \mathrm{id}_{C'} \in f^*(\phi_C(x)) \;\Leftrightarrow\; f \in \phi_C(x)$$

which shows that the classifying map ϕ is unique. We have proved the following theorem.

Theorem 0.3 *For any small category $\mathcal{C}$ the presheaf category $\widehat{\mathcal{C}}$ is a topos.*

Remark 0.4 Later on, in Example 1.35 we shall see a more general proof of this fact. A proof, moreover, which does not use any "colimits of representables". However, the proof as given here has the benefit of giving the topos structure explicitly.

Combining the subobject classifier with the cartesian closed structure, we obtain *power objects*. In a category $\mathcal{E}$ with finite products, we call an object A a *power object* of the object X, if there is a natural 1-1 correspondence

$$\mathcal{E}(Y, A) \simeq \mathrm{Sub}_{\mathcal{E}}(Y \times X)$$

The naturality means that if $f : Y \to A$ and $g : Z \to Y$ are arrows in $\mathcal{E}$ and f corresponds to the subobject U of $Y \times X$, then $fg : Z \to A$ corresponds to the subobject $(g \times \mathrm{id}_X)^*(U)$ of $Z \times X$.

Power objects are unique up to isomorphism; the power object of X, if it exists, is usually denoted $\mathcal{P}(X)$. Note the following consequence of the definition: to the identity map on $\mathcal{P}(X)$ corresponds a subobject of $\mathcal{P}(X) \times X$ which we call the "element relation" $\in_X$; it has the property that whenever $f : Y \to \mathcal{P}(X)$ corresponds to the subobject U of $Y \times X$, then $U = (f \times \mathrm{id}_X)^*(\in_X)$.

Convince yourself that power objects in the category Set are just the familiar power sets.

In a cartesian closed category with subobject classifier Ω, power objects exist: let $\mathcal{P}(X) = \Omega^X$. Clearly, the defining 1-1 correspondence is there.

$$\mathcal{P}(X)(C) \;=\; \mathrm{Sub}(y_C \times X)$$

with action $\mathcal{P}(X)(f)(U) = (y_f \times \mathrm{id}_X)^\sharp(U)$.

Exercise 7 Show that $\mathcal{P}(X)(C) = \mathrm{Sub}(y_C \times X)$ and that, for $f : C' \to C$, $\mathcal{P}(X)(f)(U) = (y_f \times \mathrm{id}_X)^*(U)$. Prove also, that the element relation, as a subpresheaf $\in_X$ of $\mathcal{P}(X) \times X$, is given by

$$(\in_X)(C) = \{(U, x) \in \mathrm{Sub}(y_C \times X) \times X(C) \mid (\mathrm{id}_C, x) \in U(C)\}$$

Exercise 8 Let $\mathcal{E}$ be a topos with subobject classifier $1 \xrightarrow{t} \Omega$. Recall that an object C of a category $\mathcal{C}$ is called *injective* if any diagram

$$\begin{tikzcd} N & \\ M \arrow[u, "m"] \arrow[r, "f"'] & C \end{tikzcd}$$

with m mono, admits an extension by an arrow $g : N \to C$ satisfying $gm = f$.

a) Prove that Ω is injective.

b) Prove that every object of the form Ω^X is injective.

c) Conclude that $\mathcal{E}$ has enough injectives.

0.2.1 Recovering the category from its presheaves?

In this short section we shall see to what extent the category $\widehat{\mathcal{C}}$ determines $\mathcal{C}$. In other words, suppose $\widehat{\mathcal{C}}$ and $\mathrm{Set}^{\mathcal{D}^{\mathrm{op}}}$ are equivalent categories; what can we say about $\mathcal{C}$ and $\mathcal{D}$?

Definition 0.5 In a regular category an object P is called (regular) *projective* if for every regular epi $f : A \to B$, any arrow $P \to B$ factors through f. Equivalently, every regular epi with codomain P has a section.

Exercise 9 Prove the equivalence claimed in definiton 0.5.

Definition 0.6 An object X is called *indecomposable* if whenever X is a coproduct $\coprod_i U_i$, then for *exactly* one i the object U_i is not initial.

Note, that an initial object is not indecomposable, just as 1 is not a prime number.

In $\widehat{\mathcal{C}}$, coproducts are *stable*, which means that they are preserved by pullback functors; this is easy to check. Another triviality is that the initial object is *strict*: the only maps into it are isomorphisms.

Proposition 0.7 *In $\widehat{\mathcal{C}}$, a presheaf X is indecomposable and projective if and only if it is a retract of a representable presheaf: there is a diagram $X \stackrel{i}{\to} y_C \stackrel{r}{\to} X$ with $ri = \mathrm{id}_X$.*

Proof. Check yourself that every retract of a projective object is again projective. Similarly, a retract of an indecomposable object is indecomposable: if $X \stackrel{i}{\to} Y \stackrel{r}{\to} X$ is such that $ri = \mathrm{id}_X$ and Y is indecomposable, any presentation of X as a coproduct $\coprod_i U_i$ can be pulled back along r to produce, by stability of coproducts, a presentation of Y as coproduct $\coprod_i V_i$ such that

$$\begin{tikzcd} V_i \arrow[r] \arrow[d] & Y \arrow[d, "r"] \\ U_i \arrow[r] & X \end{tikzcd}$$

is a pullback; for exactly one i then, V_i is non-initial; hence since r is epi and the initial object is strict, for exactly one i we have that U_i is non-initial. We see that the property of being projective and indecomposable is inherited by retracts. Moreover, every representable is indecomposable and projective, as we leave for you to check.

Conversely, assume X is indecomposable and projective. By proposition 0.2 and the standard construction of colimits from coproducts and coequalizers, there is an epi $\coprod_i y_{C_i} \to X$ from a coproduct of representables. Since X is projective, this epi has a section ι. Pulling back along ι we get a presentation of X as a coproduct $\coprod_i V_i$ such that

$$\begin{tikzcd} V_i \arrow[r] \arrow[d] & X \arrow[d, "\iota"] \\ y_{C_i} \arrow[r] & \coprod_i y_{C_i} \end{tikzcd}$$

is a pullback diagram. X was assumed indecomposable, so exactly one V_i is non-initial. But this means that X is a retract of y_{C_i}. ∎

If X is a retract of y_C, say $X \stackrel{\mu}{\to} y_C \stackrel{\nu}{\to} X$ with $\nu\mu = \mathrm{id}_X$, consider $\mu\nu : y_C \to y_C$. This arrow is *idempotent*: $(\mu\nu)(\mu\nu) = \mu(\nu\mu)\nu = \mu\nu$, and since the Yoneda embedding is full and faithful, $\mu\nu = y_e$ for an idempotent $e : C \to C$ in $\mathcal{C}$.

A category $\mathcal{C}$ is said to be *Cauchy complete* if for every idempotent $e : C \to C$ there is a diagram $D \stackrel{i}{\to} C \stackrel{r}{\to} D$ with $ri = \mathrm{id}_D$ and $ir = e$. One also says: "idempotents split". In the situation above (where X is a retract of y_C) we see that X must then be isomorphic to y_D for a retract D of C in $\mathcal{C}$. We conclude:

Theorem 0.8 *If $\mathcal{C}$ is Cauchy complete, $\mathcal{C}$ is equivalent to the full subcategory of $\widehat{\mathcal{C}}$ on the indecomposable projectives. Hence if $\mathcal{C}$ and $\mathcal{D}$ are Cauchy complete and $\widehat{\mathcal{C}}$ and $\mathrm{Set}^{\mathcal{D}^{\mathrm{op}}}$ are equivalent, so are $\mathcal{C}$ and $\mathcal{D}$.*

Exercise 10 Show that if $\mathcal{C}$ has equalizers, $\mathcal{C}$ is Cauchy complete.

0.3 Sheaves on Spaces

Given a topological space X with set of opens $\mathcal{O}_X$, we view $\mathcal{O}_X$ as a (posetal) category, and form the topos $\widehat{\mathcal{O}_X}$ of *presheaves on* X (as it is usually called). For an open $U \subseteq X$, a sieve on U can be identified with a set S of open subsets of U which is *downwards closed*: if $V \subseteq W \subseteq U$ and $W \in S$, then also $V \in S$.

Let F be a presheaf on X; an element $s \in F(U)$ is called a *local section of F at U*. For the action of F on local sections, that is: $F(V \subseteq U)(s) \in F(V)$ (where V is a subset of U and the unique morphism from V to U is denoted by the inclusion), we write $s{\restriction}V$.

Definition 0.9 A presheaf F on X is called a *sheaf* if the following holds: whenever $(U_i)_{i\in I}$ is a collection of open subsets of X with union $V = \bigcup_{i\in I} U_i$ and $(x_i)_{i\in I}$ is an I-indexed collection such that $x_i \in F(U_i)$ for all $i \in I$ and moreover, the x_i are *compatible*, that is: $x_i{\restriction}(U_i \cap U_j) = x_j{\restriction}(U_i \cap U_j)$ for every pair (i,j) of elements of I, then there exists a unique *amalgamation* of the family $(x_i)_{i\in I}$, which is an element $x \in F(V)$ such that $x{\restriction}U_i = x_i$ for all $i \in I$.

Now let F be a presheaf on the space X and x a point of X. We consider an equivalence relation on the set $\{(s,U)\,|\,x \in U, s \in F(U)\}$ of local sections defined at x, by stipulating: $(s,U) \sim_x (t,V)$ iff there is some neighbourhood W of x such that $W \subseteq U \cap V$ and $s{\restriction}W = t{\restriction}W$. An equivalence class $[(s,U)]$ is called a *germ at x* and is denoted s_x; the set of all germs at x is G_x, the *stalk of x*.

Define a topology on the disjoint union $\coprod_{x\in X} G_x$ of all the stalks: a basic open set is of the form

$$\mathcal{O}_s^U \;=\; \{(y, s_y)\,|\,y \in U\}$$

for $U \in \mathcal{O}_X$ and $s \in F(U)$. This is indeed a basis: suppose $(x,g) \in \mathcal{O}_s^U \cap \mathcal{O}_t^V$. then $g = s_x = t_x$, so there is a neighbourhood W of x such that $W \subseteq V \cap U$

and $s{\restriction}W = t{\restriction}W$. We see that

$$(x,g) \in \mathcal{O}^W_{s\restriction W} \subseteq \mathcal{O}^U_s \cap \mathcal{O}^V_t.$$

We have a map $\pi : \coprod_{x\in X} G_x \to X$, sending (x,g) to x. If $U \in \mathcal{O}_X$ and $(x,g) = (x,s_x) \in \pi^{-1}(U)$ then $s \in F(V)$ for some neighbourhood V of x; we see that $(x,s_x) \in \mathcal{O}^{U\cap V}_s \subseteq \pi^{-1}(U)$, and the map π is continuous. Moreover, $\pi(\mathcal{O}^U_s) = U$, so π is also an open map.

The map π has another important property. Let $(x,g) = (x,s_x) \in \coprod_{x\in X} G_x$. Fix some U such that $x \in U$ and $s \in F(U)$. The restriction of the map π to $\mathcal{O}^U_s$ gives a bijection from $\mathcal{O}^U_s$ to U. Since this bijection is also continuous and open, it is a homeomorphism. We conclude that every element of $\coprod_{x\in X} G_x$ has a neighbourhood such that the restriction of the map π to that neighbourhood is a homeomorphism. Such maps of topological spaces are called *local homeomorphisms*, or *étale maps*.

Let Top denote the category of topological spaces and continuous functions. For a space X let Top/X be the slice category of maps into X, and let $\mathrm{Et}(X)$ be the full subcategory of Top/X on the local homeomorphisms into X. We have the following theorem in sheaf theory:

Theorem 0.10 *The categories* $\mathrm{Et}(X)$ *and* $\mathrm{Sh}(X)$ *are equivalent.*

Proof. [Outline] For an étale map $p : Y \to X$, define a presheaf $\mathcal{F}$ on X by putting:

$$\mathcal{F}(U) \;=\; \{s : U \to Y \mid s \text{ continuous and } ps = \mathrm{id}_U\}.$$

This explains the terminology *local sections*. Then $\mathcal{F}$ is a sheaf on X. Conversely, given a sheaf F on X, define the corresponding étale map as the map $\pi : \coprod_{x\in X} G_x \to X$ constructed above. These two operations are, up to isomorphism in the respective categories, each other's inverse. ■

Exercise 11 For a nonempty set A, let F_A be the following presheaf on the real numbers $\mathbb{R}$:

$$F_A(U) \;=\; \begin{cases} A & \text{if } 0 \in U \\ \{*\} & \text{else} \end{cases}$$

Show that F_A is a sheaf, and give a concrete presentation of the étale space corresponding to F_A.

Definition 0.11 Let F be a presheaf on the space X and G a sheaf on X. Suppose that $\tau : F \to G$ is a morphism of presheaves with the following property: every morphism $\sigma : F \to H$ from F into a sheaf H factors uniquely as $\tilde{\sigma}\tau$ for a map $\tilde{\sigma} : G \to H$. In this case we call G (or, more precisely, the arrow $\tau : F \to G$) the *associated sheaf* of F.

Exercise 12 Show that for a presheaf F and the associated local homeomorphism $\pi : \coprod_{x\in X} G_x \to X$ that we have constructed, the following holds: every morphism of presheaves $F \to H$, where H is a sheaf, factors uniquely through the sheaf corresponding to $\pi : \coprod_{x\in X} G_x \to X$. Conclude that $\pi : \coprod_{x\in X} G_x \to X$ is the associated sheaf of F. Conclude that the inclusion of categories $\mathrm{Sh}(X) \to \widehat{\mathcal{O}_X}$ has a left adjoint.

Exercise 13 Show that the category $\mathrm{Sh}(X)$ is closed under finite limits in $\widehat{\mathcal{O}_X}$, and that the left adjoint of Exercise 12 preserves finite limits.

We shall see later (Example 1.49) that the category $\mathrm{Sh}(X)$ is a topos.

Next, let us consider the effect of continuous maps on categories of sheaves. First of all, given a continuous map $\phi : Y \to X$ we have the inverse image map $\phi^{-1} : \mathcal{O}_X \to \mathcal{O}_Y$ and hence a functor

$$\phi_* \;=\; \mathrm{Set}^{(\phi^{-1})^{\mathrm{op}}} : \widehat{\mathcal{O}_Y} \to \widehat{\mathcal{O}_X}$$

and the functor ϕ_* restricts to a functor $\mathrm{Sh}(Y) \to \mathrm{Sh}(X)$.

There is also a functor in the other direction: given a sheaf F on X, let $\mathcal{F} \to X$ be the corresponding étale map. It is easy to verify that étale maps are stable under pullback, so if

$$\begin{tikzcd}
\mathcal{G} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\
Y \arrow[r, "\phi"'] & X
\end{tikzcd}$$

is a pullback diagram in Top, let $\phi^*(F)$ be the sheaf on Y which corresponds to the local homeomorphism $\mathcal{G} \to Y$. This defines a functor $\mathrm{Sh}(X) \to \mathrm{Sh}(Y)$.

Proposition 0.12 *We have an adjunction $\phi^* \dashv \phi_*$; moreover, the left adjoint ϕ^* preserves finite limits.*

Definition 0.13 Let $\mathcal{E}$ and $\mathcal{F}$ be toposes. A *geometric morphism*: $\mathcal{F} \to \mathcal{E}$ consists of functors $f_* : \mathcal{F} \to \mathcal{E}$ and $f^* : \mathcal{E} \to \mathcal{F}$ satisfying: $f^* \dashv f_*$ and f^* preserves finite limits. The functor f_* is called the *direct image functor* of the geometric morphism, and f^* the *inverse image functor*.

It is clear that Definition 0.13 gives us a category $\mathcal{T}op$ of toposes and geometric morphisms, and if we believe for the moment that $\mathrm{Sh}(X)$ is always a topos, the treatment of categories of sheaves on spaces shows that we have a functor Top $\to \mathcal{T}op$ from topological spaces to toposes. This functor allows us to relate topological properties of a space to category-theoretic properties of its associated topos of sheaves.

Other examples of geometric morphisms we shall meet during this course, are:

i) Any functor $F : \mathcal{C} \to \mathcal{D}$ between small categories gives rise to a geometric morphism $\widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$.

ii) If $\mathcal{E}$ is a topos and X is an object of $\mathcal{E}$, then the slice category $\mathcal{E}/X$ is a topos; and if $f : X \to Y$ is an arrow in $\mathcal{E}$ then we will have a geometric morphism $\mathcal{E}/X \to \mathcal{E}/Y$.

iii) If $\mathcal{E}$ is a topos and $H : \mathcal{E} \to \mathcal{E}$ is a finite-limit preserving comonad on $\mathcal{E}$, then the category $\mathcal{E}_H$ of coalgebras for H in $\mathcal{E}$ is a topos, and there is a geometric morphism $\mathcal{E} \to \mathcal{E}_H$.

There is another important notion of "morphism between toposes": logical functors.

Definition 0.14 A *logical functor* between toposes is a functor which preserves the topos structure, that is: finite limits, exponentials and the subobject classifier.

Example 0.15 Let $\mathcal{G}$ be a group. In the topos $\widehat{\mathcal{G}}$ of right $\mathcal{G}$-sets we have:

i) the subobject classifier $1 \xrightarrow{t} \Omega$ is the map from $\{*\}$ to $\{0,1\}$ which sends $*$ to 1; here $\{0,1\}$ has the trivial $\mathcal{G}$-action.

ii) The exponent Y^X of two $\mathcal{G}$-sets X and Y is the set of *all* functions $X \xrightarrow{\phi} Y$, with $\mathcal{G}$-action:

$$(\phi{\cdot}g)(x) \;=\; (\phi(x{\cdot}g^{-1})){\cdot}g$$

We see at once that the forgetful functor $\widehat{\mathcal{G}} \to$ Set is logical, as is the functor Set $\to \widehat{\mathcal{G}}$ which sends a set X to the set X with trivial $\mathcal{G}$-action.

We can also consider the category $\mathrm{Set}_{\mathrm{f}}^{\mathcal{G}^{\mathrm{op}}}$ of *finite* $\mathcal{G}$-sets; and we see that this is also a topos (even if $\mathcal{G}$ itself is not finite); the inclusion functor $\mathrm{Set}_{\mathrm{f}}^{\mathcal{G}^{\mathrm{op}}} \to \widehat{\mathcal{G}}$ is logical.

0.4 Sheaves on a Site

In this section, we give a generalization of the notion "sheaves on a space". When we generalize from a topological space to an arbitrary small category, we see that what we need is the notion of a 'cover'. Because $\mathcal{C}$ is in general not a preorder, it will not do to define a 'cover of an object C' as a collection of *objects* (as in the case of $\mathcal{O}(X)$); rather, a cover of C will be a *sieve* on C.

Definition 0.16 Let $\mathcal{C}$ be a category. A *Grothendieck topology* on $\mathcal{C}$ specifies, for every object C of $\mathcal{C}$, a family $\mathrm{Cov}(C)$ of 'covering sieves' on C, in such a way that the following conditions are satisfied:

i) The maximal sieve on C, $\max(C)$, is an element of $\mathrm{Cov}(C)$

ii) If $R \in \mathrm{Cov}(C)$ then for every $f : C' \to C$, $f^*(R) \in \mathrm{Cov}(C')$

iii) If R is a sieve on C and S is a covering sieve on C, such that for every arrow $f : C' \to C$ from S we have $f^*(R) \in \mathrm{Cov}(C')$, then $R \in \mathrm{Cov}(C)$

We note an immediate consequence of the definition:

Proposition 0.17 *a) If $R \in \mathrm{Cov}(C)$, S a sieve on C and $R \subseteq S$, then $S \in \mathrm{Cov}(C)$;*

b) If $R, S \in \mathrm{Cov}(C)$ then $R \cap S \in \mathrm{Cov}(C)$

Proof. For a), just observe that for every $f \in R$, $f^*(S) = \max(C')$; apply i) and iii) of 0.16. For b), note that if $f \in R$ then $f^*(S) = f^*(R \cap S)$, and apply ii) and iii). ∎

Definition 0.18 A *universal closure operation* on $\widehat{\mathcal{C}}$ assigns to every presheaf X an operation $\overline{(\cdot)} : \mathrm{Sub}(X) \to \mathrm{Sub}(X)$ such that the following hold:

i) $A \leq \bar{A}$

ii) $\bar{A} = \bar{\bar{A}}$

iii) $A \leq B \Rightarrow \bar{A} \leq \bar{B}$

iv) For $\phi : Y \to X$ and $A \in \mathrm{Sub}(X)$, $\phi^*(\bar{A}) = \overline{\phi^*(A)}$

Every Grothendieck topology on $\mathcal{C}$ determines a universal closure operation on $\widehat{\mathcal{C}}$ (and vice versa; I defer the proof of this to a later section). Given a Grothendieck topology Cov on $\mathcal{C}$, define $J : \Omega \to \Omega$ by

$$J_C(R) = \{h : C' \to C \mid h^*(R) \in \mathrm{Cov}(C')\}.$$

Then define the operation $\bar{(\cdot)} : \mathrm{Sub}(X) \to \mathrm{Sub}(X)$ as follows: if $A \in \mathrm{Sub}(X)$ is classified by $\phi : X \to \Omega$ then $\bar{A}$ is classified by $J\phi$. So

$$\bar{A}(C) = \{x \in X(C) \mid J_C(\phi_C(x)) = \max(C)\}$$

Definition 0.19 Let Cov be a Grothendieck topology on $\mathcal{C}$, and $\bar{(\cdot)}$ the associated universal closure operation on $\widehat{\mathcal{C}}$.

A presheaf F is *separated* for Cov if for each $C \in \mathcal{C}_0$ and $x, y \in F(C)$, if the sieve $\{f : C' \to C \mid F(f)(x) = F(f)(y)\}$ covers C, then $x = y$.

A subpresheaf G of F is *closed* if $\bar{G} = G$ in $\mathrm{Sub}(F)$.

A subpresheaf G of F is *dense* if $\bar{G} = F$ in $\mathrm{Sub}(F)$.

Definition 0.20 Let F be a presheaf, C an object of $\mathcal{C}$. A *compatible family* in F at C is a family $(x_f \mid f \in R)$ indexed by a sieve R on C, of elements $x_f \in F(\mathrm{dom}(f))$, such that for $f : C' \to C$ in R and $g : C'' \to C'$ arbitrary, $x_{fg} = F(g)(x_f)$. In other words, a compatible family is an arrow $R \to F$ in $\widehat{\mathcal{C}}$. An *amalgamation* of such a compatible family is an element x of $F(C)$ such that $x_f = F(f)(x)$ for all $f \in R$. In other words, an amalgamation is an extension of the map $R \to F$ to a map $y_C \to F$.

Exercise 14 F is separated if and only if each compatible family in F, indexed by a covering sieve, has *at most one* amalgamation.

Definition 0.21 F is a *sheaf* if every compatible family in F, indexed by a covering sieve, has *exactly one* amalgamation.

Exercise 15 Suppose G is a subpresheaf of F. If G is a sheaf, then G is closed in $\mathrm{Sub}(F)$. Conversely, every closed subpresheaf of a sheaf is a sheaf.

Example 0.22 Let Y be a presheaf. Define a presheaf Z as follows: $Z(C)$ consists of all pairs (R, ϕ) such that $R \in \mathrm{Cov}(C)$ and $\phi : R \to Y$ is an arrow in $\widehat{\mathcal{C}}$. If $f : C' \to C$ then $Z(f)(R, \phi) = (f^*(R), \phi f')$ where f' is such that

$$\begin{tikzcd} f^*(R) \arrow[r, "f'"] \arrow[d] & R \arrow[d] \\ y_{C'} \arrow[r, "y_f"'] & y_C \end{tikzcd}$$

is a pullback.

Suppose we have a compatible family in Z, indexed by a covering sieve S on C. So for each $f \in S$, $f : C' \to C$ there is $R_f \in \mathrm{Cov}(C')$, $\phi_f : R_f \to Y$, such that for $g : C'' \to C'$ we have that $R_{fg} = g^*(R_f)$ and $\phi_{fg} : R_{fg} \to Y$ is $\phi_f g'$ where $g' : R_{fg} \to R_f$ is the pullback of $y_g : y_{C''} \to y_{C'}$.

Then this family has an amalgamation in Z: define $T \in \mathrm{Cov}(C)$ by $T = \{fg \mid f \in S, g \in R_f\}$. T is covering since for every $f \in S$ we have $R_f \subseteq f^*(T)$. We can define $\chi : T \to Y$ by $\chi(fg) = \phi_f(g)$. So the presheaf Z satisfies the 'existence' part of the amalgamation condition for a sheaf. It does not in general satisfy the uniqueness part.

Exercise 16 Prove that F is a sheaf if and only if for every presheaf X and every dense subpresheaf A of X, any arrow $A \to F$ has a unique extension to an arrow $X \to F$.

We denote the full subcategory of $\widehat{\mathcal{C}}$ on the sheaves for Cov by $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$. The pair $(\mathcal{C}, \mathrm{Cov})$, with Cov a Grothendieck topology on the small category $\mathcal{C}$, is called a *site*, and one also talks about "sheaves on the site $(\mathcal{C}, \mathrm{Cov})$" as the objects of $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$.

We shall see later that the inclusion from $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ into $\widehat{\mathcal{C}}$ has a left adjoint (also called "sheafification" or "the associated sheaf functor"). We shall also see that this left adjoint preserves finite limits, and that $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ is a topos. This type of toposes is very important, as there is a lot of theory based on the underlying sites.

Definition 0.23 A *Grothendieck topos* is a topos of sheaves on a site; that is, of the form $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$.

0.5 Examples of Grothendieck topologies

1. As always, there are the two trivial extremes. The smallest Grothendieck topology (corresponding to the maximal subcategory of sheaves) has $\mathrm{Cov}(C)$ equal to $\{\max(C)\}$ for all C. The only dense subpresheaves are the maximal ones; every presheaf is a sheaf.

2. The other extreme is the biggest Grothendieck topology: $\mathrm{Cov}(C) = \Omega(C)$. Every subpresheaf is dense; the only sheaf is the terminal object 1.

3. Let X be a topological space with set of opens $\mathcal{O}(X)$, regarded as a category: a poset under the inclusion order. A sieve on an open set U can be identified with a downwards closed collection R of open

subsets of U. The standard Grothendieck topology has $R \in \mathrm{Cov}(U)$ iff $\bigcup R = U$. Sheaves for this Grothendieck topology coincide with the familiar sheaves on the space X.

4. The *dense* or $\neg\neg$-topology is defined by:

$$\mathrm{Cov}(C) \;=\; \{R \in \Omega(C) \,|\, \forall f : C' \to C \exists g : C'' \to C' \,(fg \in R)\}$$

This topology corresponds to the Lawvere-Tierney topology $J : \Omega \to \Omega$ defined by

$$J_C(R) \;=\; \{h : C' \to C \,|\, \forall f : C'' \to C' \exists g : C''' \to C'' \,(hfg \in R)\}$$

This topology has the property that for every sheaf F, the collection of subsheaves of F forms a Boolean algebra.

5. For this example we assume that in the category $\mathcal{C}$, every pair of arrows with common codomain fits into a commutative square. Then the *atomic* topology takes all *nonempty* sieves as covers. This corresponds to the Lawvere-Tierney topology

$$J_C(R) \;=\; \{h : C' \to C \,|\, \exists f : C'' \to C' \,(hf \in R)\}$$

This topology has the property that for every sheaf F, the collection of subsheaves of F forms an *atomic* Boolean algebra: an *atom* in a Boolean algebra is a minimal non-bottom element. An atomic Boolean algebra is such that for every non-bottom x, there is an atom which is $\leq x$.

6. Let U be a subpresheaf of the terminal presheaf 1. With U we can associate a set of objects $\tilde{U}$ of $\mathcal{C}$ such that whenever $f : C' \to C$ is an arrow and $C \in \tilde{U}$, then $C' \in \tilde{U}$. Namely, $\tilde{U} = \{C \,|\, U(C) \neq \emptyset\}$. To such U corresponds a Grothendieck topology, the *open topology* determined by U, given by

$$\mathrm{Cov}(C) \;=\; \{R \in \Omega(C) \,|\, \forall f : C' \to C \,(C' \in \tilde{U} \Rightarrow f \in R)\}$$

and associated Lawvere-Tierney topology

$$J_C(R) \;=\; \{h : C' \to C \,|\, \forall f : C'' \to C' \,(C'' \in \tilde{U} \Rightarrow hf \in R)\}$$

Let $\mathcal{D}$ be the full subcategory of $\mathcal{C}$ on the objects in $\tilde{U}$. Then there is an equivalence of categories between $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ and $\mathrm{Set}^{\mathcal{D}^{\mathrm{op}}}$.

7. For U and $\tilde{U}$ as in the previous example, there is also the *closed* topology determined by U, given by

$$\mathrm{Cov}(C) \;=\; \{R \in \Omega(C) \,|\, C \in \tilde{U} \text{ or } R = \max(C)\}$$

   There is an equivalence between $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ and the category of presheaves on the full subcategory of $\mathcal{C}$ on the objects *not* in $\tilde{U}$.

0.6 Notions from Category Theory

First, let us deal with a subtlety which arises in basic Category Theory courses. In MacLane's book ([5]) for example, a functor $F : \mathcal{C} \to \mathcal{D}$ is said to *create limits of type* J if for every diagram $M : J \to \mathcal{C}$ and every limiting cone (D, μ) for FM in $\mathcal{D}$, there is a *unique* cone (C, ν) for M in $\mathcal{C}$ which is mapped by F to (D, μ), and moreover the cone (C, ν) is a limiting cone for M.

For an adjunction $F \dashv G : \mathcal{C} \to \mathcal{D}$ (so $G : \mathcal{C} \to \mathcal{D}, F : \mathcal{D} \to \mathcal{C}$) we have a comparison functor $K : \mathcal{C} \to \mathcal{D}^{GF}$, where $\mathcal{D}^{GF}$ is the category of algebras for the monad GF on $\mathcal{D}$. MacLane, consistently, defines the functor G to be *monadic* if K is an isomorphism of categories. It follows that every monadic functor creates limits.

However, other authors (for example, **Elephant**) call the functor G monadic if K is only an *equivalence*. And whilst the forgetful functor $U^T : \mathcal{C}^T \to \mathcal{C}$ always creates limits (here $\mathcal{C}^T$ denotes the category of algebras for a monad T), with the strict definition of MacLane this is no longer guaranteed if U^T is composed with an equivalence of categories. Yet, there are good reasons to consider "monadic" functors where the comparison is only an equivalence, and we would like to have a "creation of limits" definition which is stable under equivalence. For example, the "Crude Tripleability Theorem" (0.28) below only ensures an equivalence with the category of algebras.

Definition 0.24 (Creation of Limits) A functor $F : \mathcal{C} \to \mathcal{D}$ creates limits of type J if for any diagram $M : J \to \mathcal{C}$ and any limiting cone (X, μ) for FM in $\mathcal{D}$ the following hold:

i) There exists a cone (Y, ν) for M in $\mathcal{C}$ such that its F-image is isomorphic to (X, μ) (in the category of cones for FM).

ii) Any cone (Y, ν) for M which is mapped by F to a cone isomorphic to (X, μ), is limiting.

We say that the functor F *creates limits* if F creates limits of every small type J.

For the record:

Theorem 0.25 *Let $\mathcal{C} \overset{G}{\to} \mathcal{D}$ monadic. Then G creates limits.*

The following remark appears on the first pages of Johnstone's *Sketches of an Elephant*, and is very useful.

Remark 0.26 (Elephant A1.1.1) Let $A \underset{U}{\overset{F}{\leftrightarrows}} C$ be an adjunction with $F \dashv U$. If there is a natural isomorphism between FU and the identity on A, then the counit is a natural isomorphism. Of course, by duality a similar statement holds for units.

Definition 0.27 A parallel pair of arrows $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ is a *reflexive pair* if f and g have a common section: a morphism $s : Y \to X$ for which $fs = gs = \mathrm{id}_Y$. A category is said to have *coequalizers of reflexive pairs* if for every reflexive pair the coequalizer exists.

Theorem 0.28 (Beck's "Crude Tripleability Theorem") *Let*

$$A \underset{U}{\overset{F}{\leftrightarrows}} C$$

be an adjunction with $F \dashv U$; let $T = UF$ be the induced monad on C. Suppose that A has coequalizers of reflexive pairs, that U preserves them, and moreover that U reflects isomorphisms. Then the functor U is monadic.

Proof. We start by constructing a left adjoint L to the functor K. Recall that $K : A \to C^T$ sends an object Y of A to the T-algebra $UFUY \overset{U(\varepsilon_Y)}{\to} UY$.

Let $UFX \overset{h}{\to} X$ be a T-algebra. We have that η_X is a section of h by the axioms for an algebra, and $F(\eta_X)$ is a section of ε_{FX} by the triangular identities for an adjunction. So the parallel pair

$$FUFX \underset{\varepsilon_{FX}}{\overset{F(h)}{\rightrightarrows}} FX$$

is reflexive with common section $F(\eta_X)$; let $FX \overset{e}{\to} E$ be its coequalizer. We define $L(h)$ to be the object E. Clearly, this is functorial in h.

Let us prove that $KL(h)$ is isomorphic to h. Note that the underlying object of the T-algebra $KL(h)$ is UE. By construction of $L(h)$ and the assumptions on U, the diagram

$$UFUFX \underset{U(\varepsilon_{FX})}{\overset{UF(h)}{\rightrightarrows}} UFX \xrightarrow{U(e)} UE$$

is a coequalizer. By the associativity of the algebra h, the map h coequalizes the pair $(UF(h), U(\varepsilon_{FX}))$; so we have a unique $\xi : UE \to X$ satisfying

$$\xi \circ U(e) = h.$$

We also have the map $U(e)\circ\eta_X : X \to UE$. It is routine to check that these maps are each other's inverse, as well as that ξ is in fact an algebra map. This shows that $KL(h)$ is naturally isomorphic to h.

Let us show that $L \dashv K$. Maps in A from $E = L(h)$ to an object Y correspond, by the coequalizer property of E, to arrows $f : FX \to Y$ satisfying $f\circ F(h) = f\circ\varepsilon_{FX}$. Transposing along the adjunction $F \dashv U$, these correspond to maps $\bar{f} : X \to UY$ satisfying $\bar{f}\circ h = U(\varepsilon_Y)\circ UF(\bar{f})$; that is, to T-algebra maps from h to $K(Y)$. This establishes the adjunction and applying Johnstone's remark 0.26 we conclude that the unit of the adjunction is an isomorphism.

In order to show that also the counit of $L \dashv K$ is an isomorphism, we recall that for an object Y of A, $LK(Y)$ is the vertex of the coequalizer diagram

$$FUFUY \underset{\varepsilon_{FUY}}{\overset{FU(\varepsilon_Y)}{\rightrightarrows}} FUY \xrightarrow{w} W$$

Since also ε_Y coequalizes the parallel pair, we have a unique map $W \overset{v}{\to} Y$ satisfying $vw = \varepsilon_Y$. Itis now not too hard to prove that $U(v)$ is an isomorphism; since U reflects isomorphisms, v is an isomorphism, and we are done. ∎

The following theorem is called "Adjoint lifting theorem".

Theorem 0.29 (Adjoint Lifting Theorem; PTJ 0.15) *Let T and S be monads on categories $\mathcal{C}$ and $\mathcal{D}$ respectively. Suppose we have a commutative diagram of functors*

$$\begin{tikzcd} \mathcal{C}^T \arrow[r, "\bar{F}"] \arrow[d, "U^T"'] & \mathcal{D}^S \arrow[d, "U^S"] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

where U^T, U^S are the forgetful functors. Suppose F has a left adjoint L. Moreover, assume that the category $\mathcal{C}^T$ has coequalizers of reflexive pairs. Then the functor $\bar{F}$ also has a left adjoint.

Proof. *[Sketch]* Let (T, η, μ) and (S, ι, ν) be the respective monad structures on T and S. Our first remark is that every S-algebra is a coequalizer of a reflexive pair of arrows between free S-algebras. For an S-algebra $SX \xrightarrow{h} X$, consider the parallel pair

$$S^2X \underset{\nu_X}{\overset{Sh}{\rightrightarrows}} SX$$

This is a diagram of algebra maps $F^S(SX) \to F^S(X)$: $\nu_X S^2 h = Sh\nu_{SX}$ by naturality of ν, and $\nu_X \nu_{SX} = \nu_X S(\nu_X)$ by associativity of ν. The two arrows have a common splitting $S(\iota_X)$ which is also an algebra map since it is $F^S(\iota_X)$. That is: we have a reflexive pair in S-Alg. It is easy to see that $h : SX \to X$ coequalizes this pair: this is the associativity of h as an algebra. If $a : F^S(X) \to (\xi : SY \to Y)$ is an algebra map which coequalizes our reflexive pair then a factors through $h : F^S(X) \to (h : SX \to X)$ by $a\iota_X : (SX \xrightarrow{h} X)_{\to}(SY \xrightarrow{\xi} Y))$ and the factorization is unique because the arrow h is split epi in $\mathcal{C}$.

This construction is functorial. Given an S-algebra map $f : (SX \xrightarrow{h} X) \to (SY \xrightarrow{k} Y)$ the diagram

$$\begin{tikzcd}
S^2X \arrow[r, shift left, "\nu_X"] \arrow[r, shift right, "Sh"'] \arrow[d, "S^2 f"'] & SX \arrow[d, "Sf"] \\
S^2Y \arrow[r, shift left, "\nu_Y"] \arrow[r, shift right, "Sk"'] & SY
\end{tikzcd}$$

commutes serially (i.e., $Sf\nu_X = \nu_Y S^2 f$ and $SfSh = SkS^2 f$). So, we have a functor R from S-Alg to the category of diagrams of shape $\circ \rightrightarrows \circ$ in S-Alg, with the properties:

i) The vertices of $R(h)$ are free algebras.

ii) $R(h)$ is always a reflexive pair.

iii) The colimit of $R(h)$ is h.

Our second remark is that since $\bar{F}$ is a lifting of F ($U^S\bar{F} = FU^T$) there is a natural transformation $\lambda : SF \to FT$ constructed as follows. Consider

$F(\eta) : F \to FT = FU^TF^T = U^S\bar{F}F^T$ and let $\tilde{\lambda} : F^SF \to \bar{F}F^T$ be its transpose along $F^S \dashv U^S$. Define λ as the composite

$$SF = U^SF^SF \xrightarrow{U^S\tilde{\lambda}} U^S\bar{F}F^T = FU^TF^T = FT.$$

Claim: The natural transformation λ makes the following diagram commute:

$$\begin{tikzcd}
F \arrow[r, "\iota_F"] \arrow[dr, "F(\eta"'] & SF \arrow[d, "\lambda"'] & S^2F \arrow[l, "\nu_F"'] \arrow[d, "S\lambda"] \\
 & FT & SFT \arrow[d, "\lambda T"] \\
 & & FT^2 \arrow[ul, "F\mu"]
\end{tikzcd}$$

Now we are ready for the definition of $\bar{L}$ on objects: if $\bar{L}$ is going to be left adjoint to $\bar{F}$ then, by uniqueness of adjoints and the fact that adjoints compose, $\bar{L}F^S = F^TL$, so we know what $\bar{L}$ should do on free S-algebras F^SY. Now every S-algebra $\xi : SY \to Y$ is coequalizer of a reflexive pair of arrows between free S-algebras, and as a left adjoint, $\bar{L}$ should preserve coequalizers. Therefore we expect $\bar{L}(\xi)$ to be coequalizer of a reflexive pair

$$F^TLSY = \bar{L}F^S(SY) \underset{g_\xi}{\overset{f_\xi}{\rightrightarrows}} \bar{L}F^S(Y) = F^TLY$$

between free T-algebras. It is now our task to determine f_ξ and g_ξ.

By our first remark we have a coequalizer

$$F^S(SY) \underset{\nu_Y}{\overset{S\xi}{\rightrightarrows}} F^SY \xrightarrow{\xi} (\xi)$$

and the topmost arrow of the reflexive pair is in the image of the functor F^S, so we can take $F^TL(\xi)$ for f_ξ. The other map – ν – is not in the image of F^S and needs a bit of doctoring using the adjunction $L \dashv F$ and the natural transformation λ we constructed. Let α be the unit of the adjunction $L \dashv F$. Consider the arrow

$$SY \xrightarrow{S(\alpha_Y)} SFL(Y) \xrightarrow{\lambda_{L(Y)}} FTL(Y)$$

This transposes under $L \dashv F$ to a map $LS(Y) \to TL(Y) = U^TF^TL(Y)$, and this in turn transposes under $F^T \dashv U^T$ to a map

$$F^TLS(Y) \to F^TL(Y)$$

which we take as our g_ξ.

Note that the construction is natural in ξ, so if $k : \xi \to \zeta$ is a map of S-algebras, we obtain a natural transformation from the diagram of parallel arrows f_ξ, g_ξ to the diagram with parallel arrows f_ζ, g_ζ. Hence we also get a map from the coequalizer of the first diagram, which is $\bar{L}(\xi)$, to the coequalizer of the second one, which is $\bar{L}(\zeta)$. And this map between coequalizers will be $\bar{L}(k)$.

There is still a lot to check. This is meticulously done in Volume 2 of Borceux's *Handbook of Categorical Algebra*, section 4.5. There the proof takes 10 pages. ∎

Remark 0.30 There is a better theorem than the one we just partially proved: the *Adjoint Triangle Theorem*. It says that whenever we have functors $\mathcal{B} \stackrel{R}{\to} \mathcal{C} \stackrel{U}{\to} \mathcal{D}$ such that $\mathcal{B}$ has reflexive coequalizers and U is *of descent type* (that is: U has a left adjoint J and the comparison functor $K : \mathcal{C} \to UJ-\mathrm{Alg}$ is full and faithful), then UR has a left adjoint if and only if R has one.

Note, that given the diagram of Theorem 0.29, the diagram

$$\mathcal{C}^T \stackrel{\bar{F}}{\to} \mathcal{D}^S \stackrel{U^S}{\to} \mathcal{D}$$

satisfies the conditions of the Adjoint Triangle Theorem. Since the composition $U^S\bar{F}$, which is F^TL, has a left adjoint, we conclude that $\bar{F}$ has a left adjoint. Note in particular that we do not use that $\mathcal{C}^T$ is monadic.

Definition 0.31 A diagram $a \overset{f}{\underset{g}{\rightrightarrows}} b \xrightarrow{h} c$ in a category is called a *split fork* if $hf = hg$ and there exist maps

$$a \xleftarrow{t} b \xleftarrow{s} c$$

such that $hs = \mathrm{id}_c$, $ft = \mathrm{id}_b$ and $gt = sh$.

Exercise 17 Show that every split fork is a coequalizer diagram, and moreover a coequalizer which is preserved by any functor (this is called an *absolute* coequalizer).

Exercise 18 Suppose D_1 is the diagram $a \overset{f}{\underset{g}{\rightrightarrows}} b \xrightarrow{h} c$ in a category $\mathcal{C}$, and D_2 is the diagram $a' \overset{f'}{\underset{g'}{\rightrightarrows}} b' \xrightarrow{h'} c'$ in $\mathcal{C}$. Assume that D_2 is a retract of D_1 in the category of diagrams in $\mathcal{C}$ of type $\bullet \rightrightarrows \bullet \longrightarrow \bullet$. Prove that if D_1 is a split fork, then so is D_2.

Definition 0.32 In a category, a family of arrows $\{f_i : A_i \to B \,|\, i \in I\}$ is called epimorphic if for every parallel pair of arrows $u, v : B \to C$ the following holds: if $uf_i = vf_i$ for all $i \in I$, then $u = v$.

Exercise 19 If the ambient category has I-indexed coproducts, a family $\{f_i : A_i \to B \,|\, i \in I\}$ is epimorphic if and only if the induced arrow from the coproduct $\sum_{i\in I} A_i$ to B is an epimorphism.

We shall also have to deal with *comonads*; a comonad on a category $\mathcal{C}$ is a monad on $\mathcal{C}^{\mathrm{op}}$. Explicitly, we have a functor $G : \mathcal{C} \to \mathcal{C}$ with natural transformations $\varepsilon : G \Rightarrow \mathrm{id}_{\mathcal{C}}$ (the "counit")) and $\delta : G \Rightarrow G^2$ (the "comultiplication") which make the following (coassociativity and counitarity) diagrams commute:

$$\begin{tikzcd} G \arrow[r, "\delta"] \arrow[d, "\delta"'] & G^2 \arrow[d, "\delta G"] \\ G^2 \arrow[r, "G\delta"'] & G^3 \end{tikzcd} \qquad \begin{tikzcd} & G \arrow[d, "\delta"] & \\ G \arrow[ur, "\mathrm{id}"] & G^2 \arrow[l, "\varepsilon G"] \arrow[r, "G\varepsilon"'] & G \arrow[ul, "\mathrm{id}"'] \end{tikzcd}$$

Dual to the treatment for monads, we have the category G-Coalg of G-coalgebras, the notion of a functor being "comonadic", etcetera. We have the forgetful functor $V : G{-}\mathrm{Coalg} \to \mathcal{C}$ which has a *right* adjoint $C : \mathcal{C} \to G{-}\mathrm{Coalg}$, the "cofree coalgebra functor". Without proof we record the following theorem:

Theorem 0.33 (Eilenberg-Moore; MM V.8.1-2; PTJ 0.14) *Suppose T is a monad on a category $\mathcal{C}$, such that the functor T has a right adjoint G. Then there is a unique comonad structure (ε, δ) on G such that the categories T-Alg and G-Coalg are isomorphic by an isomorphism which commutes with the forgetful functors:*

$$\begin{tikzcd} T{-}\mathrm{Alg} \arrow[rr, "L"] \arrow[dr, "U"'] & & G{-}\mathrm{Coalg} \arrow[dl, "V"] \\ & \mathcal{C} & \end{tikzcd}$$

Proof. (Outline) We write $\mathcal{C}(-,-)$ for the functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}$ which sends (A, B) to the set $\mathcal{C}(A, B)$ of arrows from A to B. We also use $\mathcal{C}(T(-), -)$, $\mathcal{C}(-, G(-))$ for the functors $(A, B) \mapsto \mathcal{C}(TA, B), \mathcal{C}(A, GB)$, etcetera.

Let $\theta : \mathcal{C}(T(-), -) \to \mathcal{C}(-, G(-))$ be the natural isomorphism which defines the adjunction $T \dashv G$. Then θ induces, for each nonnegative integer

n, a natural isomorphism $\mathcal{C}(T^n(-), -) \to \mathcal{C}(-, G^n(-))$, which we denote by θ^n. Now suppose we have a natural transformation $\sigma : T^n \Rightarrow T^m$. Then for every object B of $\mathcal{C}$ we have a natural transformation

$$\mathcal{C}(-, G^m(B)) \stackrel{(\theta^m)^{-1}}{\to} \mathcal{C}(T^m(-), B) \stackrel{\mathcal{C}(\sigma,-)}{\to} \mathcal{C}(T^n(-), B) \stackrel{\theta^n}{\to} \mathcal{C}(-, G^n B)$$

which is a morphism of presheaves $y_{G^m B} \to y_{G^n B}$ and hence, by the Yoneda lemma, induced by a unique map $\tau_B : G^m B \to G^n B$. It is straightforward to verify, using the naturality of θ and σ, that the family of arrows $\tau = (\tau_B)_{B \in \mathcal{C}}$ is a natural transformation $G^m \Rightarrow G^n$. In this situation we say that τ is *associated to* σ.

If we apply this to the unit $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow T$ of the monad T we obtain a natural transformation $\varepsilon : G \Rightarrow \mathrm{id}_{\mathcal{C}}$, associated to η.

Similarly, associated to the multiplication $\mu : T^2 \Rightarrow T$ of T we have a natural transformation $\delta : G \Rightarrow G^2$. We claim that (G, ε, δ) is a comonad on $\mathcal{C}$.

To illustrate the proof, which I don't spell out entirely, consider the diagram of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}$:

$$\begin{tikzcd}
\mathcal{C}(T(-),-) \arrow[rr, "\theta"] \arrow[d, "\mathcal{C}(\mu,-)"'] & & \mathcal{C}(-,G(-)) \arrow[d, "\mathcal{C}(-,\delta)"] \\
\mathcal{C}(T^2(-),-) \arrow[r, "\theta"] \arrow[d, "\mathcal{C}(\mu_{T(-)},-)"'] & \mathcal{C}(T(-),G(-)) \arrow[r, "\theta"] \arrow[d, "\mathcal{C}(T(-),\delta)"] & \mathcal{C}(-),G^2(-)) \arrow[d, "\mathcal{C}(-).G\delta)"] \\
\mathcal{C}(T^3(-),-) \arrow[r, "\theta^2"'] & \mathcal{C}(T(-),G^2(-)) \arrow[r, "\theta"'] & \mathcal{C}(-,G^3(-))
\end{tikzcd}$$

The top square defines δ as associated to μ, and the lower left hand square is an instance of that. The lower right hand square commutes by naturality of θ.

Thereforee we see that $(G\delta)\circ\delta$ is associated to $\mu\circ\mu_T$. By a similar diagram we find that $\delta_G\circ\delta$ is associated to $\mu\circ T(\mu)$. Now since associates are unique, we see that the coassociativity axiom $G\delta\circ\delta = \delta_G\circ\delta$ (for G) follows from the associativity axiom $\mu\circ\mu_T = \mu\circ T(\mu)$ (for T). In a similar way we prove the counitary law for ε, using the unit law for η.

If $(TX \stackrel{h}{\to} X)$ is a T-algebra, then its transpose $(X \stackrel{\theta(h)}{\to} GX)$ is a G-coalgebra, as I leave to you to figure out. Clearly, this gives an isomorphism of categories which commutes with the forgetful functors. ∎

Corollary 0.34 *If (T, η, μ) is a monad on $\mathcal{C}$ and the functor T has a right adjoint G, then the forgetful functor $T - \mathrm{Alg} \to \mathcal{C}$ has both a left and a right adjoint.*

1 Elementary Toposes

In this chapter I discuss the basic "theory of toposes", that is: the categorical properties that follow from the definition of an elementary topos. This is largely based on Chapter 1 of **PTJ**; here and there I have expanded the proofs where I thought this might be helpful. Moreover, I have included material from sections 2.1, 2.2 and Chapter 3 of **PTJ** (topos of coalgebras, topos of internal presheaves, sheaves for Lawvere-Tierney topologies) in this chapter, because this gives us the main constructions of toposes.

1.1 Equivalence relations and partial maps

Lemma 1.1 (PTJ 1.21) *In a topos, every mono is regular.*

Proof. Every mono is a pullback of $1 \xrightarrow{t} \Omega$, and t is split mono, hence regular. ∎

Corollary 1.2 (PTJ 1.22) *Every map in a topos which is both epi and mono is an isomorphism (one says that a topos is* balanced*).*

Definition 1.3 In a category with finite limits, an *equivalence relation on an object* X is a subobject R of $X \times X$ for which the following statements hold:

i) The diagonal embedding $X \to X \times X$ factors through R.

ii) The composition $R \to X \times X \xrightarrow{\text{tw}} X \times X$ factors through R, where tw denotes the twist map

$$\langle p_1, p_0 \rangle : X \times X \to X \times X.$$

(Here $p_0, p_1 : X \times X \to X$ are the projections)

iii) The map $\langle p_0 s, p_1 t \rangle : R' \to X \times X$ factors through R, where we assume that the subobject R is represented by the arrow $\langle r_0, r_1 \rangle : R \to X \times X$, and the maps s and t are defined by the pullback diagram

$$\begin{tikzcd} R' \arrow[r, "t"] \arrow[d, "s"'] & R \arrow[d, "r_0"] \\ R \arrow[r, "r_1"'] & X \times X \end{tikzcd} .$$

The subobject R' is the "object of R-related triples".

Equivalently, a subobject R of $X \times X$ is an equivalence relation on X if and only if for every object Y, the relation

$$\{(f,g) \mid \langle f,g\rangle : Y \to X \times X \text{ factors through } R\}$$

is an equivalence relation on the set of arrows $Y \to X$.

Clearly, for every arrow $f : X \to Y$, the kernel pair of f, seen as a subobject of $X \times X$, is an equivalence relation on X. Equivalence relations which are kernel pairs are called *effective* (don't ask me why).

Proposition 1.4 (PTJ 1.23) *In a topos, every equivalence relation is effective, i.e. a kernel pair.*

Proof. Let $\phi : X \times X \to \Omega$ classify the subobject $\langle r_0, r_1\rangle : R \to X \times X$, and let $\bar{\phi} : X \to \Omega^X$ be its exponential transpose (in Set, $\bar{\phi}(x)$ will be the R-equivalence class of x). We claim that the square

$$\begin{tikzcd} R \arrow[r, "r_1"] \arrow[d, "r_0"'] & X \arrow[d, "\bar{\phi}"] \\ X \arrow[r, "\bar{\phi}"'] & \Omega^X \end{tikzcd}$$

is a pullback, so that R is the kernel pair of $\bar{\phi}$. To see that it commutes, we look at the transposes of the compositions $\bar{\phi}r_i$, which are maps

$$R \times X \xrightarrow{r_i \times \mathrm{id}} X \times X \xrightarrow{\phi} \Omega$$

Both these maps classify the object R' of R-related triples, seen as subobject of $R \times X$, so they are equal. To see that the given diagram is a pullback, suppose we have maps $f, g : U \to X$ satisfying $\bar{\phi}f = \bar{\phi}g$. Then $\phi(f \times \mathrm{id}_X) = \phi(g \times \mathrm{id}_X) : U \times X \to \Omega$. Composing with the map $\langle \mathrm{id}_U, g\rangle : U \to U \times X$ we get that the square

$$\begin{tikzcd} U \arrow[r, "\langle f,g\rangle"] \arrow[d, "\langle g,g\rangle"'] & X \times X \arrow[d, "\phi"] \\ X \times X \arrow[r, "\phi"'] & \Omega \end{tikzcd}$$

commutes. Now ϕ classifies R and by reflexivity of R the map $\langle g,g\rangle$ factors through R, so $\phi\langle g,g\rangle$ is the composite map $U \xrightarrow{!} 1 \xrightarrow{t} \Omega$; so this also holds for the other composite and therefore also $\langle f, g\rangle$ must factor through R, which says that the given diagram is indeed a pullback. ∎

The least equivalence relation on an object X is the diagonal $\delta = \langle \mathrm{id}_X, \mathrm{id}_X\rangle : X \to X \times X$. It is classified by some $\Delta : X \times X \to \Omega$. In the Introduction we defined $\{\cdot\} : X \to \Omega^X$ as the exponential transpose of Δ. The map $\{\cdot\}$ is of course thought of as the *singleton map* from X to its power object.

Definition 1.5 A *partial map* from X to Y is an arrow from a subobject of X to Y. More precisely, it is an equivalence class of diagrams (U, m, f):

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "m"'] & Y \\ X & \end{tikzcd}$$

with m mono. Two such diagrams (U, m, f) and (V, n, g) are equivalent if there is an isomorphism $s : U \to V$ such that $ns = m$ and $gs = f$.

We write $f : X \rightharpoonup Y$ to emphasize that the map is partial.

For a fixed object Y we have a presheaf $\mathrm{Part}(-, Y)$ of partial maps into Y; on objects, $\mathrm{Part}(X, Y)$ is the set of pairs (U, f) where U is a subobject of X and $f : U \to Y$ is a map; for an arrow $g : X' \to X$ and $(U, f) \in \mathrm{Part}(X, Y)$ we have $(V, f \circ m^*g) \in \mathrm{Part}(X', Y)$ where in the diagram

$$\begin{tikzcd} V \arrow[r, "m^*g"] \arrow[d, "n"'] & U \arrow[r, "f"] \arrow[d, "m"'] & Y \\ X' \arrow[r, "g"'] & X & \end{tikzcd}$$

the left-hand square is a pullback.

We say that *partial maps are representable* if each presheaf $\mathrm{Part}(-, Y)$ is representable; in other words, if for each object Y there is an object $\tilde{Y}$ such that the presheaves $\mathrm{Part}(-, Y)$ and $y_{\tilde{Y}}$ are isomorphic. In practice we often use the characterization of the following exercise:

Exercise 20 Show, that partial maps are representable if and only if for each object Y there exists a monomorphism $\eta_Y : Y \to \tilde{Y}$ with the property that for every partial map $X \xleftarrow{m} U \xrightarrow{f} Y$ from X to Y there is a *unique* arrow $\tilde{f} : X \to \tilde{Y}$, making the square

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "m"'] & Y \arrow[d, "\eta_Y"] \\ X \arrow[r, "\tilde{f}"'] & \tilde{Y} \end{tikzcd}$$

a pullback.

The object $\tilde{Y}$ (or, better, the arrow $\eta_Y : Y \to \tilde{Y}$) is called the *partial map classifier of* Y and the map $\tilde{f}$ in the diagram is said to *represent* the partial map (U, f).

Remark 1.6 Let us spell out what this means for $Y = 1$: we have an arrow $\eta : 1 \to \tilde{1}$ such that for every mono $m : U \to X$ there is a unique map $X \to \tilde{1}$ making the square

$$\begin{tikzcd} U \arrow[r] \arrow[d, "m"'] & 1 \arrow[d, "\eta_1"] \\ X \arrow[r] & \tilde{1} \end{tikzcd}$$

a pullback. But this is just means that $\eta_1 : 1 \to \tilde{1}$ is a subobject classifier; we conclude that $1 \xrightarrow{\eta} \tilde{1}$ is $1 \xrightarrow{t} \Omega$.

Theorem 1.7 (PTJ 1.26) *In a topos, partial maps are representable.*

Proof. Let $\phi : \Omega^Y \times Y \to \Omega$ classify the graph of the singleton map:

$$\begin{tikzcd} Y \arrow[r, "\langle\{\cdot\}, \mathrm{id}\rangle"] \arrow[d] & \Omega^Y \times Y \arrow[d, "\phi"] \\ 1 \arrow[r, "t"'] & \Omega \end{tikzcd}$$

and let $\bar{\phi} : \Omega^Y \to \Omega^Y$ be its exponential transpose.

Let

$$\begin{tikzcd} E \arrow[r, "e"] & \Omega^Y \arrow[r, shift left, "\bar{\phi}"] \arrow[r, shift right, "\mathrm{id}"'] & \Omega^Y \end{tikzcd}$$

be an equalizer. We shall show that we can take E for $\tilde{Y}$. Think of E as the "set"

$$\{\alpha \subseteq Y \mid \forall y (y \in \alpha \leftrightarrow \alpha = \{y\})\},$$

that is: the set of subsets of Y having at most one element. We consider the pullback diagram

$$\begin{tikzcd}
Y \arrow[r, "\mathrm{id}"] \arrow[d, "\delta"'] & Y \arrow[d, "\langle\{\cdot\},\mathrm{id}\rangle"] \\
Y \times Y \arrow[r, "\{\cdot\}\times\mathrm{id}_Y"'] & \Omega^Y \times Y
\end{tikzcd}$$

Composing this with the diagram defining ϕ, we obtain pullbacks

$$\begin{tikzcd}
Y \arrow[r, "\delta"] \arrow[d, "\mathrm{id}"'] & Y \times Y \arrow[d, "\{\cdot\}\times\mathrm{id}_Y"] \\
Y \arrow[r, "\langle\{\cdot\},\mathrm{id}_Y\rangle"] \arrow[d] & \Omega^Y \times Y \arrow[d, "\phi"] \\
1 \arrow[r, "t"'] & \Omega
\end{tikzcd}$$

from which we conclude that $\phi(\{\cdot\} \times \mathrm{id}_Y)$ classifies the diagonal map on Y; hence its exponential transpose, which is $\bar{\phi}\circ\{\cdot\} : Y \to \Omega^Y$, is equal to $\{\cdot\}$. Therefore the map $\{\cdot\} : Y \to \Omega^Y$ factors through the equalizer E above; so we have the required map $Y \to E = \tilde{Y}$ (which is monic since $\{\cdot\}$ is).

In order to show that the constructed mono $Y \to \tilde{Y}$ indeed represents partial maps into Y, let

$$\begin{tikzcd}
U \arrow[r, "f"] \arrow[d, "m"'] & Y \\
X &
\end{tikzcd}$$

be a partial map $X \rightharpoonup Y$, so m is monic. Consider the graph of f: $U \xrightarrow{\langle m,f\rangle} X \times Y$. It is classified by a map $\psi : X \times Y \to \Omega$; let $\bar{\psi} : X \to \Omega^Y$ be the exponential transpose of ψ. We have a commutative diagram

$$(*) \qquad \begin{tikzcd}
U \arrow[r, "f"] \arrow[d, "\langle m,f\rangle"'] & Y \arrow[d, "\langle\{\cdot\},\mathrm{id}\rangle"] \\
X \times Y \arrow[r, "\bar{\psi}\times\mathrm{id}"'] \arrow[d] & \Omega^Y \times Y \arrow[d] \\
X \arrow[r, "\bar{\psi}"'] & \Omega^Y
\end{tikzcd}$$

The lower square is a pullback, so the outer square is a pullback if and only if the upper square is. We prove that the outer square is a pullback. Suppose $V \xrightarrow{a} X$, $V \xrightarrow{b} Y$ are maps such that $\{\cdot\}b = \bar{\psi}a$. Then by transposing, the square

$$\begin{tikzcd}
V \times Y \arrow[r, "b\times\mathrm{id}"] \arrow[d, "a\times\mathrm{id}"'] & Y \times Y \arrow[d, "\Delta"] \\
X \times Y \arrow[r, "\psi"'] & \Omega
\end{tikzcd}$$

commutes (recall that Δ classifies the diagonal $Y \to Y \times Y$). Composing with the map $V \xrightarrow{\langle \mathrm{id}, b\rangle} V \times Y$ gives

$$\begin{array}{rcl} \psi\circ\langle a, b\rangle & = & \Delta\circ\langle b, b\rangle & = & \text{(by definition of } \Delta) \\ & & & = & V \to 1 \xrightarrow{t} \Omega \end{array}$$

So $\psi\circ\langle a, b\rangle$ factors through t, and since ψ classifies the graph of f, the map $V \xrightarrow{\langle a,b\rangle} X \times Y$ factors through U; we conclude that the outer square of $(*)$ is indeed a pullback. Hence the upper square of $(*)$ is a pullback.

Now since

$$\begin{tikzcd}
\Omega^Y \times Y \arrow[r, "\phi"] & \Omega \\
Y \arrow[r] \arrow[u, "\langle\{\cdot\},\mathrm{id}\rangle"] & 1 \arrow[u]
\end{tikzcd}$$

is a pullback by definition of ϕ, composing with the upper square of $(*)$ yields pullbacks

$$\begin{tikzcd}
X \times Y \arrow[r, "\bar{\psi}\times\mathrm{id}"] & \Omega^Y \times Y \arrow[r, "\phi"] & \Omega \\
U \arrow[r] \arrow[u, "\langle m,f\rangle"] & Y \arrow[r] \arrow[u, "\langle\{\cdot\},\mathrm{id}\rangle"'] & 1 \arrow[u, "t"']
\end{tikzcd}$$

So the graph of f is classified by $\phi\circ(\bar{\psi} \times \mathrm{id})$. It follows that $\phi\circ(\bar{\psi} \times \mathrm{id}) = \psi$, and by transposing we get $\bar{\phi}\bar{\psi} = \bar{\psi} : X \to \Omega^Y$. So $\bar{\psi} : X \to \Omega^Y$ factors through $\tilde{Y} \to \Omega^Y$ by a map $\tilde{f} : X \to \tilde{Y}$. The factorization is unique since

$\tilde{Y} \to \Omega^Y$ is monic. Summarizing, we have

$$\begin{tikzcd}
U \arrow[r, "f"] \arrow[dd, "m"'] & Y \arrow[d, "\eta_Y"] \\
 & \tilde{Y} \arrow[d] \\
X \arrow[ur, "\tilde{f}"] \arrow[r, "\bar{\psi}"'] & \Omega^Y
\end{tikzcd}$$

where the outer square is a pullback (it is the outer square of $(*)$), and since $\tilde{Y} \to \Omega^Y$ is monic the upper square is a pullback too.

From the uniqueness of $\tilde{f}$ we can prove that the assignment $Y \Rightarrow \tilde{Y}$, together with the maps $\eta_Y : Y \to \tilde{Y}$, gives a functor $\mathcal{E} \to \mathcal{E}$ (where $\mathcal{E}$ denotes the ambient topos): given a map $f : X \to Y$, let $\tilde{f} : \tilde{X} \to \tilde{Y}$ represent the partial map

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "f"'] & \tilde{X} \\
Y &
\end{tikzcd}$$

By uniqueness we see that $\tilde{g}\tilde{f} = \widetilde{gf}$. We also see that η is a natural transformation $\mathrm{id}_{\mathcal{E}} \Rightarrow \widetilde{(\cdot)}$. It has the special property that all naturality squares are pullbacks. ∎

Proposition 1.8 (PTJ 1.27) *The partial map classifiers $\tilde{Z}$ are injective.*

Proof. Given a diagram

$$\begin{tikzcd}
X' & \\
X \arrow[u, "m"] \arrow[r, "f"'] & \tilde{Z}
\end{tikzcd}$$

with m mono, we need to find a map $X' \to \tilde{Z}$ making the triangle commute. To this end, form the pullback

$$\begin{tikzcd}
X \arrow[r, "f"] & \tilde{Z} \\
Y \arrow[u, "n"] \arrow[r, "g"'] & Z \arrow[u, "\eta_Z"']
\end{tikzcd}$$

Let the partial map $X' \rightharpoonup Z$ given by $X' \xleftarrow{mn} Y \xrightarrow{g} Z$ be represented by $\tilde{g} : X' \to \tilde{Z}$. It is left to you to verify that the square

$$\begin{tikzcd} X \arrow[r, "\tilde{g}m"] & \tilde{Z} \\ Y \arrow[u, "n"] \arrow[r, "g"'] & Z \arrow[u, "\eta_Z"'] \end{tikzcd}$$

is a pullback. We see that the arrows f and $\tilde{g}m$ represent the same partial map, hence the triangle commutes. ∎

Corollary 1.9 (PTJ 1.28) *Suppose we are given a pushout square*

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ Z \arrow[r] & T \end{tikzcd}$$

with f mono. Then g is also mono, and the square is also a pullback.

Proof. Consider the partial map $Z \rightharpoonup Y$ given by the diagram $Z \xleftarrow{f} X \xrightarrow{m} Y$; let it be represented by a map $h : Z \to \tilde{Y}$. Since the original square is a pushout, we have a unique map $T \to \tilde{Y}$ making the diagram

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "f"'] & Y \arrow[d, "g"] \arrow[ddr, "\eta_Y"] & \\ Z \arrow[r] \arrow[drr, "h"'] & T \arrow[dr] & \\ & & \tilde{Y} \end{tikzcd}$$

commute. Then g is mono because η_Y is mono, and the outer square is a pullback, so the inner square is a pullback too. ∎

Remark 1.10 Proposition 1.8 shows, in particular, that a topos *has enough injectives*: that is, for every object X there is a mono from X into an injective object. The following exercise elaborates on this.

Exercise 21 a) Show that, in a topos, an object is injective if and only if it is a retract of Ω^Y for some Y.

b) Suppose $\mathcal{A} \stackrel{G}{\to} \mathcal{B}$ be a functor with left adjoint $\mathcal{B} \stackrel{F}{\to} \mathcal{A}$. Show that if F preserves monos, G preserves injectives; and that the converse holds if $\mathcal{A}$ has enough injectives.

The following exercise constructs partial map classifiers in a presheaf category.

Exercise 22 Let $\mathcal{C}$ be a small category; we work in the category $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ of presheaves on $\mathcal{C}$. Let P be such a presheaf. We define a presheaf $\tilde{P}$ as follows: for an object C of $\mathcal{C}$, $\tilde{P}(C)$ consists of those subobjects α of $y_C \times P$ which satisfy the following condition: for all arrows $f : D \to C$, the set

$$\{y \in P(D) \mid (f, y) \in \alpha(D)\}$$

has at most one element.

a) Complete the definition of $\tilde{P}$ as a presheaf.

b) Show that there is a monic map $\eta_P : P \to \tilde{P}$ with the following property: for every diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & P \\ {\scriptstyle m}\downarrow & & \\ B & & \end{array}$$

with m mono, there is a unique map $\tilde{g} : B \to \tilde{P}$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & P \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \eta_P} \\ B & \xrightarrow[\tilde{g}]{} & \tilde{P} \end{array}$$

is a pullback square.

c) Show that the assignment $P \mapsto \tilde{P}$ is part of a functor $\tilde{(\cdot)}$ in such a way that the maps η_P form a natural transformation from the identity functor to $\tilde{(\cdot)}$, and all naturality squares for η are pullbacks.

1.2 The opposite category of a topos; colimits in toposes

As usual, $\mathcal{E}$ denotes a topos. We start by considering the category $\mathcal{E}^{\mathrm{op}}$. We have a functor $P : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$: on objects, $PX = \Omega^X$ and for maps $X \xrightarrow{f} Y$ we have $Pf : \Omega^Y \to \Omega^X$, the map which is the exponential transpose of the composition $\Omega^Y \times X \xrightarrow{\mathrm{id}\times f} \Omega^Y \times Y \xrightarrow{\mathrm{ev}} \Omega$.

Note that the same data define a functor $P^* : \mathcal{E} \to \mathcal{E}^{\mathrm{op}}$, and we have:

Lemma 1.11 *We have an adjunction $P^* \dashv P$.*

Proof. We have natural bijections

$$\mathcal{E}^{\mathrm{op}}(P^*X, Y) \;\simeq\; \mathcal{E}(Y, \Omega^X)\, \mathcal{E}(X, \Omega^Y) \;=\; \mathcal{E}(X, PY).$$

$\blacksquare$

Hence, we have a monad $T = PP^*$ on $\mathcal{E}$, and thus a comparison functor $K : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}^T$.

For a mono $g : W \to Z$ we also have a map $\exists g : \Omega^W \to \Omega^Z$: it is the transpose of the map $\widetilde{\exists g} : \Omega^W \times Z \to \Omega$ which classifies the mono

$$\in_W \longrightarrow \Omega^W \times W \xrightarrow{\mathrm{id}\times g} \Omega^W \times Z$$

where $\in_W$ is the subobject of $\Omega^W \times W$ classified by the evaluation map $\mathrm{ev}_W : \Omega^W \times W \to \Omega$.

Proposition 1.12 *The maps*

$$\widetilde{\exists g} \circ (\mathrm{id} \times g) : \Omega^W \times W \to \Omega$$

and

$$\mathrm{ev}_W : \Omega^W \times W \to \Omega$$

coincide.

Proof. We have that the square

$$\begin{tikzcd}
\in_W \arrow[r] \arrow[d] & \Omega^W \times W \arrow[r, "\mathrm{id}\times g"] & \Omega^W \times Z \arrow[d, "\widetilde{\exists g}"] \\
1 \arrow[rr, "t"'] & & \Omega
\end{tikzcd}$$

is a pullback; hence, since g is mono, also the square

$$\begin{tikzcd}
\in_W \arrow[r] \arrow[dd] & \Omega^W \times W \arrow[d, "\mathrm{id}\times g"] \\
& \Omega^W \times Z \arrow[d, "\widetilde{\exists g}"] \\
1 \arrow[r, "t"'] & \Omega
\end{tikzcd}$$

is a pullback. We see that $\widetilde{\exists g}\circ(\mathrm{id}\times g)$ classifies the mono $\in_W \rightarrowtail \Omega^W \times W$, and we conclude that $\widetilde{\exists g}\circ(\mathrm{id}\times g) = \mathrm{ev}_W$. ∎

Lemma 1.13 (PTJ 1.32; "Beck Condition") *Suppose the square*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\
Z \arrow[r, "k"'] & T
\end{tikzcd}$$

is a pullback with the arrows g and h monic. Then the following square commutes:

$$\begin{tikzcd}
\Omega^Y \arrow[r, "Pf"] \arrow[d, "\exists h"'] & \Omega^X \arrow[d, "\exists g"] \\
\Omega^T \arrow[r, "Pk"'] & \Omega^Z
\end{tikzcd}$$

Proof. We look at the exponential transposes of the two compositions. For the clockwise composition $\exists g\circ Pf : \Omega^Y \to \Omega^Z$, its transpose is the top row of

$$\begin{tikzcd}
\Omega^Y \times Z \arrow[r, "Pf\times\mathrm{id}"] & \Omega^X \times Z \arrow[r, "\widetilde{\exists g}"] & \Omega \\
\Omega^Y \times X \arrow[u, "\mathrm{id}\times g"] \arrow[r, "Pf\times\mathrm{id}"'] & \Omega^X \times X \arrow[u, "\mathrm{id}\times g"'] & \\
E \arrow[u] \arrow[r] & \in_X \arrow[u] \arrow[r] & 1 \arrow[uu, "t"']
\end{tikzcd} .$$

We see that this top row classifies the subobject $E \to \Omega^Y \times X \xrightarrow{\mathrm{id}\times g} \Omega^Y \times Z$.

Since $\widetilde{\exists g} \circ (\mathrm{id} \times g) = \mathrm{ev}_X$ by Proposition 1.12, the subobject $E \to \Omega^Y \times X$ is classified by the composition $\Omega^Y \times X \xrightarrow{Pf \times \mathrm{id}} \Omega^X \times X \xrightarrow{\mathrm{ev}_X} \Omega$, which equals the composition $\Omega^Y \times X \xrightarrow{\mathrm{id} \times f} \Omega^Y \times Y \xrightarrow{\mathrm{ev}_Y} \Omega$ since both compositions are transposes of Pf. Therefore we have a pullback diagram

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & \Omega^Y \times X \arrow[d, "\mathrm{id} \times f"] \\
\in_Y \arrow[r] & \Omega^Y \times Y
\end{tikzcd}$$

For the counterclockwise composition $Pk \circ \exists h$, its transpose is $\Omega^Y \times Z \xrightarrow{\exists h \times \mathrm{id}} \Omega^T \times Z \xrightarrow{\mathrm{id} \times k} \Omega^T \times T \xrightarrow{\mathrm{ev}_T} \Omega$ which equals $\Omega^Y \times Z \xrightarrow{\mathrm{id} \times k} \Omega^Y \times T \xrightarrow{\exists h \times \mathrm{id}} \Omega^T \times T \xrightarrow{\mathrm{ev}_T} \Omega$.

Now $\mathrm{ev}_T \circ (\exists h \times \mathrm{id})$ and $\widetilde{\exists h} : \Omega^Y \times T \to \Omega$ both transpose to $\exists h$, so these maps are equal. We conclude that $Pk \circ \exists h$ transposes to the composition $\Omega^Y \times Z \xrightarrow{\mathrm{id} \times k} \Omega^Y \times T \xrightarrow{\widetilde{\exists h}} \Omega$, and we consider pullbacks

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & \Omega^Y \times X \arrow[r, "\mathrm{id} \times g"] \arrow[d, "\mathrm{id} \times f"] & \Omega^Y \times Z \arrow[d, "\mathrm{id} \times k"] \\
\in_Y \arrow[r] \arrow[d] & \Omega^Y \times Y \arrow[r, "\mathrm{id} \times h"] & \Omega Y \times T \arrow[d, "\widetilde{\exists h}"] \\
1 \arrow[rr] & & \Omega
\end{tikzcd}$$

Again using Proposition 1.12, we have $\widetilde{\exists h} \circ (\mathrm{id} \times h) = \mathrm{ev}_Y : \Omega^Y \times Y \to \Omega$ and we see that the counterclockwise composition transposes to a map which classifies the same subobject $E \to \Omega^Y \times X \xrightarrow{\mathrm{id} \times g} \Omega^Y \times Z$ as we saw for the clockwise composition.

Therefore the two compositions are equal, and the given diagram commutes. ∎

Corollary 1.14 (PTJ 1.33) *If $f : X \to Y$ is mono then $Pf \circ \exists f = \mathrm{id}_{\Omega^X}$.*

Proof. Apply 1.13 to the pullback diagram

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & X \arrow[d, "f"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

$\blacksquare$

Theorem 1.15 (PTJ 1.34) *The functor $P : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ is monadic.*

Proof. We use the Crude Tripleability Theorem (0.28). We need to verify its conditions:

1) $\mathcal{E}^{\mathrm{op}}$ has coequalizers of reflexive pairs.

2) P preserves coequalizers of reflexive pairs.

3) P reflects isomorphisms.

Verification of 1) is trivial, since coequalizers in $\mathcal{E}^{\mathrm{op}}$ are equalizers in $\mathcal{E}$, and $\mathcal{E}$ has finite limits.

For 2), let $X \xrightarrow{f} Y \underset{h}{\overset{g}{\rightrightarrows}} Z$ be a diagram in $\mathcal{E}$ which is a coequalizer of a reflexive pair in $\mathcal{E}^{\mathrm{op}}$. Since the pair (g, h) is reflexive in $\mathcal{E}^{\mathrm{op}}$ we have an arrow $Z \stackrel{d}{\to} Y$ satisfying $dg = dh = \mathrm{id}_Y$. This means that g and h are monos, and the square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[h]{} & Z \end{array}$$

is a pullback. We see that also f is mono, and applying 1.13 we find that $\exists f \circ Pf = Ph \circ \exists g$. Moreover by 1.14 we have the equalities $Pf \circ \exists f = \mathrm{id}_{\Omega^X}, Pg \circ \exists g = \mathrm{id}_{\Omega^Y}$. Using these equalities we see that the P-image of the original coequalizer diagram:

$$\Omega^Z \underset{Ph}{\overset{Pg}{\rightrightarrows}} \Omega^Y \xrightarrow{Pf} \Omega^X$$

is a split fork in $\mathcal{E}$, with splittings $\exists g : \Omega^Y \to \Omega^Z$, $\exists f : \Omega^X \to \Omega^Y$. In particular it is a coequalizer in $\mathcal{E}$.

For 3), we observe that for any morphism $f : X \to Y$ in $\mathcal{E}$, the map $Y \stackrel{\{\cdot\}}{\to} \Omega^Y \stackrel{Pf}{\to} \Omega^X$ transposes to the map $Y \times X \to \Omega$ which classifies the graph of f, i.e. the subobject represented by $\langle f, \mathrm{id}\rangle : X \to Y \times X$. Note that if the graphs of f and $g : X \to Y$ coincide then $f = g$ (Exercise 1). Therefore, $Pf = Pg$ implies $f = g$ and P is faithful, hence reflects both monos and epis. By Corollary 1.2, P reflects isomorphisms. $\blacksquare$

Corollary 1.16 (PTJ 1.36) *A topos has finite colimits.*

Proof. For a finite diagram $M : I \to \mathcal{E}$ consider $M^{\text{op}} : I^{\text{op}} \to \mathcal{E}^{\text{op}}$ and compose with $P : \mathcal{E}^{\text{op}} \to \mathcal{E}$. The diagram $P \circ M^{\text{op}}$ has a limit in $\mathcal{E}$ since $\mathcal{E}$ has finite limits. But P, being monadic, creates limits so M^{op} has a limit in $\mathcal{E}^{\text{op}}$; that is, M has a colimit in $\mathcal{E}$. ∎

Corollary 1.17 (PTJ 1.37) *Let $T : \mathcal{E} \to \mathcal{F}$ be a logical functor between toposes. Then the following hold:*

i) T preserves finite colimits.

ii) If T has a left adjoint, it also has a right adjoint.

Proof. i) Since T is logical, the diagram

$$\begin{tikzcd} \mathcal{E}^{\text{op}} \arrow[r, "T^{\text{op}}"] \arrow[d, "P"'] & \mathcal{F}^{\text{op}} \arrow[d, "P"] \\ \mathcal{E} \arrow[r, "T"'] & \mathcal{F} \end{tikzcd}$$

commutes up to isomorphism. Proving that T preserves finite colimits amounts to proving that T^{op} preserves finite limits. So let $M : I \to \mathcal{E}^{\text{op}}$ be a finite diagram, with limiting cone (D, μ) in $\mathcal{E}^{\text{op}}$. Now T and P preserve finite limits, so $TP(D, \mu)$ is a limiting cone for TPM; hence $PT^{\text{op}}(D, \mu)$ is a limiting cone for $PT^{\text{op}}M$ by commutativity of the diagram. Since P creates limits, $T^{\text{op}}(D, \mu)$ is a limitng cone for $T^{\text{op}}M$. We conclude that T^{op} preserves finite limits.

For ii), we employ the Adjoint Lifting Theorem (0.29) to the same diagram. The assumptions are readily verified, and we conclude that T^{op} has a left adjoint. But this means that T has a right adjoint. ∎

1.3 Slices of a topos; the "Fundamental Theorem of Topos Theory"

We now discuss slice categories of toposes. In any category $\mathcal{E}$, for each object X we have the category $\mathcal{E}/X$ whose objects are arrows into X and whose arrows: $(Y \stackrel{f}{\to} X) \to (Z \stackrel{g}{\to} X)$ are arrows $Y \stackrel{h}{\to} Z$ in $\mathcal{E}$ such that $f = gh$. If the category $\mathcal{E}$ has pullbacks, then for every arrow $f : Y \to X$ we have a pullback functor $f^* : \mathcal{E}/X \to \mathcal{E}/Y$, which has a left adjoint $\sum_f$; $\sum_f (Z \stackrel{g}{\to} Y = (Z \stackrel{fg}{\to} X)$. In the case of the unique arrow $X \to 1$ we write $X^* : \mathcal{E} \cong \mathcal{E}/1 \to \mathcal{E}/X$ for the pullback functor. Note that $X^*(Y)$ is the projection $Y \times X \to X$. Note also that $X \stackrel{\text{id}}{\to} X$ is a terminal object of $\mathcal{E}/X$.

The following theorem was dubbed the "Fundamental Theorem of Topos Theory" by Peter Freyd.

Theorem 1.18 (PTJ 1.42) *Let $\mathcal{E}$ be a topos and X an object of $\mathcal{E}$. Then $\mathcal{E}/X$ is a topos, and the functor $X^* : \mathcal{E} \to \mathcal{E}/X$ is logical.*

Proof. In the case $\mathcal{E} = \text{Set}$, it is useful to view objects of $\mathcal{E}/X$ as "X-indexed families of sets" rather than as functions into X. This intuition will also guide us in the general case.

Binary products in $\mathcal{E}/X$ are pullbacks over X: if we adopt the notation $Y \times_X Z$ for the vertex of the pullback diagram

$$\begin{tikzcd} Y \times_X Z \arrow[r] \arrow[d] & Z \arrow[d, "g"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

then in $\mathcal{E}/X$, the product $f \times g$ is the arrow $Y \times_X Z \to X$. Equalizers in $\mathcal{E}/X$ are just equalizers in $\mathcal{E}$. So $\mathcal{E}/X$ has finite limits, and the functor X^* preserves finite limits since it has a left adjoint $\sum_f$ as we remarked.

Monos in $\mathcal{E}/X$ are monos in $\mathcal{E}$, and the diagram

$$\begin{tikzcd} X \arrow[r, "\langle t, \mathrm{id} \rangle"] \arrow[dr, "\mathrm{id}"'] & \Omega \times X \arrow[d, "\pi"] \\ & X, \end{tikzcd}$$

seen as an arrow in $\mathcal{E}/X$, is a subobject classifier in $\mathcal{E}/X$. Note, that this map is $X^*(1 \xrightarrow{t} \Omega)$, so X^* preserves subobject classifiers.

In order to prove cartesian closure, first observe that for $\mathcal{E} = \text{Set}$, the exponent $(Z \xrightarrow{g} X)^{(Y \xrightarrow{f} X)}$ is the X-indexed family $(g^{-1}(x)^{f^{-1}(x)})_{x \in X}$, or the projection function from the set $\{(h, x) \,|\, h : f^{-1}(x) \to g^{-1}(x)\}$ to X.

We first construct the exponential $(Z \xrightarrow{g} X)^{(Y \xrightarrow{f} X)}$, then explain its meaning in intuitive terms (as if $\mathcal{E}$ were the topos Set); then we prove that it has the required universal property.

Let $\theta : X \times Y \to \tilde{X}$ represent the partial map $X \xleftarrow{f} Y \xrightarrow{\langle f, \mathrm{id} \rangle} X \times Y$, That is, let

$$\begin{tikzcd} Y \arrow[r, "\langle f, \mathrm{id} \rangle"] \arrow[d, "f"'] & X \times Y \arrow[d, "\theta"] \\ X \arrow[r, "\eta_X"'] & \tilde{X} \end{tikzcd}$$

be a pullback. Let $\bar{\theta} : X \to \tilde{X}^Y$ be the exponential transpose of θ, and let

$$\begin{tikzcd} E \arrow[r, "q"] \arrow[d, "p"'] & \tilde{Z}^Y \arrow[d, "\tilde{g}^Y"] \\ X \arrow[r, "\bar{\theta}"'] & \tilde{X}^Y \end{tikzcd}$$

be a pullback; the claim is that $E \xrightarrow{p} X$ is the required exponential.

Intuitive explanation: think of $\tilde{X}$ as the set of subsets of X having at most one element. So $\theta(x,y) = \{x \mid f(y) = x\}$. The function $\tilde{g} : \tilde{Z} \to \tilde{X}$ sends subset α of Z to $\{g(z) \mid z \in \alpha\}$. Then, the function $\tilde{g}^Y : \tilde{Z}^Y \to \tilde{X}^Y$ sends a function $h : Y \to \tilde{Z}$ to the function $y \mapsto \{g(z) \mid z \in h(y)\}$. We have $\bar{\theta}(x)(y) = \{x \mid f(y) = x\}$. So the object E can be identified with the set of pairs (x,h) satisfying:

$$\begin{array}{l} x \in X, h : Y \rightharpoonup Z \\ \mathrm{dom}(h) = f^{-1}(x) \\ \text{for all } y \in f^{-1}(x),\ h(y) \in g^{-1}(x). \end{array}$$

That is, E is isomorphic to $\{(h,x) \mid h : f^{-1}(x) \to g^{-1}(x)\}$.

Now we prove that the constructed $E \xrightarrow{p} X$ has the property of the exponential $(Z \xrightarrow{g} X)^{(Y \xrightarrow{f} X)}$; that is, maps from $(T \xrightarrow{k} X)$ to $(E \xrightarrow{p} X)$ are in natural 1-1 correspondence to maps from $(T \times_X Y \to X)$ to $(Z \xrightarrow{g} X)$. We have natural 1-1 correspondences between successive items of the following list:

1) Maps $(T \xrightarrow{k} X) \to (E \xrightarrow{p} X)$ in $\mathcal{E}/X$.

2) Maps $T \xrightarrow{l} \tilde{Z}^Y$ in $\mathcal{E}$ satisfying $\bar{\theta}k = \tilde{g}^Y l$.

3) Maps $T \times Y \xrightarrow{\bar{l}} \tilde{Z}$ in $\mathcal{E}$ satisfying $\tilde{g}\bar{l} = \theta(k \times \mathrm{id}_Y)$:

$$\begin{tikzcd} T \times Y \arrow[r, "\bar{l}"] \arrow[d, "k \times \mathrm{id}_Y"'] & \tilde{Z} \arrow[d, "\tilde{g}"] \\ X \times Y \arrow[r, "\theta"'] & \tilde{X} \end{tikzcd}$$

4) Maps $W \stackrel{u}{\to} Z$ where $W \stackrel{\langle v,w\rangle}{\longrightarrow} T \times Y$ is a mono such that the diagram

$$\begin{tikzcd} W \arrow[r, "\langle v,w\rangle"] \arrow[d, "w"'] & T \times Y \arrow[d, "k\times \mathrm{id}_Y"] \\ Y \arrow[r, "\langle f,\mathrm{id}_Y\rangle"'] & X \times Y \end{tikzcd}$$

is a pullback.

5) Maps $(T \times_X Y \to X) \to (Z \stackrel{g}{\to} X)$ in $\mathcal{E}/X$.

The correspondence from 1) to 2) is by the pullback property of E.

From 2) to 3) by the exponential adjunction.

From 3) to 4): given $T \times Y \stackrel{\bar{l}}{\to} \tilde{Z}$ as in 3), we have, for the two composite arrows $T \times Y \to \tilde{X}$ in the diagram of 3), that these represent the same partial map $T \times Y \rightharpoonup X$; say $W \to X$ for a mono $\langle v, w\rangle : W \to T \times Y$. Since this partial map is represented by $\theta(k \times \mathrm{id}_Y)$ and the square defining θ is a pullback, the map $W \to X$ factors uniquely through Y such that in the diagram

$$\begin{tikzcd} W \arrow[r, "\langle v,w\rangle"] \arrow[d] & T \times Y \arrow[d, "k\times \mathrm{id}_Y"] \\ Y \arrow[r, "\langle f,\mathrm{id}_Y\rangle"'] \arrow[d, "f"'] & X \times Y \arrow[d, "\theta"] \\ X \arrow[r, "\eta_X"'] & \tilde{X} \end{tikzcd}$$

both squares are pullbacks. Since also $\tilde{g}\bar{l}$ represents the partial map, we also have a factorization through Z, satisfying 4).

From 4) to 5): Composing the upper square in the diagram above with the diagram of projections

$$\begin{tikzcd} T \times Y \arrow[r] \arrow[d, "k\times \mathrm{id}_Y"'] & T \arrow[d, "k"] \\ X \times Y \arrow[r] & X \end{tikzcd}$$

which is a pullback, we see that $W \to X$ is actually $T \times_X Y \to X$.

■

Exercise 23 Show that X^* preserves exponentials.

Corollary 1.19 (PTJ 1.43) *For any arrow $f : X \to Y$ in $\mathcal{E}$ the pullback functor $f^* : \mathcal{E}/Y \to \mathcal{E}/X$ is logical, and has a right adjoint $\prod_f$.*

Proof. We now know that $\mathcal{E}/Y$ is a topos, so we can apply Theorem 1.18 with $\mathcal{E}/Y$ in the role of $\mathcal{E}$ and f in the role of X. We see that f^* is logical. By Corollary 1.17, f^* has a right adjoint, since it has a left adjoint $\sum_f$.

However, we can also exhibit the right adjoint $\prod_f$ directly: we do this for the case $Y = 1$. Given an object $(Y \xrightarrow{f} X)$ of $\mathcal{E}/X$ let $\ulcorner \mathrm{id} \urcorner : 1 \to X^X$ denote the exponential transpose of the identity arrow on X, and let

$$\begin{tikzcd} Z \arrow[r] & Y^X \arrow[r, shift left, "f^X"] \arrow[r, shift right, "\ulcorner \mathrm{id} \urcorner \circ !"'] & X^X \end{tikzcd}$$

be an equalizer diagram. Think of Z as the object of sections of f. Now for any object W of $\mathcal{E}$, arrows $g : X^*(W) \to f$:

$$\begin{tikzcd} W \times X \arrow[rr, "g"] \arrow[dr] & & Y \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

correspond, via the exponential adjunction, to arrows $\tilde{g} : W \to Y^X$ such that $f^X \circ \tilde{g}$ factors through $\ulcorner \mathrm{id} \urcorner$; that is to arrows $W \to Z$. Therefore Z is $\prod_X(f)$. ∎

Example 1.20 Consider the subobject classifier $1 \xrightarrow{t} \Omega$; let us calculate $\prod_t : \mathcal{E} \to \mathcal{E}/\Omega$. For an object X of $\mathcal{E}$ and an arrow $Y \xrightarrow{m} \Omega$ we have that maps from m to $\prod_t(X)$ in $\mathcal{E}/\Omega$ correspond to maps from Y' to X, where Y' is the subobject of Y classified by m. That is, to maps $g : Y \to \tilde{X}$ for which the domain (i.e. the map $g^*(\eta_X) : Y' \to Y$) is the subobject of Y classified by m. But these correspond to maps in $\mathcal{E}/\Omega$ from m to the arrow $s : \tilde{X} \to \Omega$ which classifies the mono $X \xrightarrow{\eta_X} \tilde{X}$.

Corollary 1.21 (PTJ 1.46) *Every arrow $f : X \to Y$ in $\mathcal{E}$ induces a geometric morphism*

$$f : \begin{tikzcd} \mathcal{E}/X \arrow[r, shift right, "\prod_f"'] & \mathcal{E}/Y \arrow[l, shift right, "f^*"'] \end{tikzcd} .$$

This geometric morphism has the special features that the inverse image functor f^* is logical and has a left adjoint.

Definition 1.22 A geometric morphism f for which the inverse image functor f^* has a left adjoint is called *essential*.

Without proof, I mention the following partial converse to corollary 1.21.

Theorem 1.23 (PTJ 1.47) *Let $f : \mathcal{F} \to \mathcal{E}$ be an essential geometric morphism such that f^* is logical and its left adjoint $f_!$ preserves equalizers. Then there is an object X of $\mathcal{E}$, unique up to isomorphism, such that $\mathcal{F}$ is equivalent to $\mathcal{E}/X$ and, modulo this equivalence, the geometric morphism f is isomorphic to the geometric morphism $(X^* \dashv \prod_X)$ of Corollary 1.21.*

Definition 1.24 A *regular category* is a category with finite limits, which has coequalizers of kernel pairs, and in which regular epimorphisms are stable under pullback.

In a regular category, every arrow factors, essentially uniquely, as a regular epimorphism followed by a monomorphism. The construction is as follows: given $f : X \to Y$, let $X \xrightarrow{e} E$ be the coequalizer of the kernel pair of f, and let $m : E \to Y$ be the unique factorization of f through this coequalizer.

Since pullback functors have right adjoints, they preserve regular epimorphisms, so every topos is a regular category.

Lemma 1.25 (PTJ 1.53) *In a topos, every epi is regular.*

Proof. Given an epi $f : X \to Y$, let $X \xrightarrow{e} E \xrightarrow{m} Y$ be its regular epi-mono factorization. Since f is epi, m must be epi; by 1.2, m is an isomorphism. So f is regular epi. ∎

Definition 1.26 An *exact* category is a regular category in which every equivalence relation is effective.

By 1.4 we have:

Proposition 1.27 *Every topos is an exact category.*

Proposition 1.28 (PTJ 1.56) *In a topos the initial object 0 is strict; that is, every arrow into 0 is an isomorphism.*

Proof. Given $X \xrightarrow{i} 0$, we have a pullback

$$\begin{array}{ccc} X & \xrightarrow{i} & 0 \\ {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_0} \\ X & \xrightarrow[i]{} & 0 \end{array}$$

so $\mathrm{id}_X = i^*(\mathrm{id}_0)$. Now id_0 is initial in $\mathcal{E}/0$, so id_X is initial in $\mathcal{E}/X$ (since i^*, having a right adjoint, preserves initial objects). But that means that X is initial in $\mathcal{E}$, since for any object Y of $\mathcal{E}$ there is a bijection between arrows $X \to Y$ in $\mathcal{E}$, and arrows $\mathrm{id}_X \to X^*(Y)$ in $\mathcal{E}/X$. ■

Exercise 24 Proposition 1.28 was given because its proof is a nice application of Theorem 1.18. However, you can show that in fact, in any cartesian closed category with initial object 0, this initial object is strict.

Corollary 1.29 (PTJ 1.57) *In a topos, every coprojection $X \to X+Y$ is monic. Moreover, "coproducts are disjoint": that is, the square*

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X+Y \end{tikzcd}$$

is a pullback.

Proof. From Proposition 1.28 it follows easily that every map $0 \to X$ is monic. Since the given square is always a pushout, the statement follows at once from Corollary 1.9. ■

Exercise 25 Prove that for a topos $\mathcal{E}$ and objects X, Y of $\mathcal{E}$ the categories $\mathcal{E}/(X+Y)$ and $\mathcal{E}/X \times \mathcal{E}/Y$ are equivalent.

As a consequence of regularity (and existence of coproducts) we can form *unions of subobjects*: given subobjects M, N of X, represented by monos $M \xrightarrow{m} X, N \xrightarrow{n} X$, its union $M \cup N$ (least upper bound in the poset $\mathrm{Sub}(X)$) is defined by the regular epi-mono factorization

$$M + N \to M \cup N \to X$$

of the map $\begin{bmatrix} m \\ n \end{bmatrix} : M + N \to X$. We have:

Proposition 1.30 *In a topos, for any object X the poset* $\mathrm{Sub}(X)$ *of subobjects of X is a distributive lattice. Moreover, for any arrow $X \xrightarrow{f} Y$ the pullback functor $f^* : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ between subobject lattices has both adjoints $\exists_f$ and $\forall_f$.*

Proof. Finite meets in $\mathrm{Sub}(X)$ (from now on called "intersections" of subobjects) are given by pullbacks, and unions by the construction above. Distributivity follows from the fact that pullback functors preserve coproducts and regular epimorphisms. The left adjoint $\exists f$ is constructed using the regular epi-mono factorization. The right adjoint $\forall f$ is just the restriction of $\prod_f$ to subobjects: $\prod_f$ preserves monos. ∎

The following fact will be important later on.

Proposition 1.31 (Elephant, A1.4.3) *Let $M \stackrel{m}{\to} X, N \stackrel{n}{\to} X$ be monos into X (we also write M, N for the subobjects represented by m and n). Let the intersection and union of M and N be represented by arrows $M \cap N \to X$, $M \cup N \to X$, respectively. Then the diagram*

$$\begin{tikzcd} M \cap N \arrow[r] \arrow[d] & M \arrow[d] \\ N \arrow[r] & M \cup N \end{tikzcd}$$

is both a pullback and a pushout in $\mathcal{E}$.

Proof. This proof is not the proof given in **Elephant** (that proof is far more general).

The partial order $\mathrm{Sub}(X)$ is, as a category, equivalent to the full subcategory Mon/X of the slice $\mathcal{E}/X$ on the monomorphisms into X. Since the given square is a pullback in $\mathrm{Sub}(X)$ hence in Mon/X, and the domain functor $\mathrm{Mon}/X \to \mathcal{E}$ preserves pullbacks, the square is a pullback in $\mathcal{E}$.

Let us define $\mathrm{Sub}_{\leq 1}(X)$ as the set of those subobjects $M \stackrel{m}{\to} X$ for which the unique map $M \to 1$ is a monomorphism. Note that there is a natural bijection between $\mathrm{Sub}_{\leq 1}(X)$ and $\mathcal{E}(1, \tilde{X})$, where $\tilde{X}$ is the partial map classifier of X. Writing M both for a subobject of X and for the corresponding map $1 \to \tilde{X}$, we define the subobject $\mathrm{dom}(M)$ of 1 by the pullback

$$\begin{tikzcd} \mathrm{dom}(M) \arrow[r] \arrow[d] & 1 \arrow[d, "M"] \\ X \arrow[r, "\eta_X"'] & \tilde{X} \end{tikzcd}$$

Note, that $\mathrm{dom}(M)$ is also the image of the map $M \to 1$. For a subobject c of 1, we define $M{\restriction}c$ by the pullback

$$\begin{tikzcd} M{\restriction}c \arrow[r] \arrow[d] & M \arrow[d] \\ c \arrow[r] & 1 \end{tikzcd}$$

We have the following lemma.

Lemma 1.32 *Let* $M, N \in \mathrm{Sub}_{\leq 1}(X)$*, with* $\mathrm{dom}(M) = c, \mathrm{dom}(N) = d$*. If* $M{\restriction}(c \cap d) = N{\restriction}(c \cap d)$ *as subobjects of* X*, then* $M \cup N \in \mathrm{Sub}_{\leq 1}(X)$*.*

Proof. We must prove that the map $\phi : M \cup N \to 1$ is monic. Clearly, this map factors through $c \cup d$, so it is enough to prove that $(c \cup d)^*(\phi)$ is monic in $\mathcal{E}/(c \cup d)$.

We have $c^*(M \cup N) = c^*(M) \cup c^*(N)$. Since $c^*(N)$ has domain $c^*(d) = c \cap d$ and M and N agree on $c \cap d$, we have $c^*(N) \leq c^*(M)$, so $c^*(M \cup N) = c^*(M)$ and $c^*(\phi)$ is monic. In a symmetric way, $d^*(M \cup N) = d^*(N)$ and $d^*(\phi)$ is monic.

The topos $\mathcal{E}/(c+d)$ is isomorphic to $\mathcal{E}/c \times \mathcal{E}/d$ by Exercise 25, so we see that $(c+d)^*(\phi)$ is monic. Now $c + d \to c \cup d$ is epi, so the pullback functor $\mathcal{E}/(c \cup d) \to \mathcal{E}/(c+d)$ reflects monomorphisms. We conclude that $(c \cup d)^*(\phi)$ monic, as required. This proves the lemma. ∎

Continuing the proof of Proposition 1.31: as usual, we may do as if $X = 1$. So we have subobjects c, d of 1 and we wish to prove that the square

$$\begin{tikzcd} c \cap d \arrow[r] \arrow[d] & c \arrow[d] \\ d \arrow[r] & c \cup d \end{tikzcd}$$

is a pushout. Let $M : c \to X$, $N : d \to X$ be maps which agree on $c \cap d$. Then M and N define elements of $\mathrm{Sub}_{\leq 1}(X)$ for which the hypothesis of Lemma 1.32 holds. Therefore, the map $c \cup d \to X$ which names the subobject $M \cup N$ is a mediating map, which is unique because the maps $\{c \to c \cup d, d \to c \cup d\}$ form an epimorphic family. ∎

1.4 The Topos of Coalgebras

Theorem 1.33 (MM V.8.4; PTJ 2.32) *Let* (G, δ, ε) *be a comonad on a topos* $\mathcal{E}$ *such that the functor* G *preserves finite limits. Then the category* $\mathcal{E}_G$ *of* G*-coalgebras is a topos, and there is a geometric morphism*

$$\begin{tikzcd} \mathcal{E} \arrow[r, shift right, "f_*"'] & \mathcal{E}_G \arrow[l, shift right, "f^*"'] \end{tikzcd}$$

where f^* *is the forgetful functor and* f_* *the cofree coalgebra functor.*

Proof. Finite limits are created by the forgetful functor $V : \mathcal{E}_G \to \mathcal{E}$, since G preserves finite limits; so $\mathcal{E}_G$ has finite limits. Less succinctly, let

$M : I \to \mathcal{E}_G$ be a finite diagram. Let X be a vertex of a limiting cone for $f^* \circ M : I \to \mathcal{E}$. Since G preserves finite limits, GX is (vertex of) a limiting cone for $G \circ f^* \circ M : I \to \mathcal{E}$. If $M(i)$ is the coalgebra $X_i \xrightarrow{g_i} G(X_i)$ then $f^* \circ M(i) = X_i$ and the coalgebra structures on the X_i determine a natural transformation from the constant functor $I \to \mathcal{E}$ with value X, to $G \circ f^* \circ M$. By the limiting property of GX, there is a unique mediating arrow $X \xrightarrow{g} GX$. This is a coalgebra structure on X, and the coalgebra $X \xrightarrow{g} GX$ is also limiting for M in $\mathcal{E}_G$.

Let $R : \mathcal{E} \to \mathcal{E}_G$ be the cofree coalgebra functor: $RX = GX \xrightarrow{\delta_X} G^2 X$. For coalgebras $(A, s), (B, t), (C, u)$ we have:

$$\mathcal{E}(A \times B, C) \simeq \mathcal{E}(A, C^B) \simeq \mathcal{E}_G((A, s), R(C^B))$$

where $f : A \times B \to C$ corresponds to $\tilde{f} : A \to C^B$ and to $f' = G(\tilde{f}) \circ s : A \to G(C^B)$. Note that $f = \mathrm{ev} \circ (\tilde{f} \times \mathrm{id})$.

Now $f : A \times B \to C$ is a coalgebra map if and only if the following diagram commutes:

$$\begin{tikzcd}
A \times B \arrow[r, "s \times t"] \arrow[d, "f"'] & GA \times GB \arrow[r, "\sim"] & G(A \times B) \arrow[d, "G(f)"] \\
C \arrow[rr, "u"'] & & GC
\end{tikzcd}$$

We consider the exponential transposes of both compositions in this diagram. The clockwise composition transposes to

$$(*) \quad A \xrightarrow{f'} G(C^B) \xrightarrow{\rho} GC^{GB} \xrightarrow{GC^t} GC^B$$

where ρ is the transpose of the map $G(C^B) \times GB \xrightarrow{\sim} G(C^B \times B) \xrightarrow{G(\mathrm{ev})} GC$.

The counterclockwise composition transposes to

$$(**) \quad A \xrightarrow{\tilde{f}} C^B \xrightarrow{u^B} GC^B$$

We wish to describe those maps $f : A \times B \to C$ which make these two transposes equal. Let $V : \mathcal{E}_G \to \mathcal{E}$ be the forgetful functor and R the cofree coalgebra functor; we have $V \dashv R$ and $VR = G$. Under this adjunction, the map $(*)$ corresponds to the composition

$$A \xrightarrow{f'} G(C^B) \xrightarrow{\delta} G^2(C^B) \xrightarrow{G\rho} G(GC^{GB}) \xrightarrow{G(GC^t)} G(GC^B)$$

and the map $(**)$ corresponds to the composition

$$A \xrightarrow{f'} G(C^B) \xrightarrow{G(u^B)} G(GC^B)$$

Note that both these composites are maps of coalgebras. So, the maps $f : A \times B \to C$ we are looking for, correspond to maps $\bar{f} : A \to E$, where

$$\begin{tikzcd}
 & G^2(C^B) \arrow[r, "G\rho"] & G(GC^{GB}) \arrow[dr, "G(GC^t)"] & \\
E \arrow[r] & G(C^B) \arrow[u, "\delta"] \arrow[rr, "G(u^B)"'] & & G(GC^b)
\end{tikzcd}$$

is an equalizer in $\mathcal{E}_G$ (equalizer of two maps between cofree coalgebras). So E is the exponent $(C,u)^{(B,t)}$ in $\mathcal{E}_G$.

It remains to show that $\mathcal{E}_G$ has a subobject classifier. To this end we have a look at subobjects of (A,s) in $\mathcal{E}_G$. Our first remark is that if $m : D \to A$ is a subobject of A in $\mathcal{E}$, there is at most one coalgebra structure $d : D \to GD$ on D such that m is a coalgebra map. Indeed, for m to be a coalgebra map we should have $G(m)d = sm$; now $G(m)$ is mono, so there is at most one such d.

On the other hand, if $m : D \to A$ is a subobject and $d : D \to GD$ is any map such that $G(m)d = sm$, then (D,d) is a G-coalgebra and the square

$$\begin{tikzcd}
D \arrow[r, "d"] \arrow[d, "m"'] & GD \arrow[d, "Gm"] \\
A \arrow[r, "s"'] & GA
\end{tikzcd}$$

is a pullback in $\mathcal{E}$. To see this, consider

$$\begin{tikzcd}
D \arrow[rrr, "d"] \arrow[dr, "m"] \arrow[ddd, "d"'] & & & GD \arrow[dl, "Gm"'] \arrow[ddd, "Gd"] \\
 & A \arrow[r, "s"] \arrow[d, "s"'] & GA \arrow[d, "Gs"] & \\
 & GA \arrow[r, "\delta_A"'] & G^2A & \\
GD \arrow[ur, "Gm"] \arrow[rrr, "\delta_D"'] & & & G^2D \arrow[ul, "G^2m"']
\end{tikzcd}$$

The inner square commutes since (A, s) is a coalgebra. The three upper squares commute because of the assumption $G(m)d = sm$, and the lower square is a naturality square for δ. Hence the outer square commutes, which says that the map d is coassociative. To see that d is also counitary, consider the diagram

$$\begin{tikzcd} D \arrow[r, "d"] \arrow[d, "m"'] & GD \arrow[r, "\varepsilon_D"] \arrow[d, "Gm"'] & D \arrow[d, "m"] \\ A \arrow[r, "s"'] & GA \arrow[r, "\varepsilon_A"'] & A \end{tikzcd}$$

Since $m(\varepsilon_D d) = m$ and m is mono, $\varepsilon_D d = \mathrm{id}_D$. Moreover, one sees that the left hand square is a pullback.

Now suppose $m : (D, d) \to (A, s)$ is the inclusion of a subobject in $\mathcal{E}_G$. Let $\tau : G(\Omega) \to \Omega$ be the classifying map of the mono $1 \simeq G(1) \stackrel{G(t)}{\to} G(\Omega)$. Let $h : A \to \Omega$ be the classifying map of m. In the diagram

$$\begin{tikzcd} D \arrow[r, "d"] \arrow[d, "m"'] & GD \arrow[r] \arrow[d, "Gm"'] & 1 \arrow[r] \arrow[d, "G(t)"'] & 1 \arrow[d, "t"] \\ A \arrow[r, "s"'] & GA \arrow[r, "G(h)"'] & G(\Omega) \arrow[r, "\tau"'] & \Omega \end{tikzcd}$$

all three squares are pullbacks (check!), and therefore $\tau G(h)s = h$ by uniqueness of the classifying map. Moreover, since (A, s) is a coalgebra we have $G(h)s = G(\tau)\delta_\Omega G(h)s$, so if we form an equalizer

$$\Omega_G \xrightarrow{e} G(\Omega) \underset{\mathrm{id}}{\overset{G(\tau)\delta_\Omega}{\rightrightarrows}} G(\Omega)$$

(equalizer taken in $\mathcal{E}_G$, the two maps seen as maps between cofree coalgebras), then we see that the map $G(h)s$ factors through Ω_G. Also the map $G(t) : 1 \to G(\Omega)$ factors through this equalizer by a map $e : 1 \to \Omega_G$, which is the subobject classifier of $\mathcal{E}_G$. ∎

Corollary 1.34 (MM V.7.7) *If (T, η, μ) is a monad on a topos $\mathcal{E}$ and the functor T has a right adjoint, then the category of T-algebras is again a topos.*

Proof. Combine Theorems 0.33 and 1.33. ∎

Example 1.35 To give an example, consider a monoid M: a set with an associative multiplication, for which it has a two-sided unit element. The functor $(-) \times M : \mathrm{Set} \to \mathrm{Set}$ has the structure of a monad (using the multiplication and the unit element of M). The category of algebras for this monad is the category of right M-sets, i.e. the category $\widehat{M}$. Note that the functor $(-) \times M$ has a right adjoint $(-)^M$, so we have another proof that $\widehat{M}$ is a topos.

This example can be generalized to any presheaf topos. Given a small category $\mathcal{C}$, consider the product category $\mathrm{Set}^{\mathcal{C}_0}$: the objects are $\mathcal{C}_0$-indexed families $X = (X_c)_{c\in\mathcal{C}_0}$ of sets, the arrows $X \to Y$ are $\mathcal{C}_0$-indexed families $(f_c : X_c \to Y_c)_{c\in\mathcal{C}_0}$ of functions. Clearly, $\mathrm{Set}^{\mathcal{C}_0}$ is a topos. We define an endofunctor T on $\mathrm{Set}^{\mathcal{C}_0}$ as follows: $(TX)_c$ is the set of pairs (α, x) where α is a morphism of $\mathcal{C}$ with domain c, and x is an element of $X_{\mathrm{cod}(\alpha)}$.

Exercise 26 a) Show that T has the structure of a monad on $\mathrm{Set}^{\mathcal{C}_0}$.

b) Show that the category of T-algebras is equivalent to the category $\widehat{\mathcal{C}}$ of presheaves on $\mathcal{C}$.

c) Show that the functor T has a right adjoint. [Hint: consider that the categories $\mathrm{Set}^{\mathcal{C}_0}$ and $\mathrm{Set}/\mathcal{C}_0$ are equivalent, and that modulo this equivalence, the functor T sends the object $X \stackrel{x}{\to} \mathcal{C}_0$ to the object $\mathcal{C}_1 \times_{\mathcal{C}_0} X \stackrel{\mathrm{dom}\circ p_0}{\to} \mathcal{C}_0$, where

$$\begin{tikzcd} \mathcal{C}_1 \times_{\mathcal{C}_0} X \arrow[r, "p_1"] \arrow[d, "p_0"'] & X \arrow[d, "x"] \\ \mathcal{C}_1 \arrow[r, "\mathrm{cod}"'] & \mathcal{C}_0 \end{tikzcd}$$

is a pullback. In other words, T sends $x : X \to \mathcal{C}_0$ to $\sum_{\mathrm{dom}}(\mathrm{cod}^*(x))$. And both functors $\sum_{\mathrm{dom}}$ and cod^* have right adjoints.]

Note that, in view of Corollary 1.34, Example 1.35 provides a proof of Theorem 0.3 without any mention of "colimits of representables" and so on. This should mean that there is a much more general theorem, applying to many more "toposes of sets", which we shall see in the next section.

Let me just point out that from a set theorist's point of view, the only special feature of the "topos of sets" that we make use of, is the equivalence between the categories $\mathrm{Set}^{\mathcal{C}_0}$ and $\mathrm{Set}/\mathcal{C}_0$, which needs the set-theoretic Axiom of Replacement.

Apart from this, the treatment leading up to Theorem 0.3 of course has the advantage of giving explicit constructions for the topos structure.

The geometric morphism $\mathcal{E} \to \mathcal{E}_G$ of Theorem 1.33 has a property that is important enough to deserve its own name.

Definition 1.36 A geometric morphism whose inverse image functor is faithful is called a *surjection.*

In the next chapter we shall see why the name "surjection" is appropriate for this property. We shall also see that a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is a surjection *if and only if* $\mathcal{E}$ is equivalent to the category of coalgebras for a comonad G which preserves finite limits, by an equivalence which transforms f_* into the cofree coalgebra functor and f^* into the forgetful functor.

1.5 Internal Categories and Presheaves

In this section we treat another type of constructions of toposes, generalizing the topos of presheaves on a small category. In **PTJ**, this is the starting point of an elementary theory of geometric morphisms and therefore makes up the initial sections of Chapter 2. For us, however, the elementary theory of geometric morphisms, beautiful as it is, goes outside the scope of these lecture notes. Since the following definitions work for any category with finite limits, let us assume for the time being that $\mathcal{E}$ is such a category.

Definition 1.37 An *internal category* in $\mathcal{E}$ is a structure

$$\mathbf{C} = (C_0, C_1, \mathrm{dom}, \mathrm{cod}, \mathsf{i}, \mu)$$

where C_0 and C_1 are objects of $\mathcal{E}$ (the "object of objects" and "object of arrows" of $\mathbf{C}$, respectively), and $\mathrm{dom}, \mathrm{cod} : C_1 \to C_0$, $\mathsf{i} : C_0 \to C_1$ and $\mu : C_2 \to C_1$ are morphisms of $\mathcal{E}$, where C_2 is the vertex of the pullback diagram

$$\begin{tikzcd} C_2 \arrow[r, "p_1"] \arrow[d, "p_0"'] & C_1 \arrow[d, "\mathrm{dom}"] \\ C_1 \arrow[r, "\mathrm{cod}"'] & C_0 \end{tikzcd}$$

These data should satisfy the following requirements:

1) The compositions $C_0 \xrightarrow{\mathsf{i}} C_1 \xrightarrow{\mathrm{dom}} C_0$ and $C_0 \xrightarrow{\mathsf{i}} C_1 \xrightarrow{\mathrm{cod}} C_0$ are both equal to the identity on C_0.

2) The compositions $C_2 \xrightarrow{\mu} C_1 \xrightarrow{\mathrm{cod}} C_0$ and $C_2 \xrightarrow{p_1} C_1 \xrightarrow{\mathrm{cod}} C_0$ are equal.

3) The compositions $C_2 \xrightarrow{\mu} C_1 \xrightarrow{\mathrm{dom}} C_0$ and $C_2 \xrightarrow{p_0} C_1 \xrightarrow{\mathrm{dom}} C_0$ are equal.

4) The compositions $C_1 \xrightarrow{\langle \mathsf{i}\circ\mathrm{dom}, \mathrm{id}\rangle} C_2 \xrightarrow{\mu} C_1$ and $C_1 \xrightarrow{\langle \mathrm{id}, \mathsf{i}\circ\mathrm{cod}\rangle} C_2 \xrightarrow{\mu} C_1$ are equal to the identity on C_1.

5) Let

$$\begin{tikzcd} C_3 \arrow[r, "r_1"] \arrow[d, "r_0"'] & C_2 \arrow[d, "p_0"] \\ C_2 \arrow[r, "p_0"'] & C_1 \end{tikzcd}$$

be a pullback. The compositions $C_3 \xrightarrow{\mathrm{id}\times\mu} C_2 \xrightarrow{\mu} C_1$ and $C_3 \xrightarrow{\mu\times\mathrm{id}} C_2 \xrightarrow{\mu} C_1$ are equal (note, that these arrows make sense by requirement 2)).

Definition 1.38 Let

$$\begin{array}{rcl} \mathbf{C} & = & (C_0, C_1, \mathrm{dom}, \mathrm{cod}, \mathsf{i}, \mu) \\ \mathbf{D} & = & (D_0, D_1, \mathrm{dom}, \mathrm{cod}, \mathsf{i}, \mu) \end{array}$$

be internal categories in $\mathcal{E}$ (where, for convenience, we have used the same symbols for the structure of both categories). An *internal functor* $F : \mathbf{C} \to \mathbf{D}$ consists of a pair of morphisms $F_0 : C_0 \to D_0$, $F_1 : C_1 \to D_1$ which make the following diagrams commute:

1)

$$\begin{tikzcd} C_0 \arrow[r, "\mathsf{i}"] \arrow[d, "F_0"'] & C_1 \arrow[d, "F_1"] \\ D_0 \arrow[r, "\mathsf{i}"'] & D_1 \end{tikzcd}$$

2)

$$\begin{tikzcd} C_1 \arrow[r, "F_1"] \arrow[d, "\langle\mathrm{dom},\mathrm{cod}\rangle"'] & D_1 \arrow[d, "\langle\mathrm{dom},\mathrm{cod}\rangle"] \\ C_0 \times C_0 \arrow[r, "F_0\times F_0"'] & D_0 \times D_0 \end{tikzcd}$$

3)

$$\begin{tikzcd} C_2 \arrow[r, "\mu"] \arrow[d, "F_2"'] & C_1 \arrow[d, "F_1"] \\ D_2 \arrow[r, "\mu"'] & D_1 \end{tikzcd}$$

where $F_2 : C_2 \to D_2$ is the evident map, which is well-defined by diagram 2).

Clearly, internal categories and internal functors in $\mathcal{E}$ form a category, denoted $\mathbf{cat}(\mathcal{E})$.

Definition 1.39 Let $\mathbf{C} = (C_0, C_1, \mathrm{dom}, \mathrm{cod}, \mathsf{i}, \mu)$ be an internal category in $\mathcal{E}$. An *internal presheaf* on $\mathbf{C}$ is a structure

$$\mathbf{E} = (E_0 \stackrel{e_0}{\to} C_0, E_1 \stackrel{e_1}{\to} E_0)$$

where E_0 and E_1 are objects of $\mathcal{E}$ and e_0, e_1 morphisms in $\mathcal{E}$ such that there is a pullback square

$$\begin{tikzcd} E_1 \arrow[r, "r_1"] \arrow[d, "r_0"'] & C_1 \arrow[d, "\mathrm{cod}"] \\ E_0 \arrow[r, "e_0"'] & C_0 \end{tikzcd}$$

and the following conditions hold:

i) The diagram

$$\begin{tikzcd} E_1 \arrow[r, "r_1"] \arrow[d, "e_1"'] & C_1 \arrow[d, "\mathrm{dom}"] \\ E_0 \arrow[r, "e_0"'] & C_0 \end{tikzcd}$$

commutes.

ii) The composition $E_0 \stackrel{\langle \mathrm{id}, \mathsf{i} \rangle}{\longrightarrow} E_1 \stackrel{e_1}{\to} E_0$ is the identity on E_0 (we write $\langle \mathrm{id}, \mathsf{i} \rangle : E_0 \to E_1$ for the evident factorization of this map through E_1).

iii) Let

$$\begin{tikzcd} E_2 \arrow[r, "s_1"] \arrow[d, "s_0"'] & E_1 \arrow[d, "\mathrm{dom} \circ r_1"] \\ C_1 \arrow[r, "\mathrm{cod}"'] & C_0 \end{tikzcd}$$

be a pullback. Then the two maps $\langle e_1 s_1, s_0 \rangle$ and $(\mathrm{id} \times \mu) \circ \langle r_0 s_1, \langle r_1 s_1, s_0 \rangle \rangle$ from E_2 to $E_0 \times C_1$ both factor through $E_1 \subset E_0 \times C_1$ and (using the same names for these factorizations) we require that the diagram

$$\begin{tikzcd} E_2 \arrow[rr, "\langle e_1 s_1, s_0 \rangle"] \arrow[d, "\langle r_0 s_1, \langle r_1 s_1, s_0 \rangle \rangle"'] & & E_1 \arrow[d, "e_1"] \\ E_0 \times C_2 \arrow[r, "\mathrm{id} \times \mu"'] & E_1 \arrow[r, "e_1"'] & E_0 \end{tikzcd}$$

commutes.

Definition 1.40 Let

$$\begin{aligned}\mathbf{E} &= (E_0 \stackrel{e_0}{\to} C_0, E_1 \stackrel{e_1}{\to} E_0)\\ \mathbf{F} &= (F_0 \stackrel{f_0}{\to} C_0, F_1 \stackrel{f_1}{\to} F_0)\end{aligned}$$

be internal presheaves on $\mathbf{C}$ in $\mathcal{E}$. A morphism of presheaves $\mathbf{E} \to \mathbf{F}$ is a morphism $\alpha_0 : E_0 \to F_0$ in $\mathcal{E}/C_0$ such that for the morphism $\alpha_1 : E_1 \to F_1$ induced by α_0 (given the pullbacks which define E_1 and F_1), we have that $f_1\alpha_1 = \alpha_0 e_1$. The category of internal presheaves on $\mathbf{C}$ in $\mathcal{E}$ is denoted $\mathcal{E}^{\mathbf{C}^{\mathrm{op}}}$.

The following exercise straightforwardly generalizes Example 1.35.

Exercise 27 Fix an internal category $\mathbf{C} = (C_0, C_1, \mathrm{dom}, \mathrm{cod}, \mathrm{i}, \mu)$ in $\mathcal{E}$.

i) Define a functor $T : \mathcal{E}/C_0 \to \mathcal{E}/C_0$ such that for an object $f : X \to C_0$ of $\mathcal{E}/C_0$, the object $T(f) : T(X) \to C_0$ is defined as the composition $T(X) \stackrel{a}{\to} C_1 \stackrel{\mathrm{dom}}{\to} C_0$ where the arrow $T(X) \stackrel{a}{\to} C_1$ is defined by the pullback diagram

$$\begin{tikzcd} T(X) \arrow[r, "b"] \arrow[d, "a"'] & X \arrow[d, "f"] \\ C_1 \arrow[r, "\mathrm{cod}"'] & C_0 \end{tikzcd}$$

ii) Show that the functor T has a monad structure and that the T-algebras are exactly the internal presheaves on $\mathbf{C}$.

iii) Now assume that $\mathcal{E}$ is a topos. Show that the category $\mathcal{E}^{\mathbf{C}^{\mathrm{op}}}$ is a topos.

1.6 Sheaves

We start this section by establishing an internalization of the intersection ($\cap$) operation on subobjects.

Proposition 1.41 *Let $1 \stackrel{t}{\to} \Omega$ be a subobject classifier and denote by $\wedge : \Omega \times \Omega \to \Omega$ the classifying map of the monomorphism $1 \xrightarrow{\langle t,t\rangle} \Omega \times \Omega$. Then for subobjects M, N of X we have: if M is classified by $\phi : X \to \Omega$ and N by $\psi : X \times \Omega$ then the intersection $M \cap N$ is classified by the composite*

$$X \xrightarrow{\langle \phi,\psi\rangle} \Omega \times \Omega \stackrel{\wedge}{\to} \Omega.$$

Proof. Consider maps $f : Y \to X$. If $\langle \phi, \psi \rangle \circ f : Y \to \Omega \times \Omega$ is equal to $\langle t \circ !, t \circ ! \rangle : Y \to \Omega \times \Omega$, then $\phi f = t!$ and $\psi f = t!$, so f factors both through M and through N, hence f factors through the intersection $M \cap N$. We conclude that the diagram

$$\begin{tikzcd} M \wedge N \arrow[r] \arrow[d] & X \arrow[d, "\langle \phi,\psi \rangle"] \\ 1 \arrow[r, "\langle t,t \rangle"'] & \Omega \times \Omega \end{tikzcd}$$

is a pullback, and the statement follows. ∎

Definition 1.42 A *Lawvere-Tierney topology* (MM) or simply *topology* (PTJ) in a topos $\mathcal{E}$ is an arrow $j : \Omega \to \Omega$ with the following properties:

i) $jt = t$:
$$\begin{tikzcd} 1 \arrow[r, "t"] \arrow[dr, "t"'] & \Omega \arrow[d, "j"] \\ & \Omega \end{tikzcd}$$

ii) $jj = j$:
$$\begin{tikzcd} \Omega \arrow[r, "j"] \arrow[dr, "j"'] & \Omega \arrow[d, "j"] \\ & \Omega \end{tikzcd}$$

iii) $j \circ \wedge = \wedge \circ (j \times j)$:
$$\begin{tikzcd} \Omega \times \Omega \arrow[r, "\wedge"] \arrow[d, "j \times j"'] & \Omega \arrow[d, "j"] \\ \Omega \times \Omega \arrow[r, "\wedge"'] & \Omega \end{tikzcd}$$

The following definition generalizes Definition 0.18.

Definition 1.43 (PTJ 3.13) A *universal closure operation* on a topos $\mathcal{E}$ is given by, for each object X, a map $c_X : \mathrm{Sub}(X) \to \mathrm{Sub}(X)$, which system has the following properties:

i) $M \leq c_X(M)$ for every subobject M of X (the operation is *inflationary*).

ii) $M \leq N$ implies $c_X(M) \leq c_X(N)$ for $M, N \in \mathrm{Sub}(X)$ (the operation is order-preserving).

iii) $c_X(c_X(M)) = c_X(M)$ for each $M \in \mathrm{Sub}(X)$ (the operation is idempotent).

iv) For every arrow $f : Y \to X$ and every $M \in \mathrm{Sub}(X)$ we have

$$c_Y(f^*(M)) = f^*(c_X(M))$$

(the operation is *stable*).

Instead of $c_X(M)$ we shall also sometimes write $\overline{M}$, if the subobject lattice in which we work is clear.

Exercise 28 Use the stability (requirement iv) of 1.43) to deduce that a closure operation commutes with finite intersections: $\overline{M \cap N} = \overline{M} \cap \overline{N}$.

Note that the result of Exercise 28 means that a universal closure operation is different from "closure" in Topology, where closure commutes with *union*, not with intersection of subsets.

Proposition 1.44 (MM V.1.1; PTJ 3.14) *There is a bijection between universal closure operations and Lawvere-Tierney topologies.*

Proof. If j is a Lawvere-Tierney topology, define for $M \in \mathrm{Sub}(X)$, classified by $\phi : X \to \Omega$, $\overline{M}$ as the subobject of X classified by $j\phi$. We use the letter J to denote the subobject of Ω classified by j:

$$\begin{tikzcd} J \arrow[r] \arrow[d] & \Omega \arrow[d, "j"] \\ 1 \arrow[r, "t"'] & \Omega \end{tikzcd}$$

We see that J is the closure of the subobject $(1 \xrightarrow{t} \Omega)$. We have: $\overline{M}$ is the vertex of the pullback

$$\begin{tikzcd} \overline{M} \arrow[r] \arrow[d] & X \arrow[d, "\phi"] \\ J \arrow[r] & \Omega \end{tikzcd}$$

and we conclude that $M \leq \overline{M}$. The other properties of the universal closure operation are straightforward and left to you.

In the other direction, given a universal closure operation $c_X(-)$, let j be the classifying map of $c_\Omega(1 \xrightarrow{t} \Omega)$. The verification of the properties of a Lawvere-Tierney topology, as well as that the two described operations are inverse to each other, is again left to you. ∎

Definition 1.45 Given a Lawvere-Tierney topology j with associated closure operation $c_X(-)$ (or $\overline{(-)}$), we call a subobject M of X:

dense if $\overline{M} = X$

closed if $\overline{M} = M$.

Definition 1.46 Consider, for an object X, partial maps into X with domain a dense subobject:

$$\begin{tikzcd} M' \arrow[r, "m"] \arrow[dr] & M \\ & X \end{tikzcd}$$

with $m : M' \to M$ a dense mono (i.e., the subobject represented by the mono m is dense).

> The object X is called *separated for* j if any such partial map has *at most one* extension to a map $M \to X$.
>
> The object X is called a *sheaf* for j (or a j-sheaf) if any such partial map has *exactly one* extension to a map $M \to X$.

We write $\mathrm{Sh}_j(\mathcal{E})$ for the full subcategory of $\mathcal{E}$ on the sheaves for j.

Theorem 1.47 (MM V.2.5; PTJ §3.2) *For any topos $\mathcal{E}$ with Lawvere-Tierney topology j, the category $\mathrm{Sh}_j(\mathcal{E})$ is a topos. The inclusion functor $\mathrm{Sh}_j(\mathcal{E}) \to \mathcal{E}$ preserves finite limits and exponentials, and $\mathrm{Sh}_j(\mathcal{E})$ is closed under finite limits in $\mathcal{E}$.*

Proof. Suppose $\mathcal{I}$ is a finite category and $X : \mathcal{I} \to \mathrm{Sh}_j(\mathcal{E})$ a functor with limiting cone (N, μ) in $\mathcal{E}$. So, for each object i of $\mathcal{I}$ we have an arrow $\mu_i : N \to X(i)$, and this system is natural: for an arrow $f : i \to k$ in $\mathcal{I}$ we have $X(f)\mu_i = \mu_k$.

Given a diagram

$$\begin{tikzcd} M' \arrow[r, "m"] \arrow[dr, "\phi"'] & M \\ & N \end{tikzcd}$$

with m a dense mono, the compositions $\mu_i\phi : M' \to X(i)$ extend uniquely to maps $\nu_i : M \to X(i)$, because the objects $X(i)$ are sheaves. Moreover the uniqueness of the extensions means that the maps ν_i inherit the naturality from the maps μ_i. So we have a cone ν for X with vertex M; since the cone

(N, μ) was limiting, we have a unique map $\psi : M \to N$ which is a map of cones. It follows that $\psi m = \phi$. We conclude that N is a j-sheaf.

Note that this proves that $\mathrm{Sh}_j(\mathcal{E})$ is closed under the finite limits of $\mathcal{E}$, that it has finite limits and that the inclusion preserves them.

Secondly, if F is a sheaf, then the exponential F^Y is a sheaf, for any object Y. For, given a partial map

$$\begin{tikzcd} M' \arrow[r, "m"] \arrow[dr] & M \\ & F^Y \end{tikzcd}$$

with m dense, this diagram transposes under the exponential adjunction to a partial map

$$\begin{tikzcd} M' \times Y \arrow[r, "m\times\mathrm{id}"] \arrow[dr] & M \times Y \\ & F \end{tikzcd}$$

Now by the stability of the closure operation, the subobject $M' \times Y \xrightarrow{m\times\mathrm{id}} M \times Y$ is dense. Sine F is a sheaf we have a unique extension $M \times Y \to F$, which transposes back to give a unique extension for the original diagram. We conclude that $\mathrm{Sh}_j(\mathcal{E})$ is cartesian closed and that the inlusion into $\mathcal{E}$ preserves exponentials.

For the subobject classifier of $\mathrm{Sh}_j(\mathcal{E})$ we need an intermediate result, which we have already seen in the case $\mathcal{E} = \widehat{\mathcal{C}}$.

Lemma 1.48 *Let M be a sheaf and M' a subobject of M. Then M' is a sheaf if and only if M' is closed in M.*

Proof. Suppose M' is closed in M and $M' \xleftarrow{f} N' \longrightarrow N$ is a partial map with N' dense in N. Let $i : M' \to M$ be the inclusion. Now $i \circ f$ has a unique extension $g : N \to M$. Let

$$\begin{tikzcd} L \arrow[r] \arrow[d] & N \arrow[d, "g"] \\ M' \arrow[r] & M \end{tikzcd}$$

be a pullback. Then $f : N' \to M'$ factors through $L \to M'$, so $N' \leq L$ as subobjects of N, but L is closed (since it is a pullback of $M' \to M$) and N'

is dense. We see that $N = \overline{N'} \leq \overline{L} = L$, so $L \to N$ is an isomorphism and we have $g : N \to M'$. So M' is a sheaf.

Conversely if $M' \in \mathrm{Sub}(M)$ is a sheaf, consider the partial map

$$\begin{tikzcd} M' \arrow[r] \arrow[d, "\mathrm{id}"'] & \overline{M'} \\ M' & \end{tikzcd}$$

Since $M' \to \overline{M'}$ is dense, there is a unique extension $\overline{M'} \to M'$. It follows that $M' = \overline{M'}$, so M' is closed in M. ∎

Returning to the proof of 1.47: closed subobjects of X are classified by maps of the form $j\phi$, hence their classifying maps land in the image of j, which is (by the idempotence of j) the equalizer

$$\Omega_j \longrightarrow \Omega \underset{j}{\overset{\mathrm{id}}{\rightrightarrows}} \Omega$$

Hence, Ω_j is a subobject classifier for $\mathrm{Sh}_j(\mathcal{E})$ provided we can show that it is a sheaf.

Now partial maps $\Omega_j \longleftarrow M' \longrightarrow M$ correspond to closed subobjects of M'. But given that M' is dense in M, there is an order-preserving bijection between the closed subobjects of M' and of M, given as follows: for A closed in M, we have $A \cap M'$ closed in M' and for B closed in M' we have $c_M(B)$ closed in M. To see that these operations are each other's inverse, observe that for A closed in M:

$$c_M(A \cap M') = c_M(A) \cap c_M(M') = c_M(A) = A$$

and for B closed in M' we have

$$c_M(B) \cap M' = c_{M'}(B) = B$$

The given partial map has therefore a unique extension $M \to \Omega_j$ (the classifier of the closed subobject of M corresponding to the closed subobject of M' classified by the partial map); and Ω_j is a sheaf, as desired. ∎

Example 1.49 Looking back at Section 0.3, we see almost at once that for any topological space X we have a Lawvere-Tierney topology on $\widehat{\mathcal{O}_X}$; if $F \subset G$ is a subpresheaf and U is an open subset of X then we let $\bar{F}(U)$ consist of those elements $s \in G(U)$ for which the set of opens $V \subseteq U$ satisfying $s{\restriction}V \in F(V)$ covers U. It follows at once from Theorem 1.47 that $\mathrm{Sh}(X)$ is a topos.

Next, we shall see that the embedding of sheaves in the ambient topos has a left adjoint which preserves finite limits (Theorem 1.53 below). However, Proposition 1.50 is of independent interest, since it characterizes separated objects

Proposition 1.50 *For an object X of $\mathcal{E}$ the following are equivalent:*

i) *X is j-separated.*

ii) *X is a subobject of a j-sheaf.*

iii) *X is a subobject of a sheaf of the form Ω_j^E.*

iv) *The diagonal $\delta : X \to X \times X$ is a j-closed subobject of $X \times X$.*

Proof. We prove i)⇒iv)⇒iii)⇒ii)⇒i).

For i)⇒iv): let X be separated and let $\overline{\delta}$ be the closure of δ as subobject of $X \times X$. Consider the partial map

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\mathrm{id}"'] & \overline{\delta} \\ X & \end{tikzcd}$$

If $i : \overline{\delta} \to X \times X$ is the inclusion and $p_1, p_2 : X \times X \to X$ are the projections, then both $p_1 i$ and $p_2 i$ are fillers for this diagram, so since X is separated, $p_1 i = p_2 i$. This means that $i : \overline{\delta} \to X \times X$ factors through the equalizer of p_1 and p_2, which is δ. So $\overline{\delta} = \delta$ as subobjects of $X \times X$.

For iv)⇒iii): Let $\Delta : X \times X \to \Omega$ classify the diagonal δ, and $\{\cdot\} : X \to \Omega^X$ its exponential transpose, which is a monomorphism. Since δ is closed in $X \times X$, Δ factors through Ω_j, and therefore $\{\cdot\}$ factors through Ω_j^X. So X is a subobject of Ω_j^X, which is a j-sheaf.

The implication iii)⇒ii) is trivial.

For ii)⇒i): Let $X \xrightarrow{i} F$ be mono, with F a j-sheaf. Suppose that

$$\begin{tikzcd} M' \arrow[r, "m"] \arrow[d, "k"'] & M \\ X & \end{tikzcd}$$

is a partial map with m a dense mono. If both of $f, g : M \to X$ are fillers for this diagram then $if = ig$ since F is a sheaf; hence $f = g$ since i is mono. So X is j-separated. ∎

Lemma 1.51 *Let j be a Lawvere-Tierney topology in a topos $\mathcal{E}$, and let X be an object of $\mathcal{E}$. As usual, we denote the diagonal subobject of $X \times X$ by δ and its closure by $\overline{\delta}$.*

a) *If $f, g : Z \to X$ is a parallel pair of arrows into X, then the morphism $\langle f, g\rangle : Z \to X \times X$ factors through $\overline{\delta}$ if and only if the equalizer of f and g is a j-dense subobject of Z.*

b) *The subobject $\overline{\delta}$ of $X \times X$ is an equivalence relation on X.*

c) *Let $X \to MX$ be the coequalizer of the pair $\overline{\delta} \rightrightarrows X$. Then any map $X \to L$, for a j-separated object L of $\mathcal{E}$, factors uniquely through $X \to MX$. Hence the assignment $X \mapsto MX$ induces a functor which is left adjoint to the inclusion $\mathrm{sep}_j(\mathcal{E}) \to \mathcal{E}$, where $\mathrm{sep}_j(\mathcal{E})$ denotes the full subcategory of $\mathcal{E}$ on the j-separated objects.*

Proof. a) Let $E_{fg} \to Z$ denote the equalizer of f, g. Consider the diagram:

$$\begin{tikzcd}
 & E' \arrow[dd] \arrow[dr] & \\
E_{fg} \arrow[ur] \arrow[rr] \arrow[dd] & & Z \arrow[dd, "\langle f,g\rangle"] \\
 & \overline{\delta} \arrow[dr] & \\
X \arrow[ur] \arrow[rr, "\delta"'] & & X \times X
\end{tikzcd}$$

where all the squares are pullbacks. We see that E' is the closure of E_{fg}, and we see that the map $\langle f, g\rangle$ factors through $\overline{\delta}$ if and only if $E' \to Z$ is an isomorphism, which holds if and only if E_{fg} is a dense subobject of Z.

b) We prove that for an arbitrary object Z of $\mathcal{E}$, the set of ordered pairs

$$\{(f, g) \in \mathcal{E}(Z, X)^2 \,|\, \langle f, g\rangle \text{ factors through } \overline{\delta}\}$$

is an equivalence relation on $\mathcal{E}(Z, X)$. Now reflexivity and symmetry are obvious, and using the notation above for equalizers we easily see that $E_{fg} \wedge E_{gh} \leq E_{fh}$. Since the meet of two dense subobjects is dense, we see that the relation is transitive.

c) We have to prove that any map $f : X \to L$ with L separated, coequalizes the parallel pair $r_0, r_1 : \overline{\delta} \to X$ which is the equivalence relation from part b). Now clearly for $f \times f : X \times X \to L \times L$, the composite $(f \times f) \circ \delta$ factors through the diagonal subobject $L \xrightarrow{\delta_L} L \times L$, so the composite $(f \times f) \circ \langle r_0, r_1 \rangle$ factors through the closure of δ_L. But δ_L is closed by Proposition 1.50iv), so $fr_0 = fr_1$ and f factors uniquely through $X \to MX$. The adjointness is also clear, provided we can show that MX is separated. Now δ is classified by $\Delta : X \times X \to \Omega$, which has as exponential transpose the map $\{\cdot\} : X \to \Omega^X$. So, δ is the kernel pair of $\{\cdot\}$. Now $\overline{\delta}$ is classified by $j \circ \Delta$, the exponential transpose of which is $j^X \circ \{\cdot\} : X \to \Omega_j^X$. And $\overline{\delta}$ is the kernel pair of $j^X \circ \{\cdot\}$. We see that, by the construction of epi-mono factorizations in a regular category, $X \to MX \to \Omega_j^X$ is an epi-mono factorization. So MX is a subobject of a sheaf, and therefore separated by 1.50. ■

Lemma 1.52 *Suppose we have an operation which, to any object X of $\mathcal{E}$, assigns a sheaf LX and a dense inclusion $MX \xrightarrow{i_X} LX$, where M is the functor of Lemma 1.51. Then this extends to a unique functor $L : \mathcal{E} \to \mathcal{E}$. Moreover, this functor has the property that for every X, every map from X to a sheaf factors uniquely through the composite $X \to MX \xrightarrow{i_X} LX$, so $L : \mathcal{E} \to \mathrm{Sh}_j(\mathcal{E})$ is left adjoint to the inclusion $\mathrm{Sh}_j(\mathcal{E}) \to \mathcal{E}$.*

Proof. For $f : X \to X'$, define $Lf : LX \to LX'$ as the unique filler for the partial map

$$\begin{tikzcd} MX \arrow[r, "i_X"] \arrow[d, "i_{X'} \circ Mf"'] & LX \\ LX' & \end{tikzcd}.$$

The functoriality and the adjointness follow at once. ■

Theorem 1.53 *The inclusion functor $\mathrm{Sh}_j(\mathcal{E}) \to \mathcal{E}$ has a left adjoint which preserves finite limits. Hence, we have a geometric morphism $i : \mathrm{Sh}_j(\mathcal{E}) \to \mathcal{E}$.*

Proof. Let, as before, $\Delta : X \times X \to \Omega$ classify the diagonal $\delta : X \to X \times X$. Then $j \circ \Delta : X \times X \to \Omega_j$ classifies the closure $\overline{\delta}$; let $\overline{\{\cdot\}} : X \to \Omega_j^X$ be its exponential transpose. One can easily verify that the kernel pair of $\overline{\{\cdot\}}$ is $\overline{\delta}$, so $\overline{\{\cdot\}}$ factors as $X \to MX \to \Omega_j^X$ which, since a topos is regular, is the epi-mono factorization of $\overline{\{\cdot\}}$. Let LX be the closure of the subobject MX of Ω_j^X. Then we have the assumptions of Lemma 1.52 verified, so L is a functor left adjoint to the inclusion $\mathrm{Sh}_j(\mathcal{E}) \to \mathcal{E}$. We need to prove that L preserves finite limits. The following proof is taken from **Elephant**, A4.4.7.

First of all, we have seen in the proof of Theorem 1.47 that $\mathrm{sh}_j(\mathcal{E})$ is an exponential ideal in $\mathcal{E}$ (for a sheaf F and an arbitrary X, F^X is a sheaf). From this, it follows easily that L preserves finite products: for objects A and B of $\mathcal{E}$ and a sheaf F, we have the following natural bijections:

$$\begin{array}{l}\mathcal{E}(L(A\times B),F)\simeq\mathcal{E}(A\times B,F)\simeq\mathcal{E}(A,F^B)\simeq\mathcal{E}(LA,F^B)\simeq\\ \mathcal{E}(B,F^{LA})\simeq\mathcal{E}(LB,F^{LA})\simeq\mathcal{E}(LA\times LB,F)\end{array}$$

so $L(A\times B)\simeq LA\times LB$.

Furthermore, by Exercise 21, an object in $\mathrm{sh}_j(\mathcal{E})$ is injective if and only if it is a retract of some Ω_j^X; since the inclusion $\mathrm{sh}_j(\mathcal{E})\to\mathcal{E}$ preserves exponentials and since Ω_j is a retract of Ω (hence Ω_j^X is a retract of Ω^X), we see that the inclusion preserves injective objects. Given that $\mathrm{sh}_j(\mathcal{E})$ has enough injectives, by the same exercise we have that L preserves monos.

Now we wish to show that L preserves "coreflexive equalizers". A *coreflexive pair* is a parallel pair $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ with common retraction $Y\xrightarrow{h}X$: $hf=hg=\mathrm{id}_X$. A coreflexive equalizer is an equalizer of a coreflexive pair.

In a category with finite products, every equalizer appears also as coreflexive equalizer: the arrow $E\xrightarrow{e}X$ is an equalizer of $f,g:X\to Y$ if and only if e is an equalizer of the coreflexive pair $\langle\mathrm{id}_X,f\rangle,\langle\mathrm{id}_X,g\rangle:X\to X\times Y$ (which has as common retraction the projection $X\times Y\to X$). Therefore, if coreflexive equalizers exist, all equalizers exist and if coreflexive equalizers are preserved (by a product-preserving functor) then all equalizers are preserved.

Let $f,g:X\to Y$ be a coreflexive pair. You should check that $e:E\to X$ is an equalizer of f,g if and only if the square

$$\begin{tikzcd} E \arrow[r, "e"] \arrow[d, "e"'] & X \arrow[d, "f"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

is a pullback. Therefore, if e is an equalizer of f,g then $E\to X$ is the meet (intersection) in $\mathrm{Sub}(Y)$ of the subobjects represented by f and g. We wish therefore to show that L preserves meets of subobjects.

To this end, let $M\xrightarrow{m}X, N\xrightarrow{n}X$ be monos representing subobjects M and N, and let $M\cap N$, $M\cup N$ be their intersection and union. The square

$$\begin{tikzcd} M\cap N \arrow[r] \arrow[d] & M \arrow[d] \\ N \arrow[r] & M\cup N \end{tikzcd}$$

is a pushout in $\mathcal{E}$ by Proposition 1.31. Since L is a left adjoint, the square

$$\begin{tikzcd} L(M \cap N) \arrow[r] \arrow[d] & LM \arrow[d] \\ LN \arrow[r] & L(M \cup N) \end{tikzcd}$$

is a pushout in $\mathrm{sh}_j(\mathcal{E})$. We know that L preserves monos, so $L(M \cap N) \to LN$ is mono; so Corollary 1.9 applies and the square is also a pullback. Since also $L(M \cup N) \to LX$ is mono, also the square

$$\begin{tikzcd} L(M \cap N) \arrow[r] \arrow[d] & LM \arrow[d] \\ LN \arrow[r] & X \end{tikzcd}$$

is a pullback. We conclude that $L(M \cap N) = LM \cap LN$ so L indeed preserves meets of subobjects. ∎

Definition 1.54 A geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is called an *embedding* if the direct image functor f_* is full and faithful.

The geometric morphism of Theorem 1.53 is an embedding. Moreover we shall see that every embedding is of this form (Proposition 2.21).

Examples 1.55 For our usual examples of geometric morphisms, we have:

1) Given a continuous map of topological spaces $f : X \to Y$, the associated geometric morphism $\mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ is an embedding if and only if f is an embedding of topological spaces (i.e. X is a subspace of Y).

2) For a morphism $u : X \to Y$ in a topos $\mathcal{E}$, the geometric morphism $\mathcal{E}/X \to \mathcal{E}/Y$ is an embedding if and only if u is mono.

3) For a functor $F : \mathcal{C} \to \mathcal{D}$ between small categories, the geometric morphism $\widehat{F} : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ is an embedding if and only if F is full and faithful.

1.7 Miscellaneous exercises

Exercise 29 We consider the category $\mathcal{C}$ whose objects are subsets of $\mathbb{N}$, and arrows $A \to B$ are *finite-to-one* functions, i.e. functions f satisfying the requirement that for every $b \in B$, the set $\{a \in A \mid f(a) = b\}$ is finite.

a) Show that $\mathcal{C}$ has pullbacks.

b) Define for every object A of $\mathcal{C}$ a set $\mathrm{Cov}(A)$ of sieves on A as follows: $R \in \mathrm{Cov}(A)$ if and only if R contains a finite family $\{f_1, \ldots, f_n\}$ of functions into A, which is *jointly almost surjective*, that is: the set

$$A - \bigcup_{i=1}^{n} \mathrm{Im}(f_i)$$

is finite.

Show that Cov is a Grothendieck topology.

c) Show that if $R \in \mathrm{Cov}(A)$, then R contains a family $\{f_1, \ldots, f_n\}$ which is jointly almost surjective and moreover, every f_i is injective.

d) Given a *nonempty* set X and an object A of $\mathcal{C}$, we define $F_X(A)$ as the set of equivalence classes of functions $\xi : A \to X$, where $\xi \sim \eta$ if $\xi(n) = \eta(n)$ for all but finitely many $n \in A$.

Show that this definition can be extended to the definition of a presheaf F_X on $\mathcal{C}$.

e) Show that F_X is a sheaf for Cov.

Exercise 30 In a category with finite products $\mathcal{E}$, a *monoid object* is an object A together with maps $1 \xrightarrow{e} A$ and $A \times A \xrightarrow{m} A$ such that the diagrams

$$\begin{tikzcd}
 & A \times A \arrow[d, "m"] & \\
A \arrow[ur, "{\langle e,\mathrm{id}_A \rangle}"] \arrow[r, "\mathrm{id}_A"'] & A & A \arrow[ul, "{\langle \mathrm{id}_A,e \rangle}"'] \arrow[l, "\mathrm{id}_A"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times A \times A \arrow[r, "m \times A"] \arrow[d, "A \times m"'] & A \times A \arrow[d, "m"] \\
A \times A \arrow[r, "m"'] & A
\end{tikzcd}$$

commute. We have a category $\mathrm{Mon}(\mathcal{E})$ of monoid objects (and monoid maps) in $\mathcal{E}$ and a forgetful functor $\mathrm{Mon}(\mathcal{E}) \to \mathcal{E}$. The category $\mathcal{E}$ is said to *have free monoids* if this forgetful functor has a left adjoint.

a) Prove that for every small category $\mathcal{C}$, $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ has free monoids.

b) Give an example of a small category $\mathcal{C}$ and a Grothendieck topology on $\mathcal{C}$ for which the free monoid construction of a) does not always yield a sheaf, even if we start out with a sheaf.

Exercise 31 Let $\mathcal{C}$ be a small category; we work in the category $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ of presheaves on $\mathcal{C}$.

a) (2 pts) Let U be a subobject of 1. Show that U determines a *sieve on* $\mathcal{C}$, that is: a set of objects $\mathcal{D}$ with the property that for any morphism $X \to Y$, if $Y \in \mathcal{D}$ then $X \in \mathcal{D}$.

b) (3 pts) We define, using the sieve $\mathcal{D}$ on $\mathcal{C}$ from part a), a morphism $c(U) : \Omega \to \Omega$ in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ by putting, for a sieve R on an object C:

$$c(U)_C(R) \ = \ R \cup \{f : C' \to C \,|\, C' \in \mathcal{D}\}$$

Prove that $c(U)$ is a Lawvere-Tierney topology on $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$.

c) (2 pts) Let F be a subpresheaf of a presheaf G on $\mathcal{C}$. Prove that F is dense for $c(U)$ if and only if for all $C \in \mathcal{C}_0$ and $x \in G(C)$ we have: $x \in F(C)$ or $C \in \mathcal{D}$.

d) (3 pts) Prove that the category of sheaves for $c(U)$ is equivalent to the category of presheaves on some subcategory of $\mathcal{C}$.

2 Geometric Morphisms

In this chapter, we shall limit ourselves to the theory of geometric morphisms between Grothendieck toposes (or, slightly more generally, cocomplete toposes). The material is taken mainly from **MM**. We recall that a *geometric morphism* $\mathcal{F} \to \mathcal{E}$ between toposes is an adjoint pair $f^* \dashv f_*$ with $f^* : \mathcal{E} \to \mathcal{F}$ (the *inverse image functor*), $f_* : \mathcal{F} \to \mathcal{E}$ (the *direct image functor*), with the additional property that f^* preserves finite limits.

Examples 2.1 1) In Section 0.3 we have seen that every continuous function of topological spaces $f : X \to Y$ determines a geometric morphism $\mathrm{Sh}(X) \to \mathrm{Sh}(Y)$. If the space Y is sufficiently separated (here we shall assume that Y is Hausdorff, although the weaker condition of *sober* suffices) then there is a converse to this: every geometric morphism $\mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ is induced by a unique continuous function. Indeed, let f be such a geometric morphism. In $\mathrm{Sh}(Y)$, the lattice of subobjects of 1 (the terminal object) is in 1-1, order-preserving, bijection with $\mathcal{O}(Y)$, the set of open subsets of Y. The same for X, of course. Now the inverse image f^*, preserving finite limits, preserves subobjects of 1 and therefore induces a function $f^- : \mathcal{O}(Y) \to \mathcal{O}(X)$. Since f^* preserves colimits and finite limits, the function f^- preserves the top element ($f^-(Y) = X$), finite intersections and arbitrary unions (in particular, $f^-(\emptyset) = \emptyset$).

Define a relation R from X to Y as follows: $R(x, y)$ holds if and only if $x \in f^-(V)$ for every open neighbourhood V of y. We shall show that R is in fact a function $X \to Y$, leaving the remaining details as an exercise.

i) Assume $R(x, y)$ and $R(x, y')$ both hold, and $y \neq y'$. By the Hausdorff property, y and y' have disjoint open neighbourhoods V_y and $V_{y'}$. By assumption and the preservation properties of f^- we have:

$$x \in f^-(V_y) \cap f^-(V_{y'}) \;=\; f^-(V_y \cap V_{y'}) \;=\; f^-(\emptyset) \;=\; \emptyset$$

a clear contradiction. So the relation R is single-valued.

ii) Suppose for $x \in X$ there is no $y \in Y$ satisfying $R(x, y)$. Then for every y there is a neighbourhood V_y such that $x \notin f^-(V_y)$. Then we have

$$x \notin \bigcup_{y \in Y} f^-(V_y) \;=\; f^-(\bigcup_{y \in Y} V_y) \;=\; f^-(Y) \;=\; X$$

also a clear contradiction. So the relation R is total, and therefore a function.

Exercise 32 Show that the function R just constructed is continuous, and that it induces the given geometric morphism f.

In view of this connection between topological spaces and their categories of sheaves, and the obvious equivalence between Set and the topos of sheaves on a one-point space, we have the following terminology.

Definition 2.2 A geometric morphism $\mathrm{Set} \to \mathcal{E}$ is called a *point* of $\mathcal{E}$.

2) Consider, for a group G, the category $\widehat{G}$ of right G-sets. Let $\Delta : \mathrm{Set} \to \widehat{G}$ be the functor which sends a set X to the trivial G-set X (i.e. the G-action is the identity). Note that Δ preserves finite limits. The functor Δ has a right adjoint Γ, which sends a G-set X to its subset of *G-invariant elements*, i.e. to the set

$$\{x \in X \mid xg = x \text{ for all } g \in G\}$$

Note that $\widehat{G}(\Delta(Y), X)$ is naturally isomorphic to $\mathrm{Set}(Y, \Gamma(X))$, so we have a geometric morphism $\widehat{G} \to \mathrm{Set}$. Actually, this geometric morphism is essential, because Δ also has a left adjoint: we have that $\widehat{G}(X, \Delta(Y))$ is naturally isomorphic to $\mathrm{Set}(\mathrm{Orb}(X), Y)$, where $\mathrm{Orb}(X)$ denotes the set of orbits of X under the G-action.

Exercise 33 Prove that the functor Orb does not preserve equalizers (Hint: you can do this directly (think of two maps $G \to G$), or apply Theorem 1.23).

This example can be generalized in two directions, as the following items show.

3) Let $\mathcal{E}$ be a cocomplete topos. Then there is exactly one geometric morphism $\mathcal{E} \to \mathrm{Set}$, up to natural isomorphism. For, a geometric morphism is determined by its inverse image functor, which must preserve 1 and coproducts; and since, in Set, every object X is the coproduct of X copies of 1, for $f : \mathcal{E} \to \mathrm{Set}$ we must have $f^*(X) = \sum_{x \in X} 1$. For a function $\phi : X \to Y$ we have $[\mu_{\phi(x)}]_{x \in X} : \sum_{x \in X} 1 \to \sum_{y \in Y} 1$ (where μ_i sends 1 to the i'th cofactor of the coproduct $\sum_{y \in Y} 1$) which is $f^*(\phi) : f^*(X) \to f^*(Y)$. This defines $f^* : \mathrm{Set} \to \mathcal{E}$.

Exercise 34 Show that the functor f^* preserves finite limits.

The functor f^* has a right adjoint: for a set X and object Y of $\mathcal{E}$ we have

$$\mathcal{E}(f^*(X), Y) \simeq \mathcal{E}(\sum_{x \in X} 1, Y) \simeq \prod_{x \in X} \mathcal{E}(1, Y) \simeq \mathrm{Set}(X, \mathcal{E}(1, Y))$$

so the functor which sends Y to its set of *global sections* (arrows $1 \to Y$) is right adjoint to f^*. The "global sections functor" is usually denoted by the letter Γ; its left adjoint, the "constant objects functor" by Δ.

4) Consider presheaf categories $\widehat{\mathcal{C}}, \widehat{\mathcal{D}}$, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We have a geometric morphism $\widehat{F} : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ constructed as follows. We have a functor $\widehat{F}^* : \widehat{\mathcal{D}} \to \widehat{\mathcal{C}}$ which sends a presheaf $X : \mathcal{D}^{\mathrm{op}} \to \mathrm{Set}$ to $X \circ F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. In other words,

$$\widehat{F}^*(X)(C) = X(F(C))$$

Exercise 35 Prove that the functor $\widehat{F}^*$ preserves all small limits.

A right adjoint $\widehat{F}_*$ for $\widehat{F}^*$ may be constructed using the Yoneda Lemma. Indeed, for $\widehat{F}_*$ to exist, it should satisfy:

$$\widehat{F}_*(Y)(D) \simeq \widehat{\mathcal{D}}(y_D, \widehat{F}_*(Y)) \simeq \widehat{\mathcal{C}}(\widehat{F}^*(y_D), Y)$$

so we just *define* $\widehat{F}_*$ on objects by putting $\widehat{F}_*(Y)(D) = \widehat{\mathcal{C}}(\widehat{F}^*(y_D), Y)$.

Exercise 36 Complete the definition of $\widehat{F}_*$ as a functor, and show that it is indeed a right adjoint for $\widehat{F}^*$.

The functor $\widehat{F}^* : \widehat{\mathcal{D}} \to \widehat{\mathcal{C}}$ has also a left adjoint (so the geometric morphism $\widehat{F}$ is essential). Recall from Section 0.2 that for a presheaf X on $\mathcal{C}$ we have the *category of elements of* X, denoted $\mathrm{Elts}(X)$: objects are pairs (x, C) with $x \in X(C)$, and arrows $(x, C) \to (x', C')$ are arrows $f : C \to C'$ in $\mathcal{C}$ satisfying $X(f)(x') = x$. We have the projection functor $\pi : \mathrm{Elts}(X) \to \mathcal{C}$. Define the functor $\widehat{F}_! : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ as follows: for $X \in \widehat{\mathcal{C}}$, $\widehat{F}_!(X)$ is the colimit in $\widehat{\mathcal{D}}$ of the diagram

$$\mathrm{Elts}(X) \xrightarrow{\pi} \mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{y} \widehat{\mathcal{D}}$$

We shall shortly see a more concrete presentation of such "left Kan extensions".

5) In Sections 0.4 and 1.5 we have seen that if Cov is a Grothendieck topology on a small category $\mathcal{C}$, then the category $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ of sheaves for Cov is a topos, and the inclusion functor $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov}) \to \widehat{\mathcal{C}}$ has a left adjoint (*sheafification*) which preserves finite limits; so this is also an example of a geometric morphism.

2.1 Points of $\widehat{\mathcal{C}}$

We recall from Section 0.2 that the functor $y : \mathcal{C} \to \widehat{\mathcal{C}}$ is the "free cocompletion of $\mathcal{C}$". That means the following: given an arbitrary functor F from $\mathcal{C}$ to a cocomplete category $\mathcal{E}$ there is a unique (up to natural isomorphism) colimit-preserving functor $\widetilde{F} : \widehat{\mathcal{C}} \to \mathcal{E}$ such that the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[dr, "y"'] & \mathcal{E} \\ & \widehat{\mathcal{C}} \arrow[u, "\widetilde{F}"'] \end{tikzcd}$$

commutes up to isomorphism. The functor $\widetilde{F}$ is called the "left Kan extension of F along y".

Of course, $\widetilde{F}(X)$ can be defined as the colimit in $\mathcal{E}$ of the diagram $\mathrm{Elts}(X) \stackrel{\pi}{\to} \mathcal{C} \stackrel{F}{\to} \mathcal{E}$. We wish to present this colimit as a form of "tensor product". Let us review the definition from Commutative Algebra.

If R is a commutative ring, we consider the category R-Mod of R-modules and R-module homomorphisms. If M and N are R-modules, the set $\mathrm{Hom}_R(M, N)$ of R-module homomorphisms from M to N is also an R-module (with pointwise operations), and the functor $\mathrm{Hom}_R(M, -)$: R-Mod $\to$ R-Mod has a left adjoint $(-) \otimes_R M$. For an R-module L we define an equivalence relation $\sim$ on the set $L \times M$: it is the least equivalence relation satisfying

$$(x, y{\cdot}r) \sim (x{\cdot}r, y)$$

for all $x \in L, y \in M, r \in R$. The equivalence class of (x, y) is denoted $x \otimes y$, and $L \otimes M$ is the R-module generated by all such elements $x \otimes y$, subject to the relations

$$(x + x') \otimes y = x \otimes y + x' \otimes y$$
$$x \otimes (y + y') = x \otimes y + x \otimes y'$$

and with R-action $(x \otimes y)r = (xr \otimes y) = (x \otimes ry)$. In fact, one has a coequalizer diagram of abelian groups:

$$L \times R \times M \underset{\psi}{\overset{\phi}{\rightrightarrows}} L \times M \longrightarrow L \otimes M$$

where $\phi(x,r,y) = (xr,y)$ and $\psi(x,r,y) = (x,ry)$. The R-module M is called *flat* if the functor $(-)\otimes M$ preserves exact sequences; given that this functor is a left adjoint, this is equivalent to saying that it preserves finite limits.

Something similar happens if we have a functor $A : \mathcal{C} \to \mathrm{Set}$ and a presheaf X on $\mathcal{C}$ and we wish to calculate the value of the left Kan extension $\widetilde{A}$ on X. Let $\mathcal{C}_1$ be the set of arrows of $\mathcal{C}$. On $\mathbb{A} = \sum_{C\in\mathcal{C}} A(C)$ there is a (partial) "left $\mathcal{C}_1$-action" $x \mapsto f{\cdot}x = A(f)(x)$, for $x \in A(C)$ and $f : C \to C'$. Similarly, on $\mathbb{X} = \sum_{C\in\mathcal{C}} X(C)$ there is a partial "right $\mathcal{C}_1$-action" $x \mapsto x{\cdot}f = X(f)(x)$, for $x \in X(C')$ and $f : C \to C'$. We can now represent the set $\widetilde{A}(X)$ as a coequalizer of sets

$$\textstyle\sum_{C,C'\in\mathcal{C}} X(C') \times \mathcal{C}(C,C') \times A(C) \underset{\psi}{\overset{\phi}{\rightrightarrows}} \sum_{C,C'\in\mathcal{C}} X(C) \times A(C) \longrightarrow \widetilde{A}(X)$$

where $\phi(x,f,a) = (x{\cdot}f,a)$ and $\psi(x,f,a) = (x,f{\cdot}a)$. Therefore we write, from now on, $X \otimes_{\mathcal{C}} A$ for $\widetilde{A}(X)$.

Theorem 2.3 (MM VII.2.2) *Let $A : \mathcal{C} \to \mathrm{Set}$ be a functor. Then we have an adjunction*

$$\mathrm{Set} \underset{R}{\overset{L}{\leftrightarrows}} \widehat{\mathcal{C}}$$

with $L \dashv R$, $R(Y)(C) = \mathrm{Set}(A(C),Y)$ and $L(X) = X \otimes_{\mathcal{C}} A$.

Now for geometric morphisms $\mathrm{Set} \to \widehat{\mathcal{C}}$ we need the left adjoint $(-)\otimes_{\mathcal{C}} A$ to preserve finite limits.

Definition 2.4 (MM VII.5.1) A functor $A : \mathcal{C} \to \mathrm{Set}$ is called *flat* if the functor $(-)\otimes_{\mathcal{C}} A$ preserves finite limits.

The following theorem summarizes our discussion so far.

Theorem 2.5 (MM VII.5.2) *Points of the presheaf topos $\widehat{\mathcal{C}}$ correspond to flat functors $\mathcal{C} \to \mathrm{Set}$.*

Definition 2.6 A category I is called *filtering* if the following conditions are satisfied:

i) I is nonempty.

ii) For each pair of objects (i,j) of I there is a diagram $i \leftarrow k \to j$ in I.

iii) For each parallel pair $i \underset{b}{\overset{a}{\rightrightarrows}} j$ there is an arrow $k \xrightarrow{c} i$ which equalizes the pair.

Now let $A : \mathcal{C} \to \mathrm{Set}$. We have the category $\mathrm{Elts}(A)$: objects are pairs (x, C) with $x \in A(C)$; an arrow $(x, C) \to (x', C')$ is a morphism $f : C \to C'$ in $\mathcal{C}$ such that $A(f)(x) = x'$.

Definition 2.7 *A functor $A : \mathcal{C} \to \mathrm{Set}$ is called filtering if the category $\mathrm{Elts}(A)$ is filtering.*

Exercise 37 Let P be a poset and $A : P \to \mathrm{Set}$ a filtering functor. Show that the category $\mathrm{Elts}(A)$ is isomorphic to a *filter* in P, that is: a nonempty subset $F \subseteq P$ with the following properties:

i) The set F is upwards closed: if $p \leq q$ and $p \in F$, then $q \in F$.

ii) Any two elements of F have a common lower bound in F.

The following theorem provides a concrete handle on flat functors.

Theorem 2.8 (MM VII.6.3) *A functor $A : \mathcal{C} \to \mathrm{Set}$ is flat if and only if A is filtering.*

Proof. Assume that $A : \mathcal{C} \to \mathrm{Set}$ is flat. By definition, the following diagram commutes up to isomorphism:

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "A"] \arrow[dr, "y"'] & & \mathrm{Set} \\
& \widehat{\mathcal{C}} \arrow[ur, "(-)\otimes_{\mathcal{C}} A"'] &
\end{tikzcd}$$

So, $y_C \otimes_{\mathcal{C}} A \simeq A(C)$, for objects C of $\mathcal{C}$. We check the conditions for a filtering category.

i) Since $(-) \otimes_{\mathcal{C}} A$ preserves terminal objects, $1 \otimes_{\mathcal{C}} A$ is a one-point set. This shows that A is nonempty.

ii) Since $(-) \otimes_{\mathcal{C}} A$ preserves binary products, we have that the map

$$\begin{array}{c}(y_C \times y_D) \otimes_{\mathcal{C}} A \to A(C) \times A(D) : \\ ((B \stackrel{u}{\to} C, B \stackrel{v}{\to} D), a) \mapsto (u{\cdot}a, v{\cdot}a)\end{array}$$

(for $a \in A(B), u{\cdot}a = A(u)(a), v{\cdot}a = A(v)(a)$)

must be an isomorphism; in particular it is surjective. That is condition ii) of the definition of a filtering functor.

iii) Finally, consider a parallel pair $C \overset{u}{\underset{v}{\rightrightarrows}} D$ in $\mathcal{C}$ and an element $a \in A(C)$ such that $u{\cdot}a = v{\cdot}a$ (that is, a parallel pair in $\mathrm{Elts}(A)$). Let

$$P \longrightarrow y_C \overset{y_u}{\underset{y_v}{\rightrightarrows}} y_D$$

be an equalizer diagram in $\widehat{\mathcal{C}}$. Since $(-) \otimes_{\mathcal{C}} A$ preserves equalizers, we have an equalizer diagram

$$P \otimes_{\mathcal{C}} A \xrightarrow{i} A(C) \overset{A(u)}{\underset{A(v)}{\rightrightarrows}} A(D)$$

in Set. Here, for $w \in P(B), b \in A(B)$, $i(w \otimes b) = w{\cdot}b \in A(C)$. Since $u{\cdot}a = v{\cdot}a$, there must be some pair (w, b) for which $i(w \otimes b) = a$. This gives condition iii) of the definition of a filtering functor.

For the converse, only a sketch: suppose A is filtering. Now for $R \in \widehat{\mathcal{C}}$, the set $R \otimes_{\mathcal{C}} A$ is a quotient of the sum $\sum_{C \in \mathcal{C}} R(C) \times A(C)$ by the equivalence relation $\sim$ generated by the set of equivalent pairs $((r{\cdot}g, a), (r, g{\cdot}a))$ for $r \in R(C), a \in A(C')$ and $g : C' \to C$. However, given that A is filtering this can be simplified. We have: $(r, a) \in R(C) \times A(C)$ is equivalent to $(r', a') \in R(C') \times A(C')$ if and only if there is a diagram $C \xleftarrow{u} D \xrightarrow{v} C'$ in $\mathcal{C}$ and an element $b \in A(D)$ such that the equations

$$u{\cdot}b = a \quad v{\cdot}b = a' \quad r{\cdot}u = r{\cdot}v$$

hold. From this definition, it is straightforward to prove that $(-) \otimes_{\mathcal{C}} A$ preserves finite limits. ∎

Corollary 2.9 (MM VII.6.4) *Suppose $\mathcal{C}$ is a category with finite limits. Then a functor $A : \mathcal{C} \to \mathrm{Set}$ is flat if and only if it preserves finite limits.*

Proof. Again we use that the composite functor $((-) \otimes_{\mathcal{C}} A) \circ y : \mathcal{C} \to \mathrm{Set}$ is naturally isomorphic to A. If A is flat, then $(-) \otimes_{\mathcal{C}} A$ preserves finite limits and y always preserves existing finite limits, so then A preserves all finite limits. Note, that this direction does not require $\mathcal{C}$ to have all finite limits.

Conversely, suppose $\mathcal{C}$ has finite limits and A preserves them. Then A is filtering:

i) $A(1) = 1$, so A is nonempty.

ii) We have $A(C) \times A(D) \simeq A(C \times D)$ so in condition ii) of Definition 2.6 we can take the projections $C \xleftarrow{\pi_C} C \times D \xrightarrow{\pi_D} D$ and appropriate element of $A(C \times D)$.

iii) By a similar argument, now involving an equalizer in $\mathcal{C}$.

■

Corollary 2.10 (MM VII.6.5) *Let $\mathcal{D}$ be a small category. Then the colimit functor* $\mathrm{Set}^{\mathcal{D}} \to \mathrm{Set}$ *preserves finite limits if and only if $\mathcal{D}^{\mathrm{op}}$ is filtering.*

Remark 2.11 In standard text books in category theory, for example MacLane, one finds a dual definition of "filtering" (i.e., a category is "filtering" in MacLane's sense if its opposite category is filtering in our sense). For this notion of filtering, part of Corollary 2.10 is contained in the slogan that "filtered colimits commute with finite limits in Set".

Exercise 38 Deduce Corollary 2.10.

2.2 Geometric Morphisms $\mathcal{E} \to \widehat{\mathcal{C}}$ for cocomplete $\mathcal{E}$

The universal property of the Yoneda embedding $y : \mathcal{C} \to \widehat{\mathcal{C}}$ ($\widehat{\mathcal{C}}$ being the free cocompletion of $\mathcal{C}$) holds with respect to all cocomplete categories, not just Set. Therefore, every geometric morphism $f : \mathcal{E} \to \widehat{\mathcal{C}}$ is determined by the composite functor $f^* \circ y : \mathcal{C} \to \mathcal{E}$. Again, we have a suitably defined "tensor product" $X \otimes_{\mathcal{C}} A$ (when $A : \mathcal{C} \to \mathcal{E}$ is a functor and $X \in \widehat{\mathcal{C}}$), which is now defined as a colimit in $\mathcal{E}$ rather than in Set.

We cannot write down exactly the same formula for what will be the functor $(-) \otimes_{\mathcal{C}} A$ as we did for the case of Set, as something like "$X(C') \times \mathcal{C}(C, C') \times A(C)$" is not meaningful: $X(C')$ and $\mathcal{C}(C, C')$ are sets but $A(C)$ is an object of $\mathcal{E}$. However, using the cocompleteness of $\mathcal{E}$ we have the expression $\sum_{x \in X(C'), f: C \to C'} A(C')$ which, in the case of $\mathcal{E} = \mathrm{Set}$, is the same thing. Let, for a coproduct $\sum_{i \in I} X_i$, $\mu_i : X_i \to \sum_{i \in I} X_i$ denote the i'th coprojection. Then we define $X \otimes_{\mathcal{C}} A$ as the coequalizer

$$\textstyle\sum_{C \in \mathcal{C}, x \in X(C), f: C' \to C} A(C') \overset{\theta}{\underset{\tau}{\rightrightarrows}} \sum_{C \in \mathcal{C}, x \in X(C)} A(C) \longrightarrow X \otimes_{\mathcal{C}} A$$

where $\theta = [\theta_{C,x,f}]_{C \in \mathcal{C}, x \in X(C), f: C' \to C}$; and $\theta_{C,x,f}$ is defined to be the composite

$$A(C') \xrightarrow{A(f)} A(C) \xrightarrow{\mu_{C,x}} \sum_{C \in \mathcal{C}, x \in X(C)} A(C).$$

Likewise, $\tau = [\tau_{C,x,f}]_{C\in\mathcal{C},x\in X(C),f:C'\to C}$ where $\tau_{C,x,f}$ is the map

$$A(C) \stackrel{\mu_{C',x\cdot f}}{\longrightarrow} \sum_{C\in\mathcal{C},x\in X(C)} A(C).$$

Again, we define the functor $A : \mathcal{C} \to \mathcal{E}$ to be *flat* if the functor $(-)\otimes_{\mathcal{C}} A : \widehat{\mathcal{C}} \to \mathcal{E}$ preserves finite limits. And we have a similar notion of filtering as in 2.7:

Definition 2.12 (MM VII.8.1) A functor $A : \mathcal{C} \to \mathcal{E}$ is *filtering* if the following conditions hold:

i) The family of all maps $A(C) \to 1$ is epimorphic.

ii) For objects C, D of $\mathcal{C}$, the family of maps

$$\{\langle A(u), A(v)\rangle : A(B) \to A(C)\times A(D) \,|\, u : B \to C, v : B \to D\}$$

is epimorphic.

iii) For any parallel pair of arrows $u, v : C \to D$ in $\mathcal{C}$ and equalizer diagram

$$E_{u,v} \xrightarrow{e} A(C) \overset{A(u)}{\underset{A(v)}{\rightrightarrows}} A(D)$$

in $\mathcal{E}$, the family of all arrows

$$\{A(B) \stackrel{f}{\to} E_{u,v} \,|\, \text{for some } w : B \to C \text{ in } \mathcal{C} \text{ with } uw = vw,\ ef = A(w)\}$$

is epimorphic.

Without proof, we record:

Theorem 2.13 (MM VII.9.1) *Let $\mathcal{E}$ be a cocomplete topos, and $\mathcal{C}$ a small category. Then a functor $A : \mathcal{C} \to \mathcal{E}$ is flat if and only if it is filtering.*

We see that geometric morphisms $\mathcal{E} \to \widehat{\mathcal{C}}$ correspond to filtering functors $\mathcal{C} \to \mathcal{E}$, for cocomplete $\mathcal{E}$.

2.3 Geometric morphisms to $\mathcal{E} \to \mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ for cocomplete $\mathcal{E}$

Recall that we use the word Cov to denote a general Grothendieck topology; so $\mathrm{Cov}(C)$ is a collection of covering sieves on C (where C is an object of $\mathcal{C}$). Also recall that a sieve on C can be regarded as a subobject of the representable presheaf y_C. Finally, we established that an object X of $\widehat{\mathcal{C}}$ is a sheaf for Cov, if and only if for every object C of $\mathcal{C}$ and every $R \in \mathrm{Cov}(C)$, any diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & y_C \\ X & \end{tikzcd}$$

has a *unique* filler: an arrow $y_C \to X$ making the triangle commute.
For the remainder of this section, $\mathcal{E}$ will always be a *cocomplete* topos.

Exercise 39 Let $i : \mathrm{Sh}(\mathcal{C}, \mathrm{Cov}) \to \widehat{\mathcal{C}}$ the geometric morphism where i_* is the inclusion and i^* is sheafification. Suppose $p : \mathcal{E} \to \widehat{\mathcal{C}}$ is a geometric morphism such that the direct image p_* factors through i_* by a functor $q : \mathcal{E} \to \mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$. Show that the composite p^*i_* is left adjoint to q and conclude that the inverse image p^* is isomorphic to a functor which factors through $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$.

Exercise 39 tells us that a geometric morphism $p : \mathcal{E} \to \widehat{\mathcal{C}}$ factors through $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ if and only if every object $p_*(E)$ is a sheaf for Cov. The following exercise gives us a criterion for when this is the case.

Exercise 40 Let $p : \mathcal{E} \to \widehat{\mathcal{C}}$ be a geometric morphism, and let Cov be a Grothendieck topology on $\mathcal{C}$. Then the following two statements are equivalent:

i) For every object E of $\mathcal{E}$, p_*E is a sheaf for Cov.

ii) For every Cov-covering sieve R on C, p^* sends the inclusion $R \to y_C$ to an isomorphism in $\mathcal{E}$.

Now we characterized geometric morphisms $\mathcal{E} \to \widehat{\mathcal{C}}$ by flat functors $\mathcal{C} \to \mathcal{E}$; so we would like to characterize also geometric morphisms $p : \mathcal{E} \to \mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ in terms of such functors. Every such geometric morphism determines a geometric morphism into $\widehat{\mathcal{C}}$, hence a flat functor $A : \mathcal{C} \to \mathcal{E}$; we need to see which flat functors give rise to geometric morphisms which factor through $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$. It should not be a surprise that we can characterize these functors by their behaviour on covering sieves, now seen as diagrams in $\mathcal{C}$: every sieve on C is a diagram of arrows with codomain C.

Lemma 2.14 (MM VII.7.3) *Let* Cov *be a Grothendieck topology on a small category* $\mathcal{C}$*, and let* $p : \mathcal{E} \to \widehat{\mathcal{C}}$ *be a geometric morphism. Then the following statements are equivalent:*

i) *The geometric morphism* p *factors through* $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$*.*

ii) *The composite* $p^*\circ y : \mathcal{C} \to \mathcal{E}$ *sends* Cov*-covering sieves to colimiting cocones in* $\mathcal{E}$*.*

iii) *The composite* $p^*\circ y$ *sends* Cov*-covering sieves to epimorphic families in* $\mathcal{E}$*.*

Definition 2.15 A functor $A : \mathcal{C} \to \mathcal{E}$ is called *continuous* if it has the properties of the composite $p^*\circ y$ in Lemma 2.14.

We can now state:

Theorem 2.16 (MM, Corollary VII.7.4) *There is an equivalence of categories between*

$$\mathcal{T}op(\mathcal{E}, \mathrm{Sh}(\mathcal{C}, \mathrm{Cov}))$$

and the category of flat and continuous functors $\mathcal{C} \to \mathcal{E}$*.*

Recall (Definition 1.36) that a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is called a surjection if the inverse image functor f^* is faithful.

Lemma 2.17 (MM Vii.4.3) *For a geometric morphism* $f : \mathcal{F} \to \mathcal{E}$ *the following are equivalent:*

i) *The inverse image* f^* *is faithful.*

ii) *Every component of the unit* η *of the adjunction* $f^* \dashv f_*$ *is a monomorphism.*

iii) *The functor* f^* *reflects isomorphisms.*

iv) *The functor* f^* *induces an injective homomorphism of lattices* $\mathrm{Sub}_{\mathcal{E}}(E) \to \mathrm{Sub}_{\mathcal{F}}(f^*E)$*.*

v) *The functor* f^* *reflects the order on subobjects: for* $A, B \in \mathrm{Sub}_{\mathcal{E}}(E)$*,* $f^*A \leq f^*B$ *if and only if* $A \leq B$*.*

Proof. The equivalence (i)$\Leftrightarrow$(ii) is basic Category Theory.

For (i)$\Rightarrow$(iii): a faithful functor reflects monos and epis, and a topos is balanced (1.2).

For (iii)$\Rightarrow$(iv): Since f^* preserves monos, it induces a map on subobjects. Furthermore f^* preserves images and coproducts, hence unions of subobjects; also, f^* preserves intersections. So f^* induces a lattice homomorphism. Since f^* reflects isomorphisms, it is injective.

For (iv)$\Rightarrow$(v): If $f^*A \leq f^*B$ then $f^*A = f^*A \cap f^*B = f^*(A \cap B)$ because f^* is a lattice homomorphism. Hence $A = A \cap B$ since f^* is injective; so $A \leq B$.

For (v)$\Rightarrow$(i): if $X \underset{v}{\overset{u}{\rightrightarrows}} Y$ is a parallel pair with equalizer $E \xrightarrow{e} X$, then $f^*(u) = f^*(v)$ entails (since f^* preserves equalizers) that $f^*(E)$ is the maximal subobject of f^*X. By (v), this entails that E is the maximal subobject of X; in other words, $u = v$. So f^* is faithful. ∎

Proposition 2.18 (MM VII.4.4) *A geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is a surjection if and only if $\mathcal{E}$ is equivalent to the topos of coalgebras for a finite limit preserving comonad on $\mathcal{F}$ and f is, modulo this equivalence, the cofree-forgetful geometric morphism.*

Proof. One direction is clear, since the forgetful functor is always faithful. For the other, suppose f is a surjection and consider the comonad f^*f_* on $\mathcal{F}$. Let us spell out the dual version of Beck's Crude Tripleability Theorem (0.28):

CTT$^{\mathrm{op}}$ Let $A \underset{U}{\overset{F}{\leftrightarrows}} C$ be an adjunction with $F \dashv U$. Suppose C has equalizers of coreflexive pairs, F preserves them and F reflects isomorphisms. Then the functor F is comonadic.

It is clear that for a surjection f, the conditions are satisfied. The conclusion follows. ∎

Examples 2.19 1) For a continuous map f of Hausdorff spaces, the induced geometric morphism is a surjection if and only if the map f is surjective (**MM**, start of §VII.4).

2) For a morphism $f : A \to B$ in a topos $\mathcal{E}$, the induced geometric morphism $\mathcal{E}/A \to \mathcal{E}/B$ is a surjection if and only if f is an epimorphism.

3) For a functor $F : \mathcal{C} \to \mathcal{D}$ between small categories, the induced geometric morphism $\widehat{F} : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ of Example 2.1 4) is a surjection if and

only if every object of $\mathcal{D}$ is a retract of an object in the image of F (**Elephant**, A4.2.7).

2.4 The Factorization Theorem

Theorem 2.20 (MM VII.4.6) *Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism. There exists a Lawvere-Tierney topology j in $\mathcal{E}$ such that f factors as*

$$\begin{tikzcd}
\mathcal{F} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{E} \\
& \mathrm{Sh}_j(\mathcal{E}) \arrow[ur, "i"'] &
\end{tikzcd}$$

where p is a surjection and i is the geometric morphism from Theorem 1.53. Moreover, given another factorization $\mathcal{F} \xrightarrow{q} \mathcal{G} \xrightarrow{k} \mathcal{E}$ of f with q a surjection and k an embedding, there is an equivalence $\mathcal{G} \to \mathrm{Sh}_j(\mathcal{E})$ which makes the following diagram commute:

$$\begin{tikzcd}
\mathcal{F} \arrow[rr, "f"] \arrow[dr, "p"] \arrow[ddr, "q"'] & & \mathcal{E} \\
& \mathrm{Sh}_j(\mathcal{E}) \arrow[ur, "i"'] & \\
& \mathcal{G} \arrow[u] \arrow[uur, "k"'] &
\end{tikzcd} .$$

Proof. Consider the closure operation $c_{(-)}$ on $\mathcal{E}$ defined as follows: for a subobject $U \xrightarrow{u} X$, $c_X(u)$ is the subobject of X given by the following pullback:

$$\begin{tikzcd}
c_X(u) \arrow[r] \arrow[d] & f_*f^*U \arrow[d, "f_*f^*u"] \\
X \arrow[r, "\eta_X"'] & f_*f^*X
\end{tikzcd}$$

where η is the unit of the adjunction $f^* \dashv f_*$.

Exercise 41 Check yourself that this defines a universal closure operation.

We claim that for arbitrary subobjects U, V of X the following holds: $V \leq$

$c_X(U)$ if and only if $f^*V \leq f^*U$. Indeed, consider the commuting diagram:

$$\begin{tikzcd}
V \arrow[r, "\eta"] \arrow[d] & f_*f^*V \arrow[d] \\
X \arrow[r, "\eta"] & f_*f^*X \\
c_X(U) \arrow[r] \arrow[u] & f_*f^*U \arrow[u]
\end{tikzcd}$$

where η is the unit of the adjunction $f^* \dashv f_*$. If $f^*V \leq f^*U$ then $f_*f^*V \leq f_*f^*U$ so, since the lower square is a pullback, the arrow $V \to X$ factors through $c_X(U)$; i.e., $V \leq c_X(U)$.

Conversely, if $V \leq c_X(U)$, we obtain an arrow $V \xrightarrow{\mu} f_*f^*U$ such that the following diagram commutes:

$$\begin{tikzcd}
V \arrow[r, "\mu"] \arrow[dr] & f_*f^*U \arrow[r] & f_*f^*X \\
& X \arrow[ur, "\eta"'] &
\end{tikzcd}$$

Transposing along $f^* \dashv f_*$ we get

$$\begin{tikzcd}
f^*V \arrow[r, "\hat{\mu}"] \arrow[dr] & f^*U \arrow[r] & f^*X \\
& f^*X \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

and, since $f^*V \to f^*X$ is mono, also $\hat{\mu}$ is mono, and $f^*V \leq f^*U$.

The following exercise is very similar to Exercise 40b):

Exercise 42 Suppose $\mathcal{F} \xrightarrow{f} \mathcal{E}$ is a geometric morphism and j is a Lawvere-Tierney topology in $\mathcal{E}$. Then f_* factors through the inclusion $\mathrm{sh}_j(\mathcal{E}) \to \mathcal{E}$ if and only if f^* maps j-dense monos to isomorphisms in $\mathcal{F}$.

Now if $U \xrightarrow{u} X$ is a mono which is dense for (the topology associated to) the closure operator $c_{(-)}$, then $X \leq c_X(U)$, so $f^*X \leq f^*U$ and f^*u is an isomorphism. By Exercise 42, we conclude that f_* factors through $\mathrm{sh}_j(\mathcal{E})$. And by reasoning as in Exercise 39, we obtain a factorization of geometric morphisms:

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{E} \\ & \mathrm{sh}_j(\mathcal{E} \arrow[ur, "i"'] & \end{tikzcd} .$$

Remains to see that p is a surjection. Consider subobjects $U \leq V$ of X in $\mathrm{sh}_j(\mathcal{E})$; suppose $p^*U \simeq p^*V$. Then $f^*i_*U \simeq f^*i_*V$ so, since U and V are *closed* subobjects of X, we have $i_*U \simeq i_*V$. Since i_* is full and faithful, $U \simeq V$ follows. We conclude that p^* reflects isomorphisms of subobjects; by Lemma 2.17, p is a surjection as claimed.

For the essential uniqueness of the decomposition, I refer to **MM**, Theorem VII.4.8. ∎

We can now give the promised characterization of embeddings:

Proposition 2.21 *For a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ the following statements are equivalent:*

i) f is an embedding (i.e., f_* is full and faithful).

ii) The counit $\varepsilon : f^*f_* \Rightarrow \mathrm{id}_{\mathcal{F}}$ is an isomorphism.

iii) There is a Lawvere-Tierney topology j in $\mathcal{E}$ and an equivalence $e : \mathcal{F} \to \mathrm{sh}_j(\mathcal{E})$ such that the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "f"] \arrow[dr, "e"'] & \mathcal{E} \\ & \mathrm{sh}_j(\mathcal{E}) \arrow[u] \end{tikzcd}$$

commutes up to isomorphism.

Proof. The equivalence between i) and ii) is standard Category Theory, and the implication iii)⇒i) is clear. For the converse, assume f is an embedding. By Theorem 2.20, there is a factorization

$$\begin{tikzcd} \mathcal{F} \arrow[r, "f"] \arrow[dr, "p"'] & \mathcal{E} \\ & \mathrm{sh}_j(\mathcal{E}) \arrow[u, "i"'] \end{tikzcd}$$

with p a surjection. Since i_* and f_* are full and faithful, so is p_* (check!). Therefore the counit ε for $p^* \dashv p_*$ is an isomorphism. Consider the "triangular identity" from basic Category Theory for arbitrary $E \in \mathcal{E}$:

$$\begin{tikzcd} p^*E \arrow[r, "p^*\eta_E"] \arrow[dr, "\mathrm{id}"'] & p^*p_*p^*E \arrow[d, "\varepsilon_{p^*E}"] \\ & p^*E \end{tikzcd}$$

Since ε is an isomorphism, we see that $p^*(\eta_E)$ is an isomorphism. But p is a surjection, so η_E is an isomorphism. We see that both ε and η are isomorphisms, so p is an equivalence. ∎

Examples 2.22 Let us see how standard geometric morphisms decompose:

1) Every continuous map $f : X \to Y$ of topological spaces factors as $X \to Z \to Y$, where Z is the image of X, topologized as a subspace of Y. The map $X \to Z$ is surjective, the map $Z \to Y$ is an embedding. Hence the geometric morphism $\mathrm{Sh}(X) \to \mathrm{Sh}(Z)$ is a surjection and $\mathrm{Sh}(Z) \to \mathrm{Sh}(Y)$ is an embedding.

2) Every morphism in a topos has an epi-mono factorization, as we have seen. This gives at once a surjection-embedding factorization of the geometric morphism between the slice toposes.

3) For a functor $F : \mathcal{C} \to \mathcal{D}$ between small categories, let $\mathcal{B}$ be the full subcategory of $\mathcal{D}$ on objects in the image of F; and let $\mathcal{C} \stackrel{G}{\to} \mathcal{B} \stackrel{H}{\to} \mathcal{D}$ be the evident factorization. Then G is surjective on objects and H is full and faithful; so $\widehat{\mathcal{C}} \stackrel{\widehat{G}}{\to} \widehat{\mathcal{B}} \stackrel{\widehat{H}}{\to} \widehat{\mathcal{D}}$ is a surjection-embedding factorization of $\widehat{F}$.

3 Logic in Toposes

The material for this section is taken from **MM**.

3.1 The Heyting structure on subobject lattices in a topos

Definition 3.1 A *Heyting algebra* is a poset with finite limits and colimits, which is cartesian closed as a category. So, a Heyting algebra H comes with elements $\bot, \top \in H$ (the *bottom* and *top* elements respectively), operations $\sqcap, \sqcup : H \times H \to H$ for *greatest lower bound* (or *meet*) and *least upper bound* (or *join*), respectively; and an operation $\Rightarrow : H \times H \to H$ for *Heyting implication* (the exponential in the cartesian closed structure). Spelling out the property of the exponential $y \Rightarrow z$, we get:

$$x \leq (y \Rightarrow z) \text{ if and only if } x \sqcap y \leq z$$

Note that the order on H is definable from the meet $\sqcap$ since $x \leq y$ holds if and only if $x = x \sqcap y$ so we may as well present a Heyting algebra as a set with some special elements and functions.

Exercise 43 Show that every Boolean algebra is a Heyting algebra. Show that for any topological space, the set of opens (with the inclusion ordering) is a Heyting algebra which is generally not Boolean.

Our goal in this section is to see that in a topos $\mathcal{E}$, every subobject lattice $\mathrm{Sub}(X)$ is a Heyting algebra and this holds *in a natural way*: that is, for any arrow $f : Y \to X$ in $\mathcal{E}$ the pullback map $f^* : \mathrm{Sub}(X) \to \mathrm{Sub}(Y)$ preserves the Heyting algebra structure.

In $\mathcal{E}$, $\mathrm{Sub}(X)$ is naturally isomorphic to $\mathcal{E}(X, \Omega)$ and there are constants $\mathsf{t}, \mathsf{f} : 1 \to \Omega$ and operations $\wedge, \vee, \Rightarrow : \Omega \times \Omega \to \Omega$ which induce, via this isomorphism, the Heyting structure on each $\mathrm{Sub}(X)$. Let us write these down explicitly:

> We have constants $\mathsf{t}, \mathsf{f} : 1 \to \Omega$: t is the subobject classifier t, and f classifies the least subobject of 1, which is the initial object 0. So the square
>
> $$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[d, "\mathsf{f}"] \\ 1 \arrow[r, "\mathsf{t}"'] & \Omega \end{tikzcd}$$
>
> is a pullback.

Recall from Proposition 1.41 that we have a map $\wedge : \Omega \times \Omega$ which classifies $\langle t, t\rangle : 1 \to \Omega \times \Omega$. In every subobject lattice $\mathrm{Sub}(X)$ we have *meets* (greatest lower bounds, or *intersections*) of subobjects: the meet $M \cap N$ of subobjects $m : M \to X$ and $n : N \to X$ is given by pullback:

$$\begin{tikzcd} M \cap N \arrow[r] \arrow[d] & M \arrow[d, "m"] \\ N \arrow[r, "n"'] & X \end{tikzcd}$$

Also recall from Proposition 1.41 that if M and N are classified by ϕ, ψ respectively, then $M \cap N$ is classified by the composition

$$X \xrightarrow{\langle \phi, \psi \rangle} \Omega \times \Omega \xrightarrow{\wedge} \Omega$$

In $\mathrm{Sub}(X)$ we also have *joins* (least upper bounds, or *unions*) $M \cup N$ of subobjects. The subobject $M \cup N$ can be constructed in at least two ways: we have $M \cup N$ in the epi-mono factorization of $\begin{bmatrix} m \\ n \end{bmatrix} : M + N \to X$, or define $M \cup N$ by requiring that the diagram

$$\begin{tikzcd} M \cap N \arrow[r] \arrow[d] & M \arrow[d] \\ N \arrow[r] & M \cup N \end{tikzcd}$$

be a pushout (note that the diagram is then both a pullback and a pushout).

On the level of Ω, we have the subobjects $\langle \mathrm{id}, t\rangle : \Omega \to \Omega \times \Omega$ (here we write t for the composition $\Omega \to 1 \xrightarrow{t} \Omega$) and $\langle t, \mathrm{id}\rangle : \Omega \to \Omega \times \Omega$; let $\vee : \Omega \times \Omega \to \Omega$ be the classifying map of their union.

Again, for subobjects M, N of X, classified by ϕ, ψ, their union is classified by the composition

$$X \xrightarrow{\langle \phi, \psi \rangle} \Omega \times \Omega \xrightarrow{\vee} \Omega$$

We define the subobject Ω_1 of $\Omega \times \Omega$ as the equalizer of $\wedge$ and the first projection: $\Omega \times \Omega \to \Omega$.

Exercise 44 For subobjects M, N, classified by ϕ, ψ respectively, we have that $\langle \phi, \psi \rangle : X \to \Omega \times \Omega$ factors through Ω_1 if and only if $M \leq N$.

Now, we let $\Rightarrow: \Omega \times \Omega \to \Omega$ be the map which classifies the subobject Ω_1.

For subobjects M, N of X, classified by ϕ, ψ, we have: the diagram

$$\begin{tikzcd} X \arrow[r, "\langle \phi,\psi \rangle"] \arrow[d] & \Omega \times \Omega \arrow[d, "\Rightarrow"] \\ 1 \arrow[r, "t"'] & \Omega \end{tikzcd}$$

commutes if and only if the map $\langle \phi, \psi \rangle : X \to \Omega \times \Omega$ factors through Ω_1, if and only if $M \leq N$.

As a special case of the map $\Rightarrow$ we have the *pseudocomplement* function $\neg : \Omega \to \Omega$: $\neg x = x \Rightarrow \mathsf{f}$.

Proposition 3.2 *For $M, N \in \mathrm{Sub}(X)$, classified by ϕ, ψ, and an arbitrary subobject $k{:}K \to X$ we have:*

i) The composition ϕk classifies $K \cap M$.

ii) Writing $M \Rightarrow N$ for the subobject of X classified by $\Rightarrow \circ \langle \phi, \psi \rangle$, we have:

$$K \leq (M \Rightarrow N) \text{ if and only if } (K \cap M) \leq N$$

Proof. Part i) is left to you as an exercise.
ii): By the definition of Ω_1 as the subobject of $\Omega \times \Omega$ classified by $\Rightarrow$, we see that $K \leq (M \Rightarrow N)$ if and only if there is a commutative square

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d] & X \arrow[d, "\langle \phi,\psi \rangle"] \\ \Omega_1 \arrow[r] & \Omega \times \Omega \end{tikzcd}$$

That means: if and only if $\phi k \leq \psi k$. Now by part i), this means that $(K \cap M) \leq K \cap N$, which is equivalent to $(K \cap M) \leq N$.

3.2 Quantifiers

The pullback maps $f^* : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ (for $f : X \to Y$) have both adjoints.

As in the proof of Proposition 1.31, we use the equivalence between $\mathrm{Sub}(X)$ and the category Mon/X, which is the full subcategory of the slice $\mathcal{E}/X$ on the monomorphisms into X.

Modulo this equivalence, the map f^* coincides with the pullback functor $\mathcal{E}/Y \to \mathcal{E}/X$. The pullback functor preserves monos, and restricts therefore to a functor $\mathrm{Mono}(Y) \to \mathrm{Mono}(X)$.

The same holds for the right adjoint $\prod_f : \mathcal{E}/X \to \mathcal{E}/Y$ of f^*: consider that an object of $\mathcal{E}/X$ is in $\mathrm{Mono}(X)$ if and only if its unique map to the terminal is a monomorphism. Since monos and terminal objects are preserved by right adjoints, also $\prod_f$ restricts to a functor $\mathrm{Mono}(X) \to \mathrm{Mono}(Y)$, which we call $\forall_f$.

The left adjoint to f^*, $\sum_f : \mathcal{E}/X \to \mathcal{E}/Y$ does not always yield a mono, even if its input is from $\mathrm{Mono}(X)$; therefore we take its *image along* f: the functor $\exists_f$ sends a subobject $m : M \to X$ to the image of the composition $fm : M \to Y$ (image as given by epi-mono factorization).

3.3 Interpretation of logic in toposes

Definition 3.3 (Languages) A *many-sorted* first-order language (or, *language* for short) consists of:

i) A set of *sorts* S, T, $\ldots$;

ii) For every sort S, an infinite set of *variables of sort* S: $x^S, y^S, \ldots$;

iii) For every sort S, a set of *constants of sort* S: c^S, d^S;

iv) For every $k+1$-tuple of sorts $S_1, \ldots, S_k, T$ a set of *function symbols* $f : (S_1, \ldots, S_k; T)$;

v) For every k-tuple of sorts $S_1, \ldots, S_k$ a set of *relation symbols* $R : (S_1, \ldots, S_k)$.

Languages are specified whenever one wishes to describe a particular mathematical structure. For example, if one wishes to say something about an R-module for some commutative ring R, it is natural to take a sort R for the ring, a sort M for the module, *two* symbols for addition (one for addition in the ring, one for addition in the module), and likewise two constant symbols for the neutral elements of these two additions, one function symbol for multiplication in the ring, a constant for the neutral element for this multiplication, and a function symbol of sort $(M, R) \to M$ for the action of the ring on the module.

Definition 3.4 (Terms) Given a language, one has *terms* of each sort: every variable x^S of sort S is a term of sort S; every constant symbol of sort S is a term of sort S; and if $f : (S_1, \ldots, S_k; T)$ is a function symbol and $t_1, \ldots, t_k$ are terms of sorts $S_1, \ldots, S_k$ respectively, then $f(t_1, \ldots, t_k)$ is a term of sort T.

In our example, if we have $+_R$ and $+_M$ for addition in the ring and the module, respectively, and $\cdot$ for multiplication in the ring and * for the action of the ring on the module, we have terms $x^R, y^R, x^R +_R y^R, z^M * (x^R +_R y^R), z^M * (x^R \cdot y^R)$ of sorts R, R, M and M respectively. Note that a term may contain variables, but may also be built up from constants and function symbols only.

Definition 3.5 (Formulas) Given a language L, we define what we call an *L-formula* as follows:

i) $\perp$ and $\top$ are L-formulas;

ii) If t and s are terms of sort S, then we have a formula $t = s$;

iii) If R is a relation symbol $R : (S_1, \ldots, S_k)$ and $t_1, \ldots t_k$ are terms of sorts $S_1, \ldots, S_k$ respectively, then $R(t_1, \ldots, t_k)$ is a formula;

iv) If φ and ψ are formulas then $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$ and $\neg\varphi$ are formulas;

v) if φ is a formula and x^S is a variable (of whatever sort) which occurs freely in φ, then $\exists x^S \varphi$ and $\forall x^S \varphi$ are formulas.

In the last clause of Definition 3.5 the notion of a variable occurring "freely" in a formula was used; this means that the variable is not 'captured' by a quantifier. In our example, in the formula

$$\forall z^M (z^M * (x^R \cdot y^R) = (z^M * x^R) * y^R),$$

the variables x^R and y^R occur freely. The variable z^M is "bound" by the quantifier $\forall z^M$.

Definition 3.6 (Structures) Given a topos $\mathcal{E}$ and a first-order language L, an *L-structure* in $\mathcal{E}$ consists of the following data:

i) For every sort S of L, an object $[\![S]\!]$ of $\mathcal{E}$;

ii) For every constant c^S of sort S, an arrow $1 \overset{[\![c]\!]}{\to} [\![S]\!]$ in $\mathcal{E}$;

iii) For every function symbol $f : (S_1, \ldots, S_k; T)$ an arrow

$$[\![S_1]\!] \times \cdots \times [\![S_k]\!] \xrightarrow{[\![f]\!]} [\![T]\!]$$

in $\mathcal{E}$.

Naturally, if we are given an L-structure in a topos $\mathcal{E}$, we wish to see an L-formula as some sort of statement about this structure (as we do in ordinary first-order logic in Set), which can be 'true' or 'false'. To this end, we associate to any L-formula φ a subobject $[\![\varphi]\!]$ of a suitable domain associated with φ.

First of all, we consider L-terms and L-formulas as finite lists of symbols; this means that to any L-term t we can associate a list $V(t)$ of the variables occurring in the term, and to any L-formula φ we have such a list $FV(\varphi)$ of the *free* variables in φ. We denote by $[\![V(t)]\!]$ a product of the objects $[\![S_i]\!]$ for every variable $x_i^{S_i}$ in the list $V(t)$: so if $V(t) = (x_1^{S_1}, \ldots, x_k^{S_k})$, then

$$[\![V(t)]\!] = [\![S_1]\!] \times \cdots \times [\![S_k]\!]$$

and $[\![FV(\varphi)]\!]$ is defined in a similar way. Note that if $V(t)$ is the empty sequence, then $[\![V(t)]\!] = 1$, and similar for $[\![FV(\varphi)]\!]$.

Definition 3.7 (Interpretation of terms) For every term t of sort T we define an arrow $[\![t]\!] : [\![V(t)]\!] \to [\![T]\!]$ by recursion on the term t: if t is a variable x^T then $[\![t]\!]$ is the identity arrow on T. If t is a constant c^T then $[\![t]\!] : 1 \to [\![T]\!]$ is given by the structure. If t is $f(t_1, \ldots, t_k)$ and each $[\![t_i]\!] : [\![V(t_i)]\!] \to [\![S_i]\!]$ has been defined, and we have $[\![f]\!] : [\![S_1]\!] \times \cdots \times [\![S_k]\!] \to [\![T]\!]$ given by the structure, then since every variable occurring in t_i also occurs in t, we have evident projection maps $\pi_i : [\![V(t)]\!] \to [\![V(t_i)]\!]$. So we can define $[\![t]\!]$ as the composite arrow

$$[\![V(t)]\!] \xrightarrow{\langle \pi_i \rangle_{i=1}^k} \prod_{i=1}^k [\![V(t_i)]\!] \xrightarrow{\prod_{i=1}^k [\![t_i]\!]} \prod_{i=1}^k [\![S_i]\!] \xrightarrow{[\![f]\!]} [\![T]\!]$$

Definition 3.8 (Interpretation of formulas) For every formula φ we define a subobject $[\![\varphi]\!]$ of $[\![FV(\varphi)]\!]$ as follows:

i) If $\varphi = \top$ or $\varphi = \bot$, then $[\![\varphi]\!]$ is the top element (or bottom element, respectively) of $\mathrm{Sub}([\![FV(\varphi)]\!]) = \mathrm{Sub}(1)$.

ii) If φ is the formula $t = s$ for terms t and s of the same sort S, then we have, just as in Definition 3.7, projection arrows $\pi_s : [\![FV(\varphi)]\!] \to$

$[\![V(s)]\!]$ and $\pi_t : [\![FV(\varphi)]\!] \to [\![V(t)]\!]$ and we have therefore a parallel pair $([\![s]\!] \circ \pi_s, [\![t]\!] \circ \pi_t) : [\![FV(\varphi)]\!] \to [\![S]\!]$. We let $[\![\varphi]\!]$ be the subobject of $[\![FV(\varphi)]\!]$ represented by the equalizer of this pair.

iii) Suppose φ is the formula $R(t_1, \ldots, t_k)$ for a relation symbol $R : (S_1, \ldots, S_k)$. Again, we have projections $\pi_i : [\![FV(\varphi)]\!] \to [\![V(t_i)]\!]$ and the maps $[\![t_i]\!] : [\![V(t_i)]\!] \to [\![S_i]\!]$. We define $[\![\varphi]\!]$ as the subobject of $[\![FV(\varphi)]\!]$ appearing in the following pullback diagram:

$$\begin{tikzcd}
{[\![\varphi]\!]} \arrow[r] \arrow[d] & {[\![R]\!]} \arrow[d] \\
{[\![FV(\varphi)]\!]} \arrow[r, "\langle [\![t_i]\!] \circ \pi_i \rangle_{i=1}^k"'] & \prod_{i=1}^k [\![S_i]\!]
\end{tikzcd}$$

where the right hand side vertical is a mono representing the subobject $[\![R]\!]$ given by the structure.

iv) If φ is of the form $\psi \wedge \chi$ (or $\psi \vee \chi$, or $\psi \to \chi$) then we have projections $\pi_\psi : [\![FV(\varphi)]\!] \to [\![FV(\psi)]\!]$ and $\pi_\chi : [\![FV(\varphi)]\!] \to [\![FV(\chi)]\!]$ and therefore subobjects $\pi_\psi^*([\![\psi]\!])$, $\pi_\chi^*([\![\chi]\!])$ of $[\![FV(\varphi)]\!]$. We define $[\![\psi \wedge \chi]\!]$ by

$$\pi_\psi^*([\![\psi]\!]) \cap \pi_\chi^*([\![\chi]\!])$$

and similar for $\vee$ and $\to$; using the Heyting algebra structure of $\mathrm{Sub}([\![FV(\varphi)]\!])$.

v) We take the formula $\neg\varphi$ as defined by $\varphi \to \bot$.

vi) If φ is $\exists x^S \psi$ or $\forall x^S \psi$ then we have a projection $\pi : [\![FV(\psi)]\!] \to [\![FV(\varphi)]\!]$ and we define $[\![\exists x^S \psi]\!]$ as $\exists_\pi([\![\psi]\!])$ and $[\![\forall x^S \psi]\!]$ as $\forall_\pi([\![\psi]\!])$.

Definition 3.9 (Truth) *Let φ be an L-formula, and suppose $[\![\varphi]\!]$ has been defined according to Definition 3.8 for a given structure in a topos $\mathcal{E}$. we say that φ is* true *for this interpretation if $[\![\varphi]\!]$ is the top element of $[\![FV(\varphi)]\!]$.*

3.4 Kripke-Joyal semantics in toposes

In the topos Set, when we have defined languages, structures, and interpretations as in section 3.3, we come to grips with the subset $[\![\varphi]\!]$ of $[\![FV(\varphi)]\!]$ by studying which elements $\vec{a}$ of the latter set are elements of $[\![\varphi]\!]$. We say, for such a tuple $\vec{a}$ that $\varphi[\vec{a}]$ *holds* if $\vec{a} \in [\![\varphi]\!]$. If we view $\vec{a}$ as a map from 1 to $[\![FV(\varphi)]\!]$ then we can also say: $\varphi[\vec{a}]$ holds if and only if the map $\vec{a} : 1 \to [\![FV(\varphi)]\!]$ factors through $[\![\varphi]\!]$.

This is how we shall generalize the truth definition in a general topos, except for one point: maps from the terminal object are not enough. In Set, the object 1 is a *generator*: every object of Set is a colimit (in fact, a coproduct) of a diagram of copies of 1. In a general topos this does not hold, and we consider *all* maps to $[\![\varphi]\!]$, from all possible domains.

The following definition is couched in slightly more general terms than the interpretation of a first-order language. That case can easily be extracted, and this will be done in Theorem 3.12.

Let $m : M \to X$ be a subobject. For an arbitrary arrow $\alpha : U \to X$ we write $U \Vdash M[\alpha]$ for the statement that the map α factors through M. This leads to an operational definition of truth in the topos $\mathcal{E}$, and if M is built up from more elemental subobjects ($k : K \to X, l : L \to X$) of X using the Heyting constructors or the quantifiers, we get an analysis of the statement $U \Vdash M[\alpha]$ in terms of statements $V \Vdash K[\beta]$, $W \Vdash L[\gamma]$.

Let us note that $0 \Vdash M[\alpha]$ always holds (more generally, $N \Vdash M[n]$ if $n{:}N \to X$ is a subobject of X which is $\leq M$), and that $X \Vdash M[\mathrm{id}_X]$ precisely when $U \Vdash M[\alpha]$ for all $\alpha : U \to X$, which is equivalent to M being the maximal subobject of X.

Remark 3.10 If the diagram

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "e"'] & B \arrow[d, "m"] \\ C \arrow[r, "q"'] & D \end{tikzcd}$$

commutes, with e epi and m mono, then there is a (necessarity unique) map $f : C \to B$ such that $fe = p$ and $mf = q$.

Indeed, if $p = n_1e_1$, $q = n_2e_2$ are epi-mono factorizations of p and q respectively, then both $(mn_1)e_1$ and $n_2(e_2e)$ are epi-mono factorizations of $qe = mp$, so since $\mathcal{E}$ is regular, there is an isomorphism r from the codomain of e_2 to the domain of n_1 satisfying $re_2e = e_1$ and $mn_1r = n_2$. So the arrow $f = n_1re_2 : C \to B$ has the claimed property; uniqueness follows since e is epi.

Corollary 3.11 *The relation $U \Vdash M[\alpha]$ has the following two properties:*

- **(monotonicity)** *If $U \Vdash M[\alpha]$ and $f : V \to U$ is any arrow, then $V \Vdash M[\alpha f]$.*

- **(local character)** *If* $\alpha{:}U \to X$ *is arbitrary and* $p{:}P \to U$ *is an epimorphism satisfying* $P \Vdash M[\alpha p]$, *then* $U \Vdash [\alpha]$.

Proof. Monotonicity is trivial; for local character, apply Remark 3.10 to the commutative diagram

$$\begin{array}{ccc} P & \longrightarrow & M \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle m} \\ U & \underset{\alpha}{\longrightarrow} & X \end{array}$$

The following theorem gives, for the Heyting connectives $\wedge, \vee, \Rightarrow$ and $\bot$, as well as for the quantifiers $\exists x, \forall x$, the connection between the statement $U \Vdash M[\alpha]$ and the statements $V \Vdash N[\beta]$, for subobjects N from which M is defined (using the Heyting structure).

Theorem 3.12 (MM, VI.6.1) *Let* $m{:}M \to X$ *be a subobject and* $\alpha{:}U \to X$ *an arrow in* $\mathcal{E}$.

(i) If $M = N \cap L$ *then* $U \Vdash M[\alpha]$ *if and only if both* $U \Vdash N[\alpha]$ *and* $U \Vdash L[\alpha]$.

(ii) If $M = N \cup L$ *then* $U \Vdash M[\alpha]$ *if and only if there is an epimorphism* $\begin{bmatrix} p \\ q \end{bmatrix} : P + Q \to U$ *such that* $P \Vdash N[\alpha p]$ *and* $Q \Vdash L[\alpha q]$.

(iii) If $M = N \Rightarrow L$ *then* $U \Vdash M[\alpha]$ *if and only if for every map* $f{:}V \to U$ *we have: if* $V \Vdash N[\alpha f]$ *then* $V \Vdash L[\alpha f]$.

(iv) If $M = \neg N$ *then* $U \Vdash M[\alpha]$ *if and only if for every map* $f{:}V \to U$ *we have: if* $V \Vdash N[\alpha f]$ *then* V *is initial in* $\mathcal{E}$.

(v) If $M = \exists y N$ *where* $n{:}N \to X \times Y$ *is a subobject (so* $\exists y N$ *is the image of* N *along the projection* $X \times Y \to X$*), then* $U \Vdash M[\alpha]$ *if and only if there is an epimorphism* $p{:}P \to U$ *and an arrow* $y{:}P \to Y$ *such that* $P \Vdash N[\langle \alpha p, y \rangle]$.

(vi) If $M = \forall y N$ *for* $n{:}N \to X \times Y$ *as in clause (v), then* $U \Vdash M[\alpha]$ *if and only if for every arrow* $f{:}V \to U$ *and every arrow* $y : V \to Y$ *we have* $V \Vdash N[\langle \alpha f, y \rangle]$.

Proof.

(i) is clear: α factors through $N \cap L$ if and only if α factors through both N and L.

(ii) First suppose that $\begin{bmatrix} p \\ q \end{bmatrix}$:$P + Q \to U$ is epi and $P \Vdash N[\alpha p]$ and $Q \Vdash L[\alpha q]$. Since $N \leq N \cup L$ and $L \leq N \cup L$ we have $(P+Q) \Vdash M[\alpha \begin{bmatrix} p \\ q \end{bmatrix}]$. Since $P + Q \to X$ is epi, the statement $U \Vdash M[\alpha]$ follows by **local character** (Corollary 3.11).

For the converse, assume $U \Vdash M[\alpha]$. If α':$U \to N \cup L$ is the factorization of α through M, take a pullback diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d, "r"'] & N + L \arrow[d] \\ U \arrow[r, "\alpha'"'] & N \cup L \end{tikzcd}$$

where $N + L \to N \cup L$ is the epi part of the map $N + L \to X$. By stability of coproducts in $\mathcal{E}$ we have that V is isomorphic to a coproduct $P + Q$, and the map $P + Q \to N + L$ sends P into N and Q into L. The arrow r is, modulo this isomorphism, of the form $\begin{bmatrix} p \\ q \end{bmatrix}$ for p:$P \to U$, q:$Q \to U$. It is left to you to work out that $P \Vdash N[\alpha p]$ and $Q \Vdash L[\alpha q]$, as desired.

(iii) First suppose $U \Vdash (N \Rightarrow L)[\alpha]$, and let $f : V \to U$ be such that $V \Vdash N[\alpha f]$. By **monotonicity** (Corollary 3.11) we also have that $V \Vdash (N \Rightarrow L)[\alpha f]$. By case (i) we have that $V \Vdash (N \cap (N \Rightarrow L))[\alpha f]$, but $N \cap (N \Rightarrow L) \leq L$ always, so $V \Vdash L[\alpha f]$ as claimed.

For the converse, suppose the condition holds. Consider the subobject n^*:$\alpha^*(N) \to U$ of U. Clearly, $\alpha^*(N) \Vdash N[\alpha n^*]$. By the condition, we get that also $\alpha^*(N) \Vdash L[\alpha n^*]$. That means that $\alpha^*(N) \leq \alpha^*(L)$ in $\mathrm{Sub}(U)$. Now α^* preserves all Heyting connectives, in particular $\Rightarrow$, so we have that $\alpha^*(N \Rightarrow L)$ is the maximal subobject of U, from which it readily follows that $U \Vdash (N \Rightarrow L)[\alpha]$.

(iv) This is left to you as an exercise.

(v) First suppose that $U \Vdash (\exists y N)[\alpha]$. Construct the following diagram:

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "p"'] & N \arrow[r] \arrow[d] & X \times Y \arrow[ddl] \\
U \arrow[r, "\alpha'"] \arrow[dr, "\alpha"'] & \exists y N \arrow[d] & \\
& X &
\end{tikzcd}$$

where α' is the factorization of α testifying that $U \Vdash (\exists y N)[\alpha]$, the upper left hand square is a pullback, the composition of the top row: $P \to X \times Y$ is of the form $\langle \alpha p, y \rangle$ for suitable $y : P \to Y$. The other maps are projections and images. The conclusion that $P \Vdash N[\langle \alpha p, y \rangle]$ is immediate.

Conversely, suppose that the given condition holds: we have an epi $p{:}P \to U$ and an arrow $\langle \alpha p, y \rangle{:}P \to X \times Y$ such that $P \Vdash N[\langle \alpha p, y \rangle]$. Since $\langle \alpha p, y \rangle$ factors through N, the composition of the projection $X \times Y \to X$ with this map (which is equal to αp) factors through $\exists y N$. This means that $P \Vdash \exists y N[\alpha p]$, from which we conclude that $U \Vdash \exists y N[\alpha]$, again invoking **local character**.

(vi) First suppose that for every arrow $p{:}V \to U$ and every $\beta{:}V \to Y$ we have $V \Vdash N[\langle \alpha p, \beta \rangle]$. Write $\pi{:}X \times Y \to X$ for the projection and consider the diagram

$$\begin{tikzcd}
U \times Y \arrow[r, "e"] \arrow[d] & \pi^*(\mathrm{im}(\alpha)) \arrow[r, "i"] \arrow[d] & X \times Y \arrow[d, "\pi"] \\
U \arrow[r] & \mathrm{im}(\alpha) \arrow[r] & X
\end{tikzcd}$$

where the bottom row is the epi-mono factorization of α, and the squares are pullbacks (so the top row is an epi-mono factorization too). Our assumption implies that $U \times Y \Vdash N[ie]$, which, since e is epi, by **local character** implies that $\pi^*(\mathrm{im}(\alpha)) \Vdash N[i]$. This last statement means that $\pi^*(\mathrm{im}(\alpha)) \leq N$ as subobjects of $X \times Y$; by the adjunction $\pi^* \dashv \forall y$, this gives $\mathrm{im}(\alpha) \leq \forall y N$, in other words $U \Vdash \forall y N[\alpha]$.

The converse is left to you.

Note that the following is a consequence of Theorem 3.12:

Corollary 3.13 *If $m : M \to X$ is a subobject, so that $\forall x M$ is a subobject of the terminal object* 1*, then the following three statements are equivalent:*

i) The map $\forall x M \to 1$ is epi.

ii) $1 \Vdash \forall M[\mathrm{id}]$.

iii) For every object U of $\mathcal{E}$ and every arrow $\alpha : U \to X$, we have $U \Vdash M[\alpha]$.

3.5 Application: internal posets in a topos

Let us now discuss an application. We have met the notion of an "internal poset" in a category with finite limits. It is formulated entirely within the framework of finite limits, and therefore every functor which preserves finite limits, will also preserve internal posets: if $\mathcal{C}$ and $\mathcal{D}$ have finite limits, $F : \mathcal{C} \to \mathcal{D}$ is finite-limit preserving, and (X, R) is an internal poset in $\mathcal{C}$ (i.e. the binary relation R on X satisfies the axioms for a partial order), then $(F(X), F(R))$ is an internal poset in $\mathcal{D}$.

Proposition 3.14 *Let $\mathcal{E}$ be a topos, X and object of $\mathcal{E}$ and R a subobject of $X \times X$. Then the following two statements are equivalent:*

i) The pair (X, R) is an internal poset in $\mathcal{E}$.

ii) For every object Y of $\mathcal{E}$, the relation

$$R_Y \;=\; \{(\alpha, \beta) \in \mathcal{E}(Y, X)^2 \mid \langle \alpha, \beta \rangle : Y \to X \times X \text{ factors through } R\}$$

is a partial order on the set of arrows $Y \to X$.

Proof. The implication i)⇒ii) follows directly from the remark made just before the proposition. For any object Y of $\mathcal{E}$, the functor $\mathcal{E}(Y, -) : \mathcal{E} \to$ Set preserves finite limits and therefore internal posets. So if (X, R) is an internal poset then so is the set of all arrows $Y \to X$, with the relation given in item ii).

For the converse, assume that for every object Y of $\mathcal{E}$ the set given in 3.14ii) is a partial order on $\mathcal{E}(Y, X)$. We have to prove that (X, R) is an internal poset in $\mathcal{E}$.

First, since R_X is a poset structure on $\mathcal{E}(X, X)$, hence a reflexive relation, we have that the diagonal of X, which is the map $\delta = \langle \mathrm{id}, \mathrm{id} \rangle : X \to X \times X$, factors through R. So R is internally reflexive.

Secondly, we show that R is internally antisymmetric. We do this by showing that
$$1 \Vdash \forall xy(R(x,y) \wedge R(y,x) \rightarrow x = y)[\mathrm{id}]$$
(using Corollary 3.13). So let Y be an object of $\mathcal{E}$, and α, β arrows $Y \rightarrow X$ such that $Y \Vdash (R(x,y) \wedge R(y,x))[\alpha, \beta]$. Then both (α, β) and (β, α) are elements of the relation R_Y, which is antisymmetric; so $\alpha = \beta$, or in other words $Y \Vdash (x = y)[\alpha, \beta]$.

For the third requirement, we need to show
$$1 \Vdash \forall xyz(R(x,y) \wedge R(y,z) \rightarrow R(x,z))[\mathrm{id}]$$
To this end, let again Y be an arbitrary object and $\alpha, \beta, \gamma : Y \rightarrow X$ be arrows. If $Y \Vdash (R(x,y) \wedge R(y,z))[\alpha, \beta, \gamma]$ then $(\alpha, \beta) \in R_Y$ and $(\beta . \gamma) \in R_Y$. Since R_Y is a transitive relation, we obtain $Y \Vdash R(x,z)[\alpha, \gamma]$, which gives us the required conclusion. ∎

Now let us see how we can solve the following exercise:

Exercise 45 Let $\mathcal{E}$ be a topos with subobject classifier $\mathsf{t} : 1 \rightarrow \Omega$. We define the following structure: the map $\wedge : \Omega \times \Omega \rightarrow \Omega$ classifies the subobject $\langle \mathsf{t}, \mathsf{t} \rangle : 1 \rightarrow \Omega \times \Omega$. The subobject $\Omega_1 \rightarrow \Omega \times \Omega$ is the equalizer of $\wedge$ and the first projection. The map $\vee : \Omega \times \Omega \rightarrow \Omega$ classifies the join of the subobjects $\langle \mathsf{t}, \mathrm{id} \rangle : \Omega \rightarrow \Omega \times \Omega$ and $\langle \mathrm{id}, \mathsf{t} \rangle : \Omega \rightarrow \Omega \times \Omega$. Finally, the map $\Rightarrow : \Omega \times \Omega \rightarrow \Omega$ classifies the subobject Ω_1.

(a) Show that Ω_1 is a partial order on Ω.

(b) Show that $(\Omega, \wedge, \vee)$ is an internal distributive lattice in $\mathcal{E}$.

(c) Show that $(\Omega, \wedge, \vee, \mathsf{t}, \mathsf{f}, \Rightarrow)$ is an internal Heyting algebra in $\mathcal{E}$.

Solution. First, one has to verify that if $\alpha, \beta : X \rightarrow \Omega$ classify the subobjects A and B of X, respectively, then the composition $\wedge \circ \langle \alpha, \beta \rangle : X \rightarrow \Omega$ classifies the intersection $A \cap B$ of A and B. It follows that the pair $\langle \alpha, \beta \rangle$ factors through Ω_1 if and only if $\alpha = \wedge \circ \langle \alpha, \beta \rangle$; that is, if and only if $A = A \cap B$; in other words, if $A \leq B$ as subobjects of X. This is clearly a partial order on the set of arrows $X \rightarrow \Omega$; by Proposition 3.14, Ω_1 is an internal poset relation on Ω. For part (b), one verifies that for $\alpha, \beta : X \rightarrow \Omega$ as above, the composition $\vee \circ \langle \alpha, \beta \rangle$ classifies the union $A \cup B$ of the subobjects classified by α and β. By reasoning similar to the proof of Proposition 3.14 one gets that since $\mathrm{Sub}(X)$ is a distributive lattice, $(\Omega, \wedge, \vee)$ is an internal distributive lattice.

Finally, for (c) we have to see that the map $y \Rightarrow (-)$ is internally right adjoint to the map $y \wedge (-)$; this just means that the following two inequalities hold:

$$\begin{array}{ll} (1) & x \leq (y \Rightarrow (y \wedge x)) \\ (2) & (y \wedge (y \Rightarrow x)) \leq x \end{array}$$

for arrows $x, y : X \to \Omega$.
For (1), considering the pullbacks

$$\begin{tikzcd} F \arrow[r] \arrow[d] & X \arrow[d, "\langle y, y\wedge x\rangle"] \\ \Omega_1 \arrow[r] & \Omega \times \Omega \end{tikzcd} \qquad \begin{tikzcd} \Omega_1 \arrow[r] \arrow[d] & \Omega \times \Omega \arrow[d, "\Rightarrow"] \\ 1 \arrow[r, "\mathsf{t}"'] & \Omega \end{tikzcd}$$

we see that F is classified by $y \Rightarrow (y \wedge x)$. Let $\iota : A \to X$ be classified by x and let $B \to X$ be classified by y. Since both compositions $A \xrightarrow{\iota} X \xrightarrow{y} \Omega$ and $A \xrightarrow{\iota} X \xrightarrow{y \wedge x} \Omega$ classify the inclusion $A \cap B \to A$, we see that the pair $\langle y\iota, (y \wedge x)\iota \rangle : A \to \Omega \times \Omega$ factors through Ω_1. So $\iota : A \to X$ factors through F, but that means $x \leq (y \Rightarrow (y \wedge x))$.
For (2), first we show that for a generalized element $\alpha \in \Omega$, we have $[\mathsf{t} \Rightarrow \alpha] = \alpha$. Let $\alpha : X \to \Omega$ classify $C \in \mathrm{Sub}(X)$. Now consider the diagram:

$$\begin{tikzcd} C \arrow[r] \arrow[d] & X \arrow[d, "\langle \mathsf{t}, \alpha\rangle"] \\ \Omega_1 \arrow[r] \arrow[d] & \Omega \times \Omega \arrow[d, "\Rightarrow"] \\ 1 \arrow[r, "\mathsf{t}"'] & \Omega \end{tikzcd}$$

Since

$$\begin{array}{rcl} C & = & \{x \mid \alpha(x) = \mathsf{t}\} \\ & = & \{x \mid \langle \mathsf{t}, \alpha(x)\rangle \in \Omega_1\} \\ & = & \{x \mid [\mathsf{t} \Rightarrow \alpha(x)] = \mathsf{t}\} \end{array}$$

the upper square is a pullback. Since both α and $[\mathsf{t} \Rightarrow \alpha]$ classify C, we have $\alpha = [\mathsf{t} \Rightarrow \alpha]$ as desired.

Now for $x, y : X \to \Omega$ classifying $A, B \in \mathrm{Sub}(X)$ respectively, we have for $B \xrightarrow{\tau} X$:

$$\begin{tikzcd}
B \wedge [B \Rightarrow A] \arrow[r] \arrow[d] & B \arrow[d, "\tau"] \\
{[B \Rightarrow A]} \arrow[r] \arrow[d] & X \arrow[d, "\langle y, x\rangle"] \\
\Omega_1 \arrow[r] \arrow[d] & \Omega \times \Omega \arrow[d, "\Rightarrow"] \\
1 \arrow[r, "\mathsf{t}"'] & \Omega
\end{tikzcd}$$

where $[B \Rightarrow A]$ is classified by $y \Rightarrow x$. Since y classifies $B \xrightarrow{\tau} X$, the RHS composition is equal to $\Rightarrow \circ \langle \mathsf{t}, x\tau\rangle$. Which is $\mathsf{t} \Rightarrow x\tau$, which equals $x\tau$ by the remark above; but $x\tau$ classifies the subobject $B \cap A$ of B. Since $B \cap A \subseteq B$, $x\tau \leq x$.

3.6 Kripke-Joyal in categories of sheaves

Recall from Section 3.4 that for an object X of a topos $\mathcal{E}$, a subobject M of X and an arrow $\alpha : Y \to X$, the notation $Y \Vdash M[\alpha]$ means that the arrow α factors through M. Sometimes it is expedient to prove that $Y \Vdash M[\alpha]$ holds for all objects Y and all arrows α, in which case we can conclude that M is the maximal subobject of X.

Now in the case that the topos $\mathcal{E}$ is of the form $\mathrm{Sh}(\mathcal{C}, J)$ for a small category $\mathcal{C}$ and a Grothendieck topology J on $\mathcal{C}$, we don't need to consider *all* objects of $\mathcal{E}$: if M is a subsheaf of X and M is not maximal, then there is an arrow α from a representable presheaf y_C to X which does not factor through M.

Moreover, any map from y_C to X corresponds to an element of $X(C)$ by the Yoneda Lemma, so we can reformulate the definition of Kripke-Joyal forcing, only mentioning objects of $\mathcal{C}$ and elements of the set $X(C)$.

Theorem 3.15 (MM, VI.7.1) *Let $m{:}M \to X$ be a subsheaf, C an object of the category $\mathcal{C}$ and α an element of $X(C)$.*

(i) If $M = N \cap L$ then $C \Vdash M[\alpha]$ if and only if both $C \Vdash N[\alpha]$ and $C \Vdash L[\alpha]$.

(ii) If $M = N \cup L$ then $C \Vdash M[\alpha]$ if and only if there is a J-covering sieve R of C such that for each $f : D \to C$ in R we have: either $D \Vdash N[f^*(\alpha)]$ or $D \Vdash L[f^*(\alpha)]$.

(iii) *If $M = N \Rightarrow L$ then $C \Vdash M[\alpha]$ if and only if for every arrow $f{:}D \to C$ in $\mathcal{C}$ we have: if $D \Vdash N[f^*(\alpha)]$ then $D \Vdash L[f^*(\alpha)]$.*

(iv) *If $M = \neg N$ then $C \Vdash M[\alpha]$ if and only if for every arrow $f{:}D \to C$ in $\mathcal{C}$ we have: if $D \Vdash N[f^*(\alpha)]$ then the empty sieve covers D.*

(v) *If $M = \exists y N$ where $n{:}N \to X \times Y$ is a subobject, then $C \Vdash M[\alpha]$ if and only if there is a J-covering sieve R of C such that for each $f : D \to C$ in R there is an element $\beta_f \in Y(D)$, such that $D \Vdash N[(f^*(\alpha), \beta_f)]$.*

(vi) *If $M = \forall y N$ for $n{:}N \to X \times Y$ as in clause (v), then $C \Vdash M[\alpha]$ if and only if for every arrow $f{:}D \to C$ in $\mathcal{C}$ and every element $\beta \in Y(D)$ we have $D \Vdash N[(f^*(\alpha), \beta)]$.*

This formulation, for the special case that $\mathcal{E}$ is the category of presheaves on a poset, is known as "Kripke semantics" to logicians.

3.7 First-order structures in categories of presheaves

We have a language $\mathcal{L}$, which consists of a collection of *sorts* $S, T, \ldots$, possibly *constants* c^S of sort S, *function symbols* $f : S_1, \ldots, S_n \to S$, and *relation symbols* $R \subseteq S_1, \ldots, S_n$. The definition of formula is extended with the clauses:

i) If φ and ψ are formulas then $(\varphi \vee \psi)$, $(\varphi \to \psi)$ and $\neg\varphi$ are formulas;

ii) if φ is a formula and x^S a variable of sort S then $\forall x^S \varphi$ is a formula.

For the notations $FV(t)$ and $FV(\varphi)$ we refer to the mentioned chapter 4. Again, an interpretation assigns objects $[\![S]\!]$ to the sorts S, arrows to the function symbols and subobjects to relation symbols. This then leads to the definition of the interpretation of a formula φ as a subobject $[\![\varphi]\!]$ of $[\![FV(\varphi)]\!]$, which is a chosen product of the interpretations of all the sorts of the free variables of φ: if $FV(\varphi) = \{x_1^{S_1}, \ldots, x_n^{S_n}\}$ then $[\![FV(\varphi)]\!] = [\![S_1]\!] \times \cdots \times [\![S_n]\!]$.

The definition of $[\![\varphi]\!]$ of the mentioned chapter 4 is now extended by the clauses:

i) If $[\![\varphi]\!] \to [\![FV(\varphi)]\!]$ and $[\![\psi]\!] \to [\![FV(\psi)]\!]$ are given and

$$\begin{tikzcd} {[\![FV(\varphi \wedge \psi)]\!]} \arrow[r, "\pi_1"] \arrow[dr, "\pi_2"'] & {[\![FV(\varphi)]\!]} \\ & {[\![FV(\psi)]\!]} \end{tikzcd}$$

are the projections, then

$$\begin{array}{rcll} [\![\, \varphi \vee \psi \,]\!] & = & (\pi_1)^\sharp([\![\, \varphi \,]\!]) \vee (\pi_2)^\sharp([\![\, \psi \,]\!]) & \text{in Sub}([\![\, FV(\varphi \wedge \psi) \,]\!]) \\ [\![\, \varphi \to \psi \,]\!] & = & (\pi_1)^\sharp([\![\, \varphi \,]\!]) \to (\pi_2)^\sharp([\![\, \psi \,]\!]) & \text{in Sub}([\![\, FV(\varphi \wedge \psi) \,]\!]) \\ [\![\, \neg\varphi \,]\!] & = & [\![\, \varphi \,]\!] \to \bot & \text{in Sub}([\![\, FV(\varphi) \,]\!]) \end{array}$$

(Note that $FV(\varphi \wedge \psi) = FV(\varphi \vee \psi) = FV(\varphi \to \psi)$)

ii) if $[\![\, \varphi \,]\!] \to [\![\, FV(\varphi) \,]\!]$ is given and $\pi : [\![\, FV(\varphi) \,]\!] \to [\![\, FV(\exists x\varphi) \,]\!]$ is the projection, let $FV'(\varphi) = FV(\varphi \wedge x = x)$ and $\pi' : [\![\, FV'(\varphi) \,]\!] \to [\![\, FV(\varphi) \,]\!]$ the projection. Then

$$[\![\, \forall x\varphi \,]\!] \;=\; \forall_{\pi\pi'}((\pi')^\sharp([\![\, \varphi \,]\!]))$$

We shall now write out what this means, concretely, in $\widehat{\mathcal{C}}$. For a formula φ, we have $[\![\, \varphi \,]\!]$ as a subobject of $[\![\, FV(\varphi) \,]\!]$, hence we have a classifying map $\{\varphi\} : [\![\, FV(\varphi) \,]\!] \to \Omega$ with components $\{\varphi\}_C : [\![\, FV(\varphi) \,]\!](C) \to \Omega(C)$; for $(a_1, \ldots, a_n) \in [\![\, FV(\varphi) \,]\!](C)$, $\{\varphi\}_C(a_1, \ldots, a_n)$ is a sieve on C.

Definition 3.16 For φ a formula with free variables $x_1, \ldots, x_n$, C an object of $\mathcal{C}$ and $(a_1, \ldots, a_n) \in [\![\, FV(\varphi) \,]\!](C)$, the notation $C \Vdash \varphi(a_1, \ldots, a_n)$ means that $\mathrm{id}_C \in \{\varphi\}_C(a_1, \ldots, a_n)$.

The pronunciation of "$\Vdash$" is 'forces'.

Notation. For φ a formula with free variables $x_1^{S_1}, \ldots, x_n^{S_n}$, C an object of $\mathcal{C}$ and $(a_1, \ldots, a_n) \in [\![\, FV(\varphi) \,]\!](C)$ as above, so $a_i \in [\![\, S_i \,]\!](C)$, if $f : C' \to C$ is an arrow in $\mathcal{C}$ we shall write $a_i f$ for $[\![\, S_i \,]\!](f)(a_i)$.

Note: with this notation and φ, C, $a_1, \ldots, a_n$, $f : C' \to C$ as above, we have $f \in \{\varphi\}_C(a_1, \ldots, a_n)$ if and only if $C' \Vdash \varphi(a_1 f, \ldots, a_n f)$.

Using the characterization of the Heyting structure of $\widehat{\mathcal{C}}$ given in the proof of theorem **??**, we can easily write down an inductive definition for the notion $C \Vdash \varphi(a_1, \ldots, a_n)$:

- $C \Vdash (t = s)(a_1, \ldots, a_n)$ if and only if $[\![\, t \,]\!]_C(a_1, \ldots, a_n) = [\![\, s \,]\!]_C(a_1, \ldots, a_n)$
- $C \Vdash R(t_1, \ldots, t_k)(a_1, \ldots, a_n)$ if and only if

$$([\![\, t_1 \,]\!]_C(a_1, \ldots, a_n), \ldots, [\![\, t_k \,]\!]_C(a_1, \ldots, a_n)) \in [\![\, R \,]\!](C)$$

- $C \Vdash (\varphi \wedge \psi)(a_1, \ldots, a_n)$ if and only if

$$C \Vdash \varphi(a_1, \ldots, a_n) \text{ and } C \Vdash \psi(a_1, \ldots, a_n)$$

- $C \Vdash (\varphi \vee \psi)(a_1, \ldots, a_n)$ if and only if

$$C \Vdash \varphi(a_1, \ldots, a_n) \text{ or } C \Vdash \psi(a_1, \ldots, a_n)$$

- $C \Vdash (\varphi \to \psi)(a_1, \ldots, a_n)$ if and only if for every arrow $f : C' \to C$,

$$\text{if } C' \Vdash \varphi(a_1 f, \ldots, a_n f) \text{ then } C' \Vdash \psi(a_1 f, \ldots, a_n f)$$

- $C \Vdash \neg\varphi(a_1, \ldots, a_n)$ if and only if for no arrow $f : C' \to C$, $C' \Vdash \varphi(a_1 f, \ldots, a_n f)$
- $C \Vdash \exists x^S \varphi(a_1, \ldots, a_n)$ if and only if for some $a \in [\![S]\!](C)$, $C \Vdash \varphi(a, a_1, \ldots, a_n)$
- $C \Vdash \forall x^S \varphi(a_1, \ldots, a_n)$ if and only if for every arrow $f : C' \to C$ and every $a \in [\![S]\!](C')$,

$$C' \Vdash \varphi(a, a_1 f, \ldots, a_n f)$$

Exercise 46 Prove: if $C \Vdash \varphi(a_1, \ldots, a_n)$ and $f : C' \to C$ is an arrow, then $C' \Vdash \varphi(a_1 f, \ldots, a_n f)$.

Now let ϕ be a sentence of the language, so $[\![\phi]\!]$ is a subobject of 1 in $\widehat{\mathcal{C}}$. Note: a subobject of 1 is 'the same thing' as a collection X of objects of $\mathcal{C}$ such that whenever $C \in X$ and $f : C' \to C$ is arbitrary, then $C' \in X$ also. The following theorem is straightforward.

Theorem 3.17 *For a language $\mathcal{L}$ and interpretation $[\![\cdot]\!]$ of $\mathcal{L}$ in $\widehat{\mathcal{C}}$, we have that for every $\mathcal{L}$-sentence ϕ, $[\![\phi]\!] = \{C \in \mathcal{C}_0 \,|\, C \Vdash \phi\}$. Hence, ϕ is true for the interpretation in $\widehat{\mathcal{C}}$ if and only if for every C, $C \Vdash \phi$.*

If Γ is a set of $\mathcal{L}$-sentences and ϕ an $\mathcal{L}$-sentence, we write $\Gamma \Vdash \phi$ to mean: in every interpretation in a presheaf category such that every sentence of Γ is true, ϕ is true.

We mention without proof:

Theorem 3.18 (Soundness and Completeness) *If Γ is a set of $\mathcal{L}$-sentences and ϕ an $\mathcal{L}$-sentence, we have $\Gamma \Vdash \phi$ if and only if ϕ is provable from Γ in intuitionistic predicate calculus.*

Intuitionistic predicate calculus is what one gets from classical logic by deleting the rule which infers ϕ from a proof that $\neg\phi$ implies absurdity. In a Gentzen calculus, this means that one restricts attention to those sequents $\Gamma \Rightarrow \Delta$ for which Δ consists of at most one formula.

Exercise 47 Let N denote the constant presheaf with value $\mathbb{N}$.

i) Show that there are maps $0 : 1 \to N$ and $S : N \to N$ which make N into a natural numbers object in $\widehat{\mathcal{C}}$.

ii) Accordingly, there is an interpretation of the language of first-order arithmetic in $\widehat{\mathcal{C}}$, where the unique sort is interpreted by N. Prove, that for this interpretation, a sentence in the language of arithmetic is true if and only if it is true classically in the standard model $\mathbb{N}$.

Exercise 48 Prove that for every object C of $\mathcal{C}$, the set $\Omega(C)$ of sieves on C is a Heyting algebra, and that for every map $f : C' \to C$ in $\mathcal{C}$, $\Omega(f) : \Omega(C) \to \Omega(C')$ preserves the Heyting structure. Write out explicitly the Heyting implication $(R \to S)$ of two sieves.

3.8 Two examples and applications

3.8.1 Kripke semantics

Kripke semantics is a special kind of presheaf semantics: $\mathcal{C}$ is taken to be a poset, and the sorts are interpreted by presheaves X such that for every $q \leq p$ the map $X(q \leq p) : X(p) \to X(q)$ is an inclusion of sets. Let us call these presheaves *Kripke presheaves*.

The soundness and completeness theorem 3.18 already holds for Kripke semantics. This raises the question whether the greater generality of presheaves achieves anything new. In this example, we shall see that general presheaves are richer than Kripke models if one considers *intermediate logics*: logics stronger than intuitionistic logic but weaker than classical logic.

In order to warm up, let us look at Kripke models for *propositional logic*. The propositional variables are interpreted as subobjects of 1 in $\mathrm{Set}^{\mathcal{K}^{\mathrm{op}}}$ (for a poset $(\mathcal{K}, \leq)$); that means, as downwards closed subsets of $\mathcal{K}$ (see te remark just before theorem 3.17). Let, for example, $\mathcal{K}$ be the poset:

$$\begin{tikzcd} & 0 & \\ k \arrow[ur] & & l \arrow[ul] \end{tikzcd}$$

and let $[\![\, p \,]\!] = \{k\}$. Then $0 \not\Vdash p$, $0 \not\Vdash \neg p$ (since $k \leq 0$ and $k \Vdash p$) and $0 \not\Vdash \neg\neg p$ (since $l \leq 0$ and $l \Vdash \neg p$). So $p \vee \neg p \vee \neg\neg p$ is not true for this interpretation. Even simpler, if $\mathcal{K} = \{0 \leq 1\}$ and $[\![\, p \,]\!] = \{0\}$, then $1 \not\Vdash p \vee \neg p$. However, if $\mathcal{K}$ is a linear order, then $(p \to q) \vee (q \to p)$ is always true on $\mathcal{K}$, since if $\mathcal{K}$ is linear, then so is the poset of its downwards closed subsets. From this one can conclude that if one adds to intuitionistic propositional logic the axiom scheme

$$(\phi \to \psi) \vee (\psi \to \phi)$$

one gets a logic which is strictly between intuitionistic and classical logic.

Exercise 49 Prove that $(p \to q) \vee (q \to p)$ is always true on $\mathcal{K}$ if and only if $\mathcal{K}$ has the property that for every $x \in \mathcal{K}$, the set $\downarrow x = \{y \in \mathcal{K} \mid y \leq x\}$ is linearly ordered.

Prove also, that $\neg p \vee \neg\neg p$ is always true on $\mathcal{K}$ if and only if $\mathcal{K}$ has the following property: whenever two elements have an upper bound, they also have a lower bound.

Not only certain properties of posets can be characterized by the propositional logic they satisfy in the sense of exercise 49, also properties of presheaves.

Exercise 50 Let X be a Kripke presheaf on a poset $\mathcal{K}$. Show that the following axiom scheme of predicate logic:

$$\text{D} \qquad \forall x(A(x) \vee B) \to (\forall x A(x) \vee B)$$

(where A and B may contain additional variables, but the variable x is not allowed to occur in B) is always true in X, if and only if for every $k' \leq k$ in $\mathcal{K}$, the map $X(k) \to X(k')$ is the identity.

Suppose now one considers the logic D-J, which is intuitionistic logic extended with the axiom schemes $\neg\phi \vee \neg\neg\phi$ and the axiom scheme D from exercise 50. One might expect (in view of exercises 49 and 50) that this logic is complete with respect to constant presheaves on posets $\mathcal{K}$ which have the property that whenever two elements have an upper bound, they also have a lower bound. However, this is not the case!

Proposition 3.19 *Suppose X is a constant presheaf on a poset $\mathcal{K}$ which has the property that whenever two elements have an upper bound, they also have a lower bound. Then the following axiom scheme is always true on X:*

$$\begin{array}{c}\forall x[(R \to (S \vee A(x))) \vee (S \to (R \vee A(x)))] \wedge \neg\forall x A(x) \\ \to \\ [(R \to S) \vee (S \to R)]\end{array}$$

Exercise 51 Prove proposition 3.19.

However, the axiom scheme in proposition 3.19 is not a consequence of the logic D-J, which fact can be shown using presheaves. This was also shown by Ghilardi. We give the relevant statements without proof; the interested reader is referred to *Arch.Math.Logic* **29** (1989), 125–136.

Proposition 3.20 *i) The axiom scheme $\neg\phi \vee \neg\neg\phi$ is true in every interpretation in $\widehat{\mathcal{C}}$ if and only if the category $\mathcal{C}$ has the property that every pair of arrows with common codomain fits into a commutative square.*

ii) Let X be a presheaf on a category $\mathcal{C}$. Suppose X has the property that for all $f : C' \to C$ in $\mathcal{C}$, all $n \geq 0$, all $x_1, \ldots, x_n \in X(C)$ and all $y \in X(C')$ there is $f' : C' \to C$ and $x \in X(C)$ such that $xf = y$ and $x_1 f = x_1 f', \ldots, x_n f = x_n f'$. Then for every interpretation on X the axiom scheme D of exercise 50 is true.

iii) There exist a category $\mathcal{C}$ satisfying the property of i), and a presheaf X on $\mathcal{C}$ satisfying the property of ii), and an interpretation on X for which an instance of the axiom scheme of proposition 3.19 is not true.

3.8.2 Failure of the Axiom of Choice

In this example, due to M. Fourman and A. Scedrov (*Manuscr. Math.* **38** (1982), 325–332), we explore a bit the higher-order structure of a presheaf category. Recall that the Axiom of Choice says: if X is a set consisting of nonempty sets, there is a function $F : X \to \bigcup X$ such that $F(x) \in x$ for every $x \in X$. This axiom is not provable in Zermelo-Fraenkel set theory, but it is classically totally unproblematic for *finite* X (induction on the cardinality of X).

We exhibit here a category $\mathcal{C}$, a presheaf Y on $\mathcal{C}$, and a subpresheaf X of the power object $\mathcal{P}(Y)$ such that the following statements are true in $\widehat{\mathcal{C}}$:

$\forall \alpha\beta \in X(\alpha = \beta)$ ("X has at most one element")

$\forall \alpha \in X \exists xy \in Y(x \neq y \wedge \forall z \in Y(z \in \alpha \leftrightarrow z = x \vee z = y))$ ("every element of X has exactly two elements")

There is no arrow $X \to \bigcup X$ (this is stronger than: X has no choice function).

Consider the category $\mathcal{C}$ with two objects and two non-identity arrows:

$$\begin{tikzcd} D \arrow[loop left, "\beta"] \arrow[r, "\alpha"] & E \end{tikzcd}$$

subject to the equations $\beta^2 = \mathrm{id}_D$ and $\alpha\beta = \alpha$.

We calculate the representables y_D and y_E, and the map $y_\alpha : y_D \to y_E$:

$y_D(E) = \emptyset$	$(y_\alpha)_D(\mathrm{id}_D) = \alpha$
$y_D(D) = \{\mathrm{id}_D, \beta\}$	$(y_\alpha)_D(\beta) = \alpha$
$y_D(\alpha)$ is the empty function	$(y_\alpha)_E$ is the empty function
$y_D(\beta)(\mathrm{id}_D) = \beta$	$y_D(\beta)(\beta) = \mathrm{id}_D$

Since E is terminal in $\mathcal{C}$, y_E is a terminal object in $\widehat{\mathcal{C}}$:

$$y_E(E) = \{\mathrm{id}_E\},\ y_E(D) = \{\alpha\},\ y_E(\alpha)(\mathrm{id}_E) = \alpha,\ y_E(\beta)(\alpha) = \alpha$$

Now let us calculate the power object $\mathcal{P}(y_D)$. According to the explicit construction of power objects in presheaf categories, we have

$$\begin{array}{rcl} \mathcal{P}(y_D)(E) & = & \mathrm{Sub}(y_E \times y_D) \\ \mathcal{P}(y_D)(D) & = & \mathrm{Sub}(y_D \times y_D) \end{array}$$

$(y_E \times y_D)(D)$ is the two-element set $\{(\alpha, \mathrm{id}_D), (\alpha, \beta)\}$ which are permuted by the action of β, and $(y_E \times y_D)(E) = \emptyset$. So we see that $\mathrm{Sub}(y_E \times y_D)$ has two elements: $\emptyset$ (the empty presheaf) and $y_E \times y_D$ itself. $(y_D \times y_D)(D)$ has 4 elements: $(\mathrm{id}_D, \beta), (\beta, \mathrm{id}_D), (\beta, \beta), (\mathrm{id}_D, \mathrm{id}_D)$ and we have: $(\mathrm{id}_D, \beta)\beta = (\beta, \mathrm{id}_D)$ and $(\beta, \beta)\beta = (\mathrm{id}_D, \mathrm{id}_D)$.

So $\mathrm{Sub}(y_D \times y_D)$ has 4 elements: $\emptyset, y_D \times y_D, A, B$ where A and B are such that

$$\begin{array}{ll} A(E) = \emptyset & A(D) = \{(\mathrm{id}_D, \beta), (\beta, \mathrm{id}_D)\} \\ B(E) = \emptyset & B(D) = \{(\beta, \beta), (\mathrm{id}_D, \mathrm{id}_D)\} \end{array}$$

Summarizing: we have $\mathcal{P}(y_D)(E) = \{\emptyset, y_E \times y_D\}$, $\mathcal{P}(y_D)(D) = \{\emptyset, y_D \times y_D, A, B\}$. The map $\mathcal{P}(y_D)(\alpha)$ is given by pullback along $y_\alpha \times \mathrm{id}_{y_D}$ and sends therefore $\emptyset$ to $\emptyset$ and $y_E \times y_D$ to $y_D \times y_D$. $\mathcal{P}(y_D)(\beta)$ is by pullback along $y_\beta \times \mathrm{id}_{y_D}$ and sends $\emptyset$ to $\emptyset$, $y_D \times y_D$ to $y_D \times y_D$, and permutes A and B.

Now let X be the subpresheaf of $\mathcal{P}(y_D)$ given by:

$$X(E) = \emptyset \quad X(D) = \{y_D \times y_D\}$$

Then X is a 'set of sets' (a subobject of a power object), and clearly, in X, the sentence $\forall xy(x = y)$ is true. So X 'has at most one element'. We have the element relation $\in_{y_D}$ as a subobject of $\mathcal{P}(y_D) \times y_D$, and its restriction to a subobject of $X \times y_D$. This is the presheaf Z with $Z(E) = \emptyset$ and $Z(D) = \{(y_D \times y_D, \mathrm{id}_D), (y_D \times y_D, \beta)\}$. So we see that the sentence expressing 'every element of X has exactly two elements' is true. The presheaf $\bigcup X$ of 'elements of elements of X' is the presheaf $(\bigcup X)(E) = \emptyset$, $(\bigcup X)(D) = \{\mathrm{id}_D, \beta\}$ as subpresheaf of y_D. Now there cannot be any arrow in $\widehat{\mathcal{C}}$ from X to $\bigcup X$, because, in $X(D)$, the unique element is fixed by the action of β; however, in $(\bigcup X)(D)$ there is no fixed point for the action of β. Hence there is no 'choice function'.

3.9 Sheaves

3.10 Structure of the category of sheaves

In this section we shall see, among other things, that also the category $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ is a topos.

Proposition 3.21 $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ *is closed under arbitrary limits in* $\widehat{\mathcal{C}}$*.*

Proof. This is rather immediate from the defining property of sheaves and the way (point-wise) limits are calculated in $\widehat{\mathcal{C}}$. Suppose $F : I \to \widehat{\mathcal{C}}$ is a diagram of sheaves with limiting cone $(X, (\mu_i : X \to F(i)))$ in $\widehat{\mathcal{C}}$. We show that X is a sheaf.

Suppose $R \in \mathrm{Cov}(C)$ and $\phi : R \to X$ is a map in $\widehat{\mathcal{C}}$. Since every $F(i)$ is a sheaf, every composite $\mu_i \phi : R \to F(i)$ has a unique amalgamation $y_i \in F(i)(C)$, and by uniqueness these satisfy, for every map $k : i \to j$ in the index category I, the equality $(F(k))_C(y_i) = y_j$. Since $X(C)$ is the vertex of a limiting cone for the diagram $F(\cdot)(C) : I \to \mathrm{Set}$, there is a unique $x \in X(C)$ such that $(\mu_i)_C(x) = y_i$ for each i. But this means that x is an amalgamation (and the unique such) for $R \xrightarrow{\phi} X$. ■

Proposition 3.22 *Let X be a presheaf, Y a sheaf. Then Y^X is a sheaf.*

Proof. Suppose $A \to Z$ is a dense subobject, and $A \xrightarrow{\phi} Y^X$ a map. By exercise 16 we have to see that ϕ has a unique extension to a map $Z \to Y^X$. Now ϕ transposes to a map $\tilde{\phi} : A \times X \to Y$. By stability of the closure operation, if $A \to Z$ is dense then so is $A \times X \to Z \times X$. Since Y is a sheaf, $\tilde{\phi}$ has a unique extension to $\psi : Z \times X \to Y$. Transposing back gives $\bar{\psi} : Z \to Y^X$, which is the required extension of ϕ. ■

Corollary 3.23 *The category* $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ *is cartesian closed.*

Now we turn to the subobject classifier in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$. Let $J : \Omega \to \Omega$ be the associated Lawvere-Tierney topology. Sieves on C which are in the image of J_C are called *closed*. This is good terminology, since a closed sieve on C is the same thing as a closed subpresheaf of y_C.

By exercise 15 we know that subsheaves of a sheaf are the closed subpresheaves, and from exercise **??**i) we know that a subpresheaf is closed if and only if its classifying map takes values in the image of J. This is a subobject of Ω; let us call it Ω_J. So subobjects in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ admit unique classifying maps into Ω_J; note that the map $1 \xrightarrow{t} \Omega$, which picks out the maximal sieve on any C, factors through Ω_J since every maximal sieve is closed. So $1 \xrightarrow{t} \Omega_J$ is a subobject classifier in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ provided we can show that it is a map between sheaves. It is easy to see (and a special case of 3.21) that 1 is a sheaf. For Ω_J this requires a little argument.

Proposition 3.24 *The presheaf* Ω_J *is a sheaf.*

Proof. We have seen that the arrow $1 \xrightarrow{t} \Omega_J$ classifies closed subobjects. Therefore, in order to show that Ω_J has the unique-extension property w.r.t. dense inclusions, it is enough to see that whenever X is a dense subpresheaf of Y there is a bijection between the closed subpresheaves of X and the closed subpresheaves of Y.

For a closed subpresheaf A of X let $k(A)$ be the closure of A in $\mathrm{Sub}(Y)$. For a closed subpresheaf B of Y let $l(B) = B \cap X$; this is a closed subpresheaf of X.

Now $kl(B) = k(B \cap X) = \overline{B \cap X} = \bar{B} \cap \bar{X} = \bar{B} = B$ since X is dense and B closed. Conversely, $lk(A) = \bar{A} \cap X$ which is (by stability of closure) the closure of A in X. But A was closed, so this is A. Hence the maps k and l are inverse to each other, which finishes the proof. ∎

Corollary 3.25 *The category* $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ *is a topos.*

Definition 3.26 A pair $(\mathcal{C}, \mathrm{Cov})$ of a small category and a Grothendieck topology on it is called a *site*. For a sheaf on $\mathcal{C}$ for Cov, we also say that it is a *sheaf on the site* $(\mathcal{C}, \mathrm{Cov})$. A *Grothendieck topos* is a category of sheaves on a site.

Not every topos is a Grothendieck topos. For the moment, there is only one simple example to give of a topos that is not Grothendieck: the category of finite sets. It is not a Grothendieck topos, for example because it does not have all small limits.

Exercise 52 The terminal category **1** is a topos. Is it a Grothendieck topos?

Let us say something about power objects and the natural numbers in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$.

For power objects there is not much more to say than this: for a sheaf X, its power object in $\mathrm{Sh}(\mathcal{C},\mathrm{Cov})$ is Ω_J^X; we shall also write $\mathcal{P}_J(X)$. By the Yoneda Lemma we have a natural 1-1 correspondence between $\mathcal{P}_J(X)(C)$ and the set of closed subpresheaves of $y_C \times X$; for $f : C' \to C$ and A a closed subpresheaf of $y_C \times X$, $\mathcal{P}_J(X)(f)(A)$ is given by $(y_f \times \mathrm{id}_X)^\sharp(A)$.

Next, let us discuss natural numbers. We use exercise 47 which says that the constant presheaf with value $\mathbb{N}$ is a natural numbers object in $\widehat{\mathcal{C}}$, and we also use the following result:

Exercise 53 Suppose $\mathcal{E}$ has a natural numbers object and $F : \mathcal{E} \to \mathcal{F}$ is a functor which has a right adjoint and preserves the terminal object. Then F preserves the natural numbers object.

So the natural numbers object in $\mathrm{Sh}(\mathcal{C},\mathrm{Cov})$ is N^{++}, where N is the constant presheaf with value $\mathbb{N}$. In fact, we don't have to apply the 'plus' construction twice, because N is 'almost' separated: clearly, if n,m are two distinct natural numbers and $R \in \mathrm{Cov}(C)$ is such that for all $f \in R$ we have $nf = mf$, then $R = \emptyset$. So the only way that N can fail to be separated is that for some objects C we have $\emptyset \in \mathrm{Cov}(C)$. Now define the presheaf N' as follows:

$$N'(C) \;=\; \begin{cases} \mathbb{N} & \text{if } \emptyset \notin \mathrm{Cov}(C) \\ \{*\} & \text{if } \emptyset \in \mathrm{Cov}(C) \end{cases}$$

Exercise 54 Prove:

a) N' is separated

b) $\zeta_N : N \to N^+$ factors through N'

c) $N^{++} \simeq (N')^+$

Colimits in $\mathrm{Sh}(\mathcal{C},\mathrm{Cov})$ are calculated as follows: take the colimit in $\widehat{\mathcal{C}}$, then apply the associated sheaf functor. For coproducts of sheaves, we have a simplification comparable to that of N. We write $\bigsqcup$ for the coproduct in $\widehat{\mathcal{C}}$ and $\bigsqcup_J$ for the coproduct in $\mathrm{Sh}(\mathcal{C},\mathrm{Cov})$. So $\bigsqcup_J F_i = \mathbf{a}(\bigsqcup F_i)$, but if we define $\bigsqcup' F_i$ by

$$(\bigsqcup{}' F_i)(C) \;=\; \begin{cases} \bigsqcup F_i(C) & \text{if } \emptyset \notin \mathrm{Cov}(C) \\ \{*\} & \text{if } \emptyset \in \mathrm{Cov}(C) \end{cases}$$

then it is not too hard to show that $\bigsqcup_J F_i \simeq (\bigsqcup' F_i)^+$. Concretely, a compatible family in $\bigsqcup' F_i$ indexed by a covering sieve R on C, i.e./ a map

$\phi : R \to \bigsqcup' F_i$, gives for each i a sub-sieve R_i and a map $\phi_i : R_i \to F_i$. The system of subsieves R_i has the property that if $h : C' \to C$ is an element of $R_i \cap R_j$ and $i \neq j$, then $\emptyset \in \mathrm{Cov}(C')$. Of course, such compatible families are still subject to the equivalence relation defining $(\bigsqcup' F_i)^+$.

Exercise 55 Prove:

i) Coproducts are stable in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$

ii) For any sheaf F, $F^{N_J} \simeq \prod_{n \in \mathbb{N}} F$

Images in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$: given a map $\phi : F \to G$ between sheaves, the image of ϕ (as subsheaf of G) is the closure of the image in $\widehat{\mathcal{C}}$ of the same map. The arrow ϕ is an epimorphism in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ if and only if for each C and each $x \in G(C)$, the sieve $\{f : C' \to C \mid \exists y \in F(C')(\phi_{C'}(y) = xf)\}$ covers C.

Exercise 56 Prove this characterization of epis in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$. Prove also that in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$, an arrow which is both mono and epi is an isomorphism.

Regarding the structure of the lattice of subobjects in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ of a sheaf F, we know that these are the closed subpresheaves, so the fixed points of the closure operation. That the subobjects again form a Heyting algebra is then a consequence of the following exercise.

Exercise 57 Suppose H is a Heyting algebra with operations $\bot, \top, \wedge, \vee, \to$ and let $j : H \to H$ be order-preserving, idempotent, inflationary (that is: $x \leq j(x)$ for all $x \in H$), and such that $j(x \wedge y) = j(x) \wedge j(y)$. Let H_j be the set of fixed points of j. Then H_j is a Heyting algebra with operations:

$$\begin{array}{rr} \top_j = \top & \bot_j = j(\bot) \\ x \wedge_j y = x \wedge y & x \vee_j y = j(x \vee y) \\ & x \to_j y = x \to y \end{array}$$

Exercise 58 If H is a Heyting algebra, show that the map $\neg\neg : x \mapsto (x \to \bot) \to \bot$ satisfies the requirements of the map j in exercise 57. Show also that $H_{\neg\neg}$ is a Boolean algebra.

Exercise 59 Let J be the Lawvere-Tierney topology corresponding to the dense topology (see section 0.5). Show that in the Heyting algebra $\Omega(C)$, J_C is the map $\neg\neg$ of exercise 58.

As for presheaves, we can express the interpretation of first-order languages in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ in terms of a 'forcing' definition. The basic setup is the same;

only now, of course, we take sheaves as interpretation of the sorts, and closed subpresheaves (subsheaves) as interpretation of the relation symbols. We then define $[\![\,\varphi\,]\!]$ as a subsheaf of $[\![\,FV(\varphi)\,]\!]$ and let $\{\varphi\} : [\![\,FV(\varphi)\,]\!] \to \Omega_J$ be its classifying map. The notation $C \Vdash_J \varphi(a_1, \ldots, a_n)$ again means that $\{\varphi\}_C(a_1, \ldots, a_n)$ is the maximal sieve on C. This relation then again admits a definition by recursion on the formula φ. The inductive clauses of the definition of $\Vdash_J$ are the same for $\Vdash$ for the cases: atomic formula, $\wedge$, $\to$ and $\forall$, and we put:

- $C \Vdash_J \neg\varphi(a_1, \ldots, a_n)$ if and only if for every arrow $g : D \to C$ in $\mathcal{C}$ we have: if $D \Vdash_J \varphi(a_1 g, \ldots, a_n g)$ then $\emptyset$ covers D;
- $C \Vdash_J (\varphi \vee \psi)(a_1, \ldots, a_n)$ if and only if the sieve $\{g : C' \to C \,|\, C' \Vdash_J \varphi(a_1 g, \ldots, a_n g)$ or $C' \Vdash_J \psi(a_1 g, \ldots, a_n g)\}$ covers C;
- $C \Vdash_J \exists x \varphi(x, a_1, \ldots, a_n)$ if and only if the sieve $\{g : C' \to C \,|\, \exists x \in F(C')\, C' \Vdash_J \varphi(x, a_1 g, \ldots, a_n g)\}$ covers C (where F is the interpretation of the sort of x).

That this works should be no surprise in view of our characterisation of images in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ and our treatment of the Heyting structure on the subsheaves of a sheaf. We have the following properties of the relation $\Vdash_J$:

Theorem 3.27 *i) If $C \Vdash_J \varphi(a_1, \ldots, a_n)$ then for each arrow $f : C' \to C$, $C' \Vdash_J \varphi(a_1 f, \ldots, a_n f)$;*

ii) if R is a covering sieve on C and for every arrow $f : C' \to C$ in R we have $C' \Vdash_J \varphi(a_1 f, \ldots, a_n f)$, then $C \Vdash_J \varphi(a_1, \ldots, a_n)$.

Exercise 60 Let N_J be the natural numbers object in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$. Prove the same result as we had in exercise 47, that is: for the standard interpretation of te language of arithmetic in N_J, a sentence is true if and only it is true in the (classical) standard model of natural numbers.

Exercise 61 We assume that we have a site $(\mathcal{C}, \mathrm{Cov})$ and an object I of $\mathcal{C}$ satisfying the following conditions:

i) $\emptyset \notin \mathrm{Cov}(I)$

ii) If there is no arrow $I \to A$ then $\emptyset \in \mathrm{Cov}(A)$

iii) If there is an arrow $I \to A$ then every arrow $A \to I$ is split epi

We call a sheaf F in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ $\neg\neg$-*separated* if for every object A of $\mathcal{C}$ and all $x, y \in F(A)$,

$$A \Vdash_J \neg\neg(x = y) \to x = y$$

Prove that the following two assertions are equivalent, for a sheaf F:

a) F is $\neg\neg$-separated

b) For every object A of $\mathcal{C}$ and all $x, y \in F(A)$ the following holds: if for every arrow $\phi : I \to A$ we have $x\phi = y\phi$ in $F(I)$, then $x = y$

3.11 Application: a model for the independence of the Axiom of Choice

In this section we treat a model, due to P. Freyd, which shows that in toposes where *classical* logic always holds, the axiom of choice need not be valid. Specifically, we construct a topos $\mathcal{F} = \mathrm{Sh}(F, \mathrm{Cov})$ and in $\mathcal{F}$ a subobject E of $N_J \times \mathcal{P}_J(N_J)$ with the properties:

i) $\mathcal{F}$ is *Boolean*, that is: every subobject lattice is a Boolean algebra;

ii) $\Vdash_J \forall n \exists \alpha((n, \alpha) \in E)$

iii) $\Vdash \neg\exists f \in \mathcal{P}_J(N_J)^{N_J} \forall n\,((n, f(n)) \in E)$

So, E is an N_J-indexed collection of nonempty (in a strong sense) subsets of $\mathcal{P}_J(N_J)$, but admits no choice function.

Let $\mathbb{F}$ be the following category: it has objects $\bar{n}$ for each natural number n, and an arrow $f : \bar{m} \to \bar{n}$ is a function $\{0, \ldots, m\} \to \{0, \ldots, n\}$ such that $f(i) = i$ for every i with $0 \le i \le n$. It is understood that there are no morphisms $\bar{m} \to \bar{n}$ for $m < n$. Note, that $\bar{0}$ is a terminal object in this category.

On $\mathbb{F}$ we let Cov be the dense topology, so a sieve R on $\bar{m}$ covers $\bar{m}$ if and only if for every arrow $g : \bar{n} \to \bar{m}$ there is an arrow $h : \bar{k} \to \bar{n}$ such that $gh \in R$. We shall work in the topos $\mathcal{F} = \mathrm{Sh}(\mathbb{F}, \mathrm{Cov})$, the *Freyd topos*. Let E_n be the object $\mathbf{a}(y_{\bar{n}})$, the sheafification of the representable presheaf on $\bar{n}$.

Lemma 3.28 Cov *has the following properties:*

a) Every covering sieve is nonempty

b) Every nonempty sieve on $\bar{0}$ *is a cover* *c)* *Every representable presheaf is separated*

d) $y_{\bar{0}}$ *has only two closed subobjects*

Proof. For a), apply the definition of 'R covers $\bar{m}$' to the identity on $\bar{m}$; it follows that there is an arrow $h : \bar{k} \to \bar{m}$ such that $h \in R$.

For b), suppose S is a sieve on $\bar{0}$ and $\bar{k} \xrightarrow{f} \bar{0}$ is in S. Since $\bar{0}$ is terminal, for any $\bar{m} \xrightarrow{g} \bar{0}$ and any maps $\overline{m+k} \to \bar{k}$, $\overline{m+k} \to \bar{m}$, the square

$$\begin{tikzcd}
\overline{m+k} \arrow[r] \arrow[d] & \bar{k} \arrow[d, "f"] \\
\bar{m} \arrow[r, "g"'] & \bar{0}
\end{tikzcd}$$

commutes, so for any such g there is an h with $gh \in R$, hence R covers $\bar{0}$.

For c), suppose $g, g' : \bar{k} \to \bar{n}$ are such that for a cover R of $\bar{k}$ we have $gf = g'f$ for all $f \in R$. We need to see that $g = g'$. Pick $i \leq k$. Let $h : \overline{k+1} \to \bar{k}$ be such that $h(k+1) = i$. Since R covers $\bar{k}$ there is $u : \bar{l} \to \overline{k+1}$ such that $hu \in R$. Then $ghu = g'hu$, which means that $g(i) = ghu(k+1) = g'hu(k+1) = g'(i)$. So $g = g'$, as desired.

Finally, d) follows directly from b): suppose R is a closed sieve on $\bar{0}$. If $R \neq \emptyset$, then R is covering by b), hence (being also closed) equal to $\max(\bar{0})$. Hence the only closed sieves are $\emptyset$ and $\max(\bar{0})$. ∎

Proposition 3.29 *The unique map* $E_n \to 1$ *is an epimorphism.*

Proof. By lemma 3.28d), $1 = \mathbf{a}(y_{\bar{0}})$ has only two subobjects and $y_{\bar{n}}$ is nonempty, so the image of $E_n \to 1$ is 1. ∎

Proposition 3.30 *If* $n > m$ *then* $E_n(\bar{m}) = \emptyset$*.*

Proof. Since $y_{\bar{n}}$ is separated by 3.28c), $E_n = (y_{\bar{n}})^+$, so $E_n(\bar{m})$ is an equivalence class of morphisms $\tau : S \to y_{\bar{n}}$ in $\mathrm{Set}^{\mathbb{F}^{\mathrm{op}}}$, for a cover S of $\bar{m}$. We claim that such τ don't exist.

For, since such S is nonempty (3.28a)), pick $s : \bar{k} \to \bar{m}$ in S and let $f = \tau_{\bar{k}}(s)$, so $f : \bar{k} \to \bar{n}$. Let $g, h : \overline{k+1} \to \bar{k}$ be such that $g(k+1) = n$, and $h(k+1) = s(n) \leq m < n$. Then $sg = sh$ (check!). So

$$fg = \tau_{\bar{k}}(s)g = \tau_{\overline{k+1}}(sg) = \tau_{\overline{k+1}}(sh) = \tau_{\bar{k}}(s)h = fh$$

However, $fg(k+1) = f(n) = n$, whereas $fh(k+1) = f(s(n+1)) = s(n)$. Contradiction. ∎

Corollary 3.31 *The product sheaf $\prod_{n \in \mathbb{N}} E_n$ is empty.*

Proof. For, if $(\prod_n E_n)(\bar{m}) \neq \emptyset$ then by applying the projection $\prod_n E_n \to E_{\overline{m+1}}$ we would have $E_{m+1}(\bar{m}) \neq \emptyset$, contradicting 3.30. ∎

Proposition 3.32 *For each n there is a monomorphism $E_n \to \mathcal{P}_J(N_J)$.*

Proof. Since $E_n = \mathbf{a}(y_{\bar{n}})$ and $\mathcal{P}_J(N_J)$ is a sheaf, it is enough to construct a monomorphism $y_{\bar{n}} \to \mathcal{P}_J(N_J)$, which gives then a unique extension to a map from E_n; since $\mathbf{a}$ preserves monos, the extension will be mono if the given map is.

Fix n for the rest of the proof. Let $(g_k)_{k \in \mathbb{N}}$ be a 1-1 enumeration of all the arrows in $\mathbb{F}$ with codomain $\bar{n}$. For each g_i, let C_i be the smallest closed sieve on $\bar{n}$ containing g_i (i.e., C_i is the $J_{\bar{n}}$-image of the sieve generated by g_i).

$\mathcal{P}_J(N_J)(\bar{m})$ is the set of closed subpresheaves of $y_{\bar{m}} \times N_J$. Elements of $(y_{\bar{m}} \times N_J)(\bar{k})$ are pairs $(h, (S_i)_{i \in \mathbb{N}})$ where $h : \bar{k} \to \bar{m}$ and $(S_i)_i$ is an $\mathbb{N}$-indexed collection of sieves on $\bar{k}$, such that $S_i \cap S_j = \emptyset$ for $i \neq j$, and $\bigcup_i S_i$ covers $\bar{k}$.

Define $\mu_{\bar{m}} : y_{\bar{n}}(\bar{m}) \to \mathcal{P}_J(N_J)(\bar{m})$ as follows. For $f : \bar{m} \to \bar{n}$, $\mu_{\bar{m}}(f)$ is the subpresheaf of $y_{\bar{m}} \times N_J$ given by: $(h, (S_i)_i) \in \mu_{\bar{m}}(f)(\bar{k})$ iff for each i, $S_i \subseteq (fh)^*(C_i)$. It is easily seen that $\mu_{\bar{m}}(f)$ is a closed subpresheaf of $y_{\bar{m}} \times N_J$.

Let us first see that μ is a natural transformation. Suppose $g : \bar{l} \to \bar{m}$. For $h' : \bar{k} \to \bar{l}$ we have:

$$\begin{array}{ll} & (h', (S_i)_i) \in (y_g \times \mathrm{id}_{N_J})^\sharp(\mu_{\bar{m}}(f))(\bar{k}) \\ \text{iff} & (gh', (S_i)_i) \in \mu_{\bar{m}}(f)(\bar{k}) \\ \text{iff} & \forall i(\, S_i \subseteq (fgh')^*(C_i)) \\ \text{iff} & (h', (S_i)_i) \in \mu_{\bar{l}}(fg)(\bar{k}) \end{array}$$

Next, let us prove that μ is mono. Suppose $\mu_{\bar{m}}(f) = \mu_{\bar{m}}(f')$ for $f, f' : \bar{m} \to \bar{n}$. Let j and j' be such that in our enumeration, $f = g_j$ and $f' = g_{j'}$. Now consider the pair $\xi = (\mathrm{id}_{\bar{m}}, (S_i)_i)$, where S_i is the empty sieve if $i \neq j$, and $S_j = \max(\bar{m})$. Then ξ is easily seen to be an element of $\mu_{\bar{m}}(f)(\bar{m})$, so it must also be an element of $\mu_{\bar{m}}(f')(\bar{m})$, which means that $f' \in C_j$. So $C_j \cap C_{j'} \neq \emptyset$. But this means that we must have a commutative square in $\mathbb{F}$:

$$\begin{tikzcd} \bar{l} \arrow[r] \arrow[d] & \bar{m} \arrow[d, "f'"] \\ \bar{m} \arrow[r, "f"'] & \bar{n} \end{tikzcd}$$

It is easy to conclude from this that $f = f'$. ∎

3.12 Application: a model for "every function from reals to reals is continuous"

In 1924, L.E.J. Brouwer published a paper: *Beweis, dass jede volle Funktion gleichmässig stetig ist* (Proof, that every total function is uniformly continuous), *Nederl. Akad. Wetensch. Proc.* **27**, pp.189–193. His lucubrations on intuitionistic mathematics had led him to the conclusion that every function from $\mathbb{R}$ to $\mathbb{R}$ must be continuous. Among present-day researchers of constructive mathematics, this statement is known as *Brouwer's Principle* (although die-hard intuitionists still refer to it as Brouwer's Theorem).

The principle can be made plausible in a number of ways; one is, to look at the reals from a computational point of view. If a computer, which can only deal with finite approximations of reals, computes a function, then for every required precision for $f(x)$ it must be able to approximate x closely enough and from there calculate $f(x)$ within the prescribed precision; this just means that f must be continuous.

In this section we shall show that the principle is *consistent* with higher-order intuitionistic type theory, by exhibiting a topos in which it holds, for the standard real numbers. In order to do this, we have of course to say what the "object of real numbers" in a topos is. That will be done in the course of the construction.

We shall work with a full subcategory $\mathbb{T}$ of the category Top of topological spaces and continuous functions. It doesn't really matter so much what $\mathbb{T}$ exactly is, but we require that:

- $\mathbb{T}$ is closed under finite products and open subspaces
- $\mathbb{T}$ contains the space $\mathbb{R}$ (with the euclidean topology)

We specify a Grothendieck topology on $\mathbb{T}$ by defining, for an object T of $\mathbb{T}$, that a sieve R on T covers T, if the set of open subsets U of T for which the inclusion $U \to T$ is in R, forms an open covering of T. It is easy to verify that this is a Grothendieck topology.

The first thing to note is that for this topology (we call it Cov), every representable presheaf is a sheaf, because it is a presheaf of (continuous) functions: given a compatible family $R \to y_T$ for R a covering sieve on X, this family contains maps $f_U : U \to T$ for every open U contained in a covering of X; and these maps agree on intersections, because we have a sieve. So they have a unique amalgamation to a continuous map $f : X \to T$, i.e. an element of $y_T(X)$.

Also for spaces S not necessarily in the category $\mathbb{T}$ we have sheaves $y_S = \mathrm{Cts}(-, S)$.

Recall that the Yoneda embedding preserves existing exponents in $\mathbb{T}$. This also extends to exponents which exist in Top but are not in $\mathbb{T}$. If T is a locally compact space, then for any space X we have an exponent X^T in Top: it is the set of continuous functions $T \to X$, equipped with the compact-open topology (a subbase for this topology is given by the sets $\mathcal{C}(C, U)$ of those continuous functions that map C into U, for a compact subset C of T and an open subset U of X). Thus, even if X is not an object of $\mathbb{T}$, we still have in $\mathrm{Sh}(\mathbb{T}, \mathrm{Cov})$:

$$y_{X^T} \simeq (y_X)^{(Y_T)}$$

Exercise 62 Prove this fact.

From now on, we shall denote the category $\mathrm{Sh}(\mathbb{T}, \mathrm{Cov})$ by $\mathcal{T}$.
Notation: in this section we shall dispense with all subscripts $(\cdot)_J$, since we shall only work in $\mathcal{T}$. So, N denotes the *sheaf* of natural numbers, $\mathcal{P}(X)$ is the power *sheaf* of X, $\Vdash$ refers to forcing in sheaves, etc.

The natural numbers are given by the constant sheaf N, the $\mathbb{N}$-fold coproduct of copies of 1. The rational numbers are formed as a quotient of $N \times N$ by an equivalence relation which can be defined in a quantifier-free way, and hence is also a constant sheaf; therefore the object of rational numbers Q is the constant sheaf on the classical rational numbers $\mathbb{Q}$, and therefore the $\mathbb{Q}$-fold coproduct of copies of 1.

Proposition 3.33 *In $\mathcal{T}$, N and Q are isomorphic to the representable sheaves $y_\mathbb{N}$, $y_\mathbb{Q}$ respectively, where $\mathbb{N}$ and $\mathbb{Q}$ are endowed with the discrete topology.*

Proof. We shall do this for N; the proof for Q is similar. An element of $y_\mathbb{N}(X)$ is a continuous function from X to the discrete space $\mathbb{N}$; this is the same thing as an open covering $\{U_n \,|\, n \in \mathbb{N}\}$ of pairwise disjoint sets; which in turn is the same thing as an (equivalence class of an) $\mathbb{N}$-indexed collection $\{R_n \,|\, n \in \mathbb{N}\}$ of sieves on X such that whenever for $n \neq m$, $f : Y \to X$ is in $R_n \cap R_m$, $Y = \emptyset$; and moreover the sieve $\bigcup_n R_n$ covers X. But that last thing is just an element of $(\coprod_n 1)(X)$. ∎

Under this isomorphism, the order on N and Q corresponds to the pointwise ordering on functions.

Exercise 63 Show that in $\mathcal{T}$, the objects N and Q are linearly ordered, that is: for every space X in $\mathbb{T}$, $X \Vdash \forall rs \in Q\,(r < s \vee r = r \vee s < r)$.

We now construct the *object of Dedekind reals* R_d. Just as in the classical definition, a real number is a Dedekind cut of rational numbers, that is: a pair (L, R) of subsets of Q satisfying:

i) $\forall q \in Q \neg(q \in L \wedge q \in R)$

ii) $\exists q(q \in L) \wedge \exists r(r \in R)$

iii) $\forall qr(q < r \wedge r \in L \to q \in L) \ \wedge \ \forall st(s < t \wedge s \in R \to t \in R)$

iv) $\forall q \in L \exists r(q < r \wedge r \in L) \ \wedge \ \forall s \in R \exists t(t < s \wedge t \in R)$

v) $\forall qr(q < r \to q \in L \vee r \in R)$

Write $\mathrm{Cut}(L, R)$ for the conjunction of these formulas. So the object of reals R_d is the subsheaf of $\mathcal{P}(Q) \times \mathcal{P}(Q)$ given by:

$$R_d(X) \ = \ \{(L, R) \in (\mathcal{P}(Q) \times \mathcal{P}(Q))(X) \mid X \Vdash \mathrm{Cut}(L, R)\}$$

This is always a sheaf, by theorem 3.27ii).

Proposition 3.34 *The sheaf R_d is isomorphic to the representable sheaf $y_{\mathbb{R}}$.*

Proof. Let W be an object of $\mathbb{T}$ and $(L, R) \in R_d(W)$. Then L and R are subsheaves of $y_W \times Q$, which is isomorphic to $y_{W \times \mathbb{Q}}$. So both L and R consist of pairs of maps (α, p) with $\alpha : Y \to W$, $p : Y \to \mathbb{Q}$ continuous. Since L and R are sub*sheaves* we have: if $(\alpha, p) \in L(Y)$ then for any $f : V \to Y$, $(\alpha f, pf) \in L(V)$, and if $(\alpha \restriction V_i, p \restriction V_i) \in L(V_i)$ for an open cover $\{V_i\}_i$ of Y, then $(\alpha, p) \in L(Y)$ (and similar for R, of course).

Now for such $(L, R) \in \mathcal{P}(\mathbb{Q})(W) \times \mathcal{P}(\mathbb{Q})(W)$ we have $(L, R) \in R_d(W)$ if and only if $W \Vdash \mathrm{Cut}(L, R)$. We are now going to spell out what this means, and see that such (L, R) uniquely determine a continuous function $W \to \mathbb{R}$.

i)$'$ For $\beta : W' \to W$ and $q : W' \to \mathbb{Q}$, not both $(\beta, q) \in L(W')$ and $(\beta, q) \in R(W')$

ii)$'$ There is an open covering $\{W_i\}$ of W such that for each i there are $W_i \xrightarrow{l_i} \mathbb{Q}$ and $W_i \xrightarrow{r_i} \mathbb{Q}$ with $(W_i \to W, W_i \xrightarrow{l_i} \mathbb{Q}) \in L(W_i)$, and $(W_i \to W, W_i \xrightarrow{r_i} \mathbb{Q}) \in R(W_i)$

iii)$'$ For any map $\beta : W' \to W$ and any $q, r : W' \to \mathbb{Q}$: if $(\beta, r) \in L(W)$ and $q(x) < r(x)$ for all $x \in W'$, then $(\beta, q) \in L(W')$, and similar for R

iv)′ For any $\beta : W' \to W$ ad $q : W' \to \mathbb{Q}$: if $(\beta, q) \in L(W')$ there is an open covering $\{W'_i\}$ of W', and maps $r_i : W'_i \to \mathbb{Q}$ such that $(\beta \restriction W'_i, r_i) \in L(W'_i)$, and $r_i(x) > q(x)$ for all $x \in W'_i$. And similar for R

v)′ For any $\beta : W' \to W$ and $q, r : W' \to \mathbb{Q}$ satisfying $q(x) < r(x)$ for all $x \in W'$, there is an open covering $\{W'_i\}$ of W' such that for each i, either $(\beta \restriction W'_i, q \restriction W'_i) \in L(W'_i)$ or $(\beta \restriction W'_i, q \restriction W'_i) \in R(W'_i)$.

Let $\hat{q} : W \to \mathbb{Q}$ be the constant function with value q. For every $x \in W$ we define:

$$\begin{array}{rcl} L_x & = & \{q \in \mathbb{Q} \,|\, \exists \text{open } V \subseteq W (x \in V \wedge (V \to W, \hat{q} \restriction V) \in L(V))\} \\ R_x & = & \{q \in \mathbb{Q} \,|\, \exists \text{open } V \subseteq W (x \in V \wedge (V \to W, \hat{q} \restriction V) \in R(V))\} \end{array}$$

Then you should verify that (L_x, R_x) form a Dedekind cut in Set, hence determine a real number $f_{L,R}(x)$.

By definition of L_x and R_x, if q, r are rational numbers then $q < f_{L,R}(x) < r$ holds if and only if $q \in L_x$ and $r \in R_x$; so the preimage of the open interval (q, r) under $f_{L,R}$ is open; that is, $f_{L,R}$ is continuous. We have therefore defined a map $(L, R) \mapsto f_{L,R} : R_d(W) \to y_{\mathbb{R}}(W)$. It is easy to verify that this gives a map of sheaves: $R_d \to y_{\mathbb{R}}$.

For the other direction, if $f : W \to \mathbb{R}$ is continuous, one defines subsheaves L_f, R_f of $y_{W \times \mathbb{Q}}$ as follows: for $\beta : W' \to W, p : W' \to \mathbb{Q}$ put

$$\begin{array}{lll} (\beta, p) \in L_f(W') & \text{iff} & \forall x \in W'(p(x) < f(\beta(x))) \\ (\beta, p) \in R_f(W') & \text{iff} & \forall x \in W'(p(x) > f(\beta(x))) \end{array}$$

We leave it to you to verify that then $W \Vdash \mathrm{Cut}(L_f, R_f)$ and that the two given operations between $y_{\mathbb{R}}(W)$ and $R_d(W)$ are inverse to each other. You should observe that every continuous function $f : W \to \mathbb{Q}$ is locally constant, as $\mathbb{Q}$ is discrete. ∎

Corollary 3.35 *The exponential $(R_d)^{R_d}$ is isomorphic to $y_{\mathbb{R}^{\mathbb{R}}}$, where $\mathbb{R}^{\mathbb{R}}$ is the set of continuous maps $\mathbb{R} \to \mathbb{R}$ with the compact-open topology.*

Proof. This follows at once from proposition 3.34, the observation that y preserves exponents, and the fact that $\mathbb{R}$ is locally compact. ∎

From the corollary we see at once that arrows $R_d \to R_d$ in $\mathcal{T}$ correspond bijectively to continuous functions $\mathbb{R} \to \mathbb{R}$, but this is not yet quite Brouwer's statement that all functions (defined, possibly, with extra parameters) from R_d to R_d are continuous. So we prove that now.

Theorem 3.36 $\mathcal{T} \Vdash$ *"All functions $R_d \to R_d$ are continuous"*

Proof. . In other words, we have to prove that the sentence

$$\begin{aligned}&\forall f \in (R_d)^{R_d} \forall x \in R_d \forall \epsilon \in R_d (\epsilon > 0 \to \exists \delta \in R_d (\delta > 0 \wedge \\ &\quad \forall y \in R_d (x - \delta < y < x + \delta \to f(x) - \epsilon < f(y) < f(x) + \epsilon)))\end{aligned}$$

is true in $\mathcal{T}$.

We can work in $y_{\mathbb{R}^{\mathbb{R}}}$ for $(R_d)^{R_d}$, so $(R_d)^{R_d}(W) = \mathrm{Cts}(W \times \mathbb{R}, \mathbb{R})$. Take $f \in (R_d)^{R_d}(W)$ and $a, \epsilon \in R_d(W)$ such that $W \Vdash \epsilon > 0$. So $f : W \times \mathbb{R} \to \mathbb{R}$, and $a, \epsilon : W \to \mathbb{R}$, $\epsilon(x) > 0$ for all $x \in W$. We have to show:

(∗) $W \Vdash \exists \delta \in R_d (\delta > 0 \wedge \forall y \in R_d (a - y < \delta < a + \delta \to$
$f(a) - \epsilon < f(y) < f(a) + \epsilon))$

Now f and ϵ are continuous, so for each $x \in W$ there is an open neighborhood $W_x \subseteq W$ of x, and a $\delta_x > 0$ such that for each $\xi \in W_x$ and $t \in (a(x) - \delta_x, a(x) + \delta_x)$:

(1) $|a(\xi) - a(x)| < \frac{1}{2}\delta_x$

(2) $|f(\xi, t) - f(\xi, a(x))| < \frac{1}{2}\epsilon(\xi)$

We claim:

$$W_x \Vdash \forall y (a - \frac{1}{2}\delta_x < y < a + \frac{1}{2}\delta_x \to f(a) - \epsilon < f(y) < f(a) + \epsilon)$$

Note that this establishes what we want to prove.

To prove the claim, choose $\beta : V \to W_x$, $b : V \to \mathbb{R}$ such that

$$V \Vdash a\beta - \frac{1}{2}\delta_x < b < a\beta + \frac{1}{2}\delta_x$$

Then for all $\zeta \in V$, $|a\beta(\zeta) - b(\zeta)| < \frac{1}{2}\delta_x$, so by (1),

$$|a(x) - b(\zeta)| < \delta_x$$

Therefore we can substitute $\beta\zeta$ for ξ, and $b(\zeta)$ for t in (2) to obtain

$$|f(\beta(\zeta), b(\zeta)) - f(x, a(x))| < \tfrac{1}{2}\epsilon\beta(\zeta)$$
and
$$|f(\beta(\zeta), a\beta(\zeta)) - f(x, a(x))| < \tfrac{1}{2}\epsilon\beta(\zeta)$$

We conclude that $|f(\beta(\zeta)), b(\zeta)) - f(\beta(\zeta), a\beta(\zeta))| < \epsilon\beta(\zeta)$. Hence,

$$V \Vdash (f\beta)(a\beta) - \epsilon\beta < (f\beta)(b) < (f\beta)(a\beta) + \epsilon\beta$$

which proves the claim and we are done. ∎

4 Classifying Toposes

4.1 Examples

Example 4.1 (Torsors) Let G be a group and suppose $\gamma : \mathcal{E} \to \mathrm{Set}$ is a geometric morphism (we speak of a "topos over Set", i.e. a topos with a geometric morphism to Set). Then $\gamma^*(G)$ is a group object in $\mathcal{E}$. A *G-torsor* over $\mathcal{E}$ is an object T of $\mathcal{E}$ equipped with a left group action

$$\mu : \gamma^*(G) \times T \to T$$

which, apart from the axioms for a group action, satisfies the following conditions:

i) $T \to 1$ is an epimorphism.

ii) The action μ induces an isomorphism

$$\langle \mu, p_1 \rangle : \gamma^*(G) \times T \to T \times T$$

(recall that p_1 denotes the projection on the *second* coordinate)

In the topos $\widehat{G}$ of right G-sets, we have a torsor whose underlying set is G itself, with its canonical action on the left (note that the actions on the left and on the right commute with each other, so the left action is a map of G-sets). We call this G-torsor U_G.

The G-torsors in $\mathcal{E}$ form a category $\mathrm{Tor}(\mathcal{E}, G)$, whose objects are G-torsors over $\mathcal{E}$ and whose morphisms are morphisms of left G-sets in $\mathcal{E}$. Since for cocomplete toposes, the geometric morphism to Set is essentially unique, we have, for a geometric morphism $f : \mathcal{E} \to \widehat{G}$, a diagram

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "f"] \arrow[dr, "\gamma"'] & & \widehat{G} \arrow[dl, "g"] \\ & \mathrm{Set} & \end{tikzcd}$$

which commutes up to isomorphism (where g is the geometric morphism we have already seen).

Clearly, the structures of a G-torsor and of a map between G-torsors are preserved by inverse images of geometric morphisms, so any geometric morphism $f : \mathcal{F} \to \mathcal{E}$ gives rise to a functor $f^* : \mathrm{Tor}(\mathcal{E}, G) \to \mathrm{Tor}(\mathcal{F}, G)$.

For the following theorem we should state the 2-dimensional character of the category $\mathcal{T}op$: for two geometric morphisms $f, g : \mathcal{F} \to \mathcal{E}$ we can also consider natural transformations $f^* \to g^*$. In this way we have, for any two toposes $\mathcal{F}, \mathcal{G}$ a category $\mathcal{T}op(\mathcal{F}, \mathcal{E})$.

Theorem 4.2 (MM VIII.2.7) *For a topos $\mathcal{E}$ over* Set *there is an equivalence of categories*

$$\mathcal{T}op(\mathcal{E}, \widehat{G}) \ \simeq \ \mathrm{Tor}(\mathcal{E}, G).$$

This equivalence is, on objects, induced by the operation which sends the geometric morphism $g : \mathcal{E} \to \widehat{G}$ to the G-torsor $g^*(U_G)$ and is therefore natural in $\mathcal{E}$.

This example is an instance of a general phenomenon. We consider, for a topos $\mathcal{E}$, the category $\mathcal{E}_T$ of "structures of a type T" in $\mathcal{E}$. For the moment, let us not worry about what these structures are or what the morphisms could be, except that we suppose that when M is such a structure in $\mathcal{E}$ and $f : \mathcal{F} \to \mathcal{E}$ is a geometric morphism, then f^*M is such a structure in $\mathcal{F}$; and similarly, if we have an arrow $\mu : M \to N$ in $\mathcal{E}_T$ then $f^*(\mu)$ is an arrow $f^*M \to f^*N$ in $\mathcal{F}_T$, so that we have a functor $f^* : \mathcal{E}_T \to \mathcal{F}_T$.

Definition 4.3 A *classifying topos* for structures of type T is a topos $\mathcal{B}(T)$ over Set, for which there is a natural equivalence of categories

$$\mathcal{T}op(\mathcal{E}, \mathcal{B}(T)) \to \mathcal{E}_T$$

Applying the equivalence to the identity geometric morphism on $\mathcal{B}(T)$ and reasoning like in the Yoneda Lemma, we see that there is a structure U_T of type T in $\mathcal{B}(T)$ (the *universal* T-structure), such that the equivalence of Definition 4.3 is given by: $f \mapsto f^*(U_T)$.

We shall later specify what "structures of type T" will be (models of a certain logical theory); for now, we continue with some more examples.

Example 4.4 (Objects) The simplest "structure of type T" is: just an object. If $\mathcal{B}$ is a classifying topos for objects, we have an equivalence of categories

$$\mathcal{T}op(\mathcal{E}, \mathcal{B}) \to \mathcal{E}$$

given by $f \mapsto f^*(U)$ for some "universal object" U of $\mathcal{B}$.

Lemma 4.5 (MM VIII.4.1) *Let* Set_f *be the category of finite sets. Then* Set_f *is the free category with finite colimits generated by one object.*

Proof. The statement of the lemma means: there is a finite set X such that for every category $\mathcal{C}$ with finite colimits and every object C of $\mathcal{C}$, there is an essentially unique functor $F_C : \mathrm{Set}_f \to \mathcal{C}$ which preserves finite colimits and sends X to C. Indeed, let X be a one-element set. For an arbitrary finite set E, let

$$F_C(E) \ = \ \sum_{e \in E} C$$

Clearly, $F_C(X) = C$. Moreover, F_C preserves all finite colimits (see MM VIII.4.1 for details). ∎

Dual to Lemma 4.5 we have:

Lemma 4.6 (MM VIII.4.2) *The category* $\mathrm{Set}_f^{\mathrm{op}}$ *is the free category with finite limits, generated by one object.*

Now we have a chain of equivalences:

$$\begin{array}{rl} \text{Geometric morphisms } \mathcal{E} \to \mathrm{Set}^{\mathrm{Set}_f} & \simeq \\ \text{Flat functors } \mathrm{Set}_f^{\mathrm{op}} \to \mathcal{E} & \simeq \\ \text{Finite limit preserving functors } \mathrm{Set}_f^{\mathrm{op}} \to \mathcal{E} & \simeq \\ \mathcal{E} & \end{array}$$

So, the classifying topos for objects is $\mathrm{Set}^{\mathrm{Set}_f}$.

Exercise 64 What is the "universal object" in $\mathrm{Set}^{\mathrm{Set}_f}$?

Example 4.7 (Rings) Our next example concerns commutative rings, here just called rings. In a category $\mathcal{C}$ with finite limits, a *ring object* is a diagram

$$1 \underset{1}{\overset{0}{\rightrightarrows}} R \underset{\cdot}{\overset{+}{\leftleftarrows}} R \times R$$

for which the axioms for rings (expressed by commuting diagrams) hold. We have an obvious definition of homomorphism of ring objects in $\mathcal{C}$, and hence a category $\mathrm{ring}(\mathcal{C})$. Any finite limit preserving functor $F : \mathcal{C} \to \mathcal{D}$ induces a functor $\mathrm{ring}(\mathcal{C}) \to \mathrm{ring}(\mathcal{D})$.

Definition 4.8 *A ring is* finitely presented *if it is isomorphic to*

$$\mathbb{Z}[X_1, \dots, X_n]/I$$

where $\mathbb{Z}[X_1, \dots, X_n]$ *is the ring of polynomials in* n *variables with integer coefficients, and* I *is an ideal. Since* $\mathbb{Z}[X_1, \dots, X_n]$ *is Noetherian, the ideal* I *can be written as* $(P_1, \dots, P_k)$ *for elements* $P_1, \dots, P_k$ *of* $\mathbb{Z}[X_1, \dots, X_n]$.

Let **fp-rings** be the full subcategory of the category of rings on the finitely presented rings. A morphism

$$\alpha : \mathbb{Z}[X_1, \dots, X_n]/(P_1, \dots, P_k) \to \mathbb{Z}[Y_1, \dots, Y_m]/(Q_1, \dots, Q_l)$$

is given by an n-tuple $(\alpha(X_1), \dots, \alpha(X_n))$ of polynomials in $Y_1, \dots, Y_m$, such that the polynomials

$$P_j(\alpha(X_1), \dots, \alpha(X_n))$$

are elements of the ideal $(Q_1, \ldots, Q_l)$.

The category **fp-rings** has finite coproducts: the initial object is $\mathbb{Z}$, and the sum

$$\mathbb{Z}[X_1, \ldots, X_n]/(P_1 \ldots, P_k) \,+\, \mathbb{Z}[Y_1, \ldots, Y_m]/(Q_1, \ldots, Q_l)$$

(where we assume that the strings of variables $\vec{X}$ and $\vec{Y}$ are disjoint) is the ring

$$\mathbb{Z}[X_1, \ldots, X_n, Y_1, \ldots, Y_m]/(P_1, \ldots, P_k, Q_1, \ldots, Q_l)$$

Moreover, the category **fp-rings** has coequalizers: given a parallel pair of arrows

$$\mathbb{Z}[\vec{X}]/(\vec{P}) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \mathbb{Z}[\vec{Y}]/(\vec{Q})$$

its coequalizer is the quotient ring

$$\mathbb{Z}[\vec{Y}]/(\vec{Q}, \alpha(X_1) - \beta(X_1), \ldots, \alpha(X_n) - \beta(X_n))$$

with the evident quotient map.

Now, we consider $\textbf{fp-rings}^{\mathrm{op}}$. This is a category with finite limits. Note that $\mathbb{Z}$ is terminal in $\textbf{fp-rings}^{\mathrm{op}}$. A ring object in $\textbf{fp-rings}^{\mathrm{op}}$ is a diagram

$$\mathbb{Z} \underset{1}{\overset{0}{\leftleftarrows}} R \rightrightarrows R + R$$

in **fp-rings**, subject to the duals of the axioms for rings. An example of such a structure in **fp-rings** is the ring $\mathbb{Z}[X]$ with maps $0, 1 : \mathbb{Z}[X] \to \mathbb{Z}$ sending a polynomial P to $P(0)$ and to $P(1)$ respectively; and $+, \cdot : \mathbb{Z}[X] \to \mathbb{Z}[X, Y]$ (note that $\mathbb{Z}[X, Y] = \mathbb{Z}[X] + \mathbb{Z}[X]$ in fp-rings, sending $P(X)$ to $P(X + Y)$ and to $P(XY)$ respectively.

Lemma 4.9 (MM VIII.5.1) *The category* $\textbf{fp-rings}^{\mathrm{op}}$*, together with the ring object* $\mathbb{Z}[X]$ *as just described, is the free category with finite limits and a ring object.*

The statement of the lemma means: for any category $\mathcal{C}$ with finite limits and ring object R, there is an essentially unique finite limit preserving functor from $\textbf{fp-rings}^{\mathrm{op}}$ to $\mathcal{C}$ which sends $\mathbb{Z}[X]$ to R.

We can now argue in exactly the same way as in the two previous examples: ring objects in a topos $\mathcal{E}$ correspond to flat, that is: finite limit preserving, functors from $\textbf{fp-rings}^{\mathrm{op}}$ to $\mathcal{E}$, which correspond to geometric morphisms from $\mathcal{E}$ to $\mathrm{Set}^{\textbf{fp-rings}}$; the latter therefore being the "classsifying topos for rings".

Example 4.10 (Posets) In this example we shall show that the functor category $\mathrm{Set}^{\mathrm{Pos}_f}$ is a classifying topos for posets; here, Pos_f denotes the category of *finite* posets and order-preserving maps.

Let us look at both a poset object in a category with finite limits and the dual notion, a *co-poset object* in a category with finite colimits.

A *poset object* in a category with finite limits consists of an object P and a monomorphism $\langle r_0, r_1 \rangle : R \to P \times P$, satisfying the conditions:

(R) Reflexifity: the diagonal $P \to P \times P$ factors through R.

(A) Antisymmetry: let R^{op} be the subobject $\langle r_1, r_0 \rangle : R \to P \times P$. Then the intersection of R and R^{op} (as subobjects of $P \times P$) is the diagonal $P \to P \times P$.

(T) Transitivity: let

$$\begin{tikzcd} R_1 \arrow[r, "s"] \arrow[d, "q"'] & R \arrow[d, "r_0"] \\ R \arrow[r, "r_1"'] & P \end{tikzcd}$$

be a pullback. Then the map $\langle r_0 q, r_1 s \rangle : R_1 \to P \times P$ factors through $\langle r_0, r_1 \rangle : R \to P \times P$.

Dually, a *co-poset object* in a category with finite colimits consists of an object P and an epimorphism $\begin{bmatrix} s_0 \\ s_1 \end{bmatrix} : P + P \to S$, satisfying the conditions:

(co-R) Co-reflexivity: the codiagonal $\begin{bmatrix} \mathrm{id} \\ \mathrm{id} \end{bmatrix} : P + P \to P$ factors through $P + P \to S$.

(co-A) Co-antisymmetry: there is a pushout diagram

$$\begin{tikzcd} P + P \arrow[r, "\begin{bmatrix} s_0 \\ s_1 \end{bmatrix}"] \arrow[d, "\begin{bmatrix} s_1 \\ s_0 \end{bmatrix}"'] & S \arrow[d] \\ S \arrow[r] & P \end{tikzcd}$$

where the composite $P + P \to P$ is the codiagonal.

(co-T) Co-transitivity: given a pushout diagram

$$\begin{tikzcd} P \arrow[r, "s_0"] \arrow[d, "s_1"'] & S \arrow[d, "\sigma"] \\ S \arrow[r, "\tau"'] & S_1 \end{tikzcd}$$

the map $\begin{bmatrix} \tau s_0 \\ \sigma s_1 \end{bmatrix} : P+P \to S_1$ factors through the map $\begin{bmatrix} s_0 \\ s_1 \end{bmatrix} : P+P \to S$.

Now consider the category Pos_f of finite posets; this is a category with finite colimits. We have the posets $\mathbf{1} = \{*\}$ and $\mathbf{2} = \{a, b\}$ with $a < b$. We have the maps $s_0, s_1 : \mathbf{1} \to \mathbf{2}$ given by $s_0(*) = a$, $s_1(*) = b$. Clearly, the map $\begin{bmatrix} s_0 \\ s_1 \end{bmatrix} : \mathbf{1}+\mathbf{1} \to \mathbf{2}$ is an epimorphism; we claim that this defines a co-poset structure on $\mathbf{1}$.

Clearly, co-reflexivity holds since $\mathbf{1}$ is terminal in Pos_f.

For co-antisymmetry, suppose the diagram

$$\begin{tikzcd} \mathbf{1}+\mathbf{1} \arrow[r, "\begin{bmatrix} s_0 \\ s_1 \end{bmatrix}"] \arrow[d, "\begin{bmatrix} s_1 \\ s_0 \end{bmatrix}"'] & \mathbf{2} \arrow[d, "f"] \\ \mathbf{2} \arrow[r, "g"'] & X \end{tikzcd}$$

commutes. Let $\mathbf{1}+\mathbf{1} = \{x, y\}$ with $\begin{bmatrix} s_0 \\ s_1 \end{bmatrix}(x) = a$ and $\begin{bmatrix} s_0 \\ s_1 \end{bmatrix}(y) = b$.

Then we have the equations:

$$\begin{aligned} f(a) &= f \begin{bmatrix} s_0 \\ s_1 \end{bmatrix}(x) = g \begin{bmatrix} s_1 \\ s_0 \end{bmatrix}(x) = g(b) \\ f(b) &= f \begin{bmatrix} s_0 \\ s_1 \end{bmatrix}(y) = g \begin{bmatrix} s_1 \\ s_0 \end{bmatrix}(y) = g(a) \end{aligned}$$

We conclude, by the monotonicity of f and g, that $f(a) \leq f(b) = g(a) \leq g(b) = f(a)$, so the diagram

$$\begin{tikzcd} \mathbf{1}+\mathbf{1} \arrow[r, "\begin{bmatrix} s_0 \\ s_1 \end{bmatrix}"] \arrow[d, "\begin{bmatrix} s_1 \\ s_0 \end{bmatrix}"'] & \mathbf{2} \arrow[d] \\ \mathbf{2} \arrow[r] & \mathbf{1} \end{tikzcd}$$

is a pushout, and co-antisymmetry holds.

For co-transitivity, we see that in Pos_f the diagram

$$\begin{tikzcd} \mathbf{1} \arrow[r, "s_0"] \arrow[d, "s_1"'] & \mathbf{2} \arrow[d, "\sigma"] \\ \mathbf{2} \arrow[r, "\tau"'] & \mathbf{3} \end{tikzcd}$$

is a pushout, where $\mathbf{3}$ is the poset $u < v < w$ and $\begin{bmatrix} \tau s_0 \\ \sigma s_1 \end{bmatrix} : \mathbf{1} + \mathbf{1} \to \mathbf{3}$ satisfies $\begin{bmatrix} \tau s_0 \\ \sigma s_1 \end{bmatrix}(x) = u$ and $\begin{bmatrix} \tau s_0 \\ \sigma s_1 \end{bmatrix}(y) = w$. By transitivity in $\mathbf{3}$ we have a map $\mathbf{2} \to \mathbf{3}$ (sending a to u and b to w), so that we have a factorization $\mathbf{1} + \mathbf{1} \to \mathbf{2} \to \mathbf{3}$, as required.

We conclude that we have a co-poset object in Pos_f. Moreover, every object of Pos_f is a finite colimit of a diagram of copies of $\mathbf{1}$ and $\mathbf{2}$. Therefore, we have:

> *The category* Pos_f *with the co-poset object* $\begin{bmatrix} s_0 \\ s_1 \end{bmatrix} : \mathbf{1} + \mathbf{1} \to \mathbf{2}$ *is the free category with finite colimits and a co-poset object.*

This means: for any category $\mathcal{C}$ with finite colimits and a co-poset object $P + P \to S$ there is an essentially unique functor $\mathrm{Pos}_f \to \mathcal{C}$ which preserves finite colimits and sends $\mathbf{1} + \mathbf{1} \to \mathbf{2}$ to $P + P \to S$.

Dually then, for every category $\mathcal{E}$ (in particular, a topos) with finite limits and a poset object $R \to P \times P$ we have an essentially unique functor from $\mathrm{Pos}_f^{\mathrm{op}}$ to $\mathcal{E}$ which preserves finite limits (hence is flat) and sends the poset object $\mathbf{2} \to \mathbf{1} \times \mathbf{1}$ (product in $\mathrm{Pos}_f^{\mathrm{op}}$!) to $R \to P \times P$. Therefore, if $\mathcal{E}$ s a Grothendieck topos with poset object, we have an essentially unique geometric morphism $\mathcal{E} \xrightarrow{f} \mathrm{Set}^{\mathrm{Pos}_f}$, such that f^* sends the generic poset in $\mathrm{Set}^{\mathrm{Pos}_f}$ to the given one in $\mathcal{E}$. So $\mathrm{Set}^{\mathrm{Pos}_f}$ is a classifying topos for posets.

In each of the four examples we have just seen, the classifying topos was a presheaf topos. That is because of the "algebraic character" of the type of structures we considered: the structure is given by a number of operations and the axioms are equations. Not every structure which admits a classifying topos is of such a simple kind. But let us now define what kind of structures we have in mind: structures for *geometric logic*.

4.2 Geometric Logic

We consider a *multi-sorted language*. That is: we have a set of *sorts*, a stock of variables for each sort (we write x^S in order to indicate that the variable x has sort S), and constants, function symbols and relation symbols with also specified sorts. We write:

> c^S to indicate that the constant c is of sort S;
>
> $f : S_1, \ldots, S_n \to T$ to indicate that the function symbol f takes arguments of sorts $S_1, \ldots, S_n$, and then yields something of sort T;
>
> $R \subseteq S_1, \ldots, S_n$ to indicate that the relation symbol R takes arguments of sorts $S_1, \ldots, S_n$.

All terms of the language have a specified sort: for a variable x^S of sort S, x^S is a term of sort S. Every constant of sort S is a term of sort S. If $f : S_1, \ldots, S_n \to T$ is a function symbol and $t_1, \ldots, t_n$ are terms of sorts $S_1, \ldots, S_n$ respectively, then $f(t_1, \ldots, t_n)$ is a term of sort T.

An *atomic formula* is an expression of one of three forms: it is the symbol $\top$ (for "true"), it is an equation $t = s$ where t and s are terms of the same sort, or it is an expression $R(t_1, \ldots, t_n)$, where $R \subseteq S_1, \ldots, S_n$ is a relation symbol and t_i is a term of sort S_i for $i = 1, \ldots, n$.

The class of *geometric formulas* (for a given language) is defined as follows:

> Every atomic formula is a geometric formula;
>
> If ϕ and ψ are geometric formulas, then $\phi \wedge \psi$ is a geometric formula;
>
> If ϕ is a geometric formula and x^S is a variable, then $\exists x^S \phi$ is a geometric formula;
>
> If X is a set of geometric formulas and X contains only finitely many free variables, then $\bigvee X$ is a geometric formula.

If $\mathcal{E}$ is a cocomplete topos, then there is a straightforward definition of what a *structure* for a language in $\mathcal{E}$ should be: for every sort S, we have an object $[\![S]\!]$ of $\mathcal{E}$; for every function symbol $f : S_1, \ldots, S_n \to T$ we have a morphism $[\![f]\!] : [\![S_1]\!] \times \cdots \times [\![S_n]\!] \to [\![T]\!]$ in $\mathcal{E}$; for every relation symbol $R \subseteq S_1, \ldots, S_n$ we have a subobject $[\![R]\!]$ of $[\![S_1]\!] \times \cdots \times [\![S_n]\!]$.

Just as straightforwardly, one now obtains, for any formula ϕ with free variables $x_1^{S_1}, \ldots, x_n^{S_n}$, a subobject $[\![\phi]\!]$ of $[\![S_1]\!] \times \cdots \times [\![S_n]\!]$. For the case when ϕ is of the form $\bigvee X$, we use of course the cocompleteness of $\mathcal{E}$, which implies that subobject lattices are complete (have arbitrary joins).

A *geometric sequent* is an expression of the form $\phi \vdash_{\vec{x}} \psi$, where ϕ and ψ are geometric formulas, and $\vec{x}$ is a finite list of variables which contains every variable which appears freely in ϕ or ψ.

If a structure for the language is given, let us write $[\![\vec{x}]\!]$ for the product $\prod_{i=1}^{n} [\![S_i]\!]$ if $\vec{x} = (x_1^{S_1}, \ldots, x_n^{S_n})$. If $\vec{y}_\phi$ is the list of variables appearing freely in ϕ and $\vec{y}_\psi$ the list of those in ψ, then we have evident projections $p_\phi : [\![\vec{x}]\!] \to [\![\vec{y}_\phi]\!]$ and $p_\psi : [\![\vec{x}]\!] \to [\![\vec{y}_\psi]\!]$, and hence subobjects $[\![\phi]\!]_{\vec{x}} = p_\phi^*([\![\phi]\!])$ and $[\![\psi]\!]_{\vec{x}} = p_\psi^*([\![\psi]\!])$ of $[\![\vec{x}]\!]$.

We say that the sequent $\phi \vdash_{\vec{x}} \psi$ is *true* in the given structure, if $[\![\phi]\!]_{\vec{x}} \leq [\![\psi]\!]_{\vec{x}}$ in $\mathrm{Sub}([\![\vec{x}]\!])$. We think of the sequent $\phi \vdash_{\vec{x}} \psi$ as of the "formula"

$$\forall \vec{x}(\phi \Rightarrow \psi)$$

For instance, if for one of the variables x^S in $\vec{x}$ we have that the object $[\![S]\!]$ is initial, then the sequent $\phi \vdash_{\vec{x}} \psi$ is always true.

Let us denote a structure for a given language by $\mathcal{M}$. So we have the interpretation $[\![\cdot]\!]^{\mathcal{M}}$ of the sorts, function symbols, constants and relation symbols in some topos $\mathcal{E}$. If $f : \mathcal{F} \to \mathcal{E}$ is a geometric morphism, we have a structure $f^*\mathcal{M}$ in $\mathcal{F}$ by applying the inverse image functor f^* to all the data of $\mathcal{M}$. We now have interpretations $[\![\phi]\!]^{\mathcal{M}}$ in $\mathcal{E}$ and $[\![\phi]\!]^{f^*\mathcal{M}}$ in $\mathcal{F}$.

Proposition 4.11 *Let $\mathcal{M}$ be a structure for a language in a topos $\mathcal{E}$, and suppose $f : \mathcal{F} \to \mathcal{E}$ is a geometric morphism. Then we have:*

a) For any formula ϕ of the language, $[\![\phi]\!]^{f^*\mathcal{M}} = f^*([\![\phi]\!]^{\mathcal{M}})$.

b) If the sequent $\phi \vdash_{\vec{x}} \psi$ is true with respect to the structure $\mathcal{M}$, then it is also true with respect to $f^*\mathcal{M}$.

c) If the geometric morphism f is a surjection, then the converse of b) holds: if $\phi \vdash_{\vec{x}} \psi$ is true with respect to the structure $f^*\mathcal{M}$ then it is true with respect to $\mathcal{M}$.

A *geometric theory* in a given language is a set of geometric sequents in that language. If $\mathcal{M}$ is a structure in which every sequent of a theory is true, then $\mathcal{M}$ is called a *model* of the theory.

Now we can be more precise about the "structures of a type T" mentioned in Definition 4.3: they are, in fact, models of a geometric theory. One advantage of making this notion precise is, that we can investigate geometric theories also syntactically, and, much as in classical Model Theory, study relations between syntactic properties of theories and topos-theoretic properties of their classifying toposes.

For example, in the examples we have discussed so far, the classifying toposes were presheaf toposes (as we already remarked). This is connected to the fact that the respective theories are all *universal*: no $\bigvee$ and no existential quantifier (you might object by saying that in the theory of rings we need to express that every element has an additive inverse, and that we need an existential quantifier for this; however, since the additive inverse is unique this existential quantifier is not essential and we could expand the language with an extra function symbol).

Example 4.12 (Flat functors) Let us now consider a theory where the use of existential quantifiers and (possibly infinite) disjunctions is necessary: the *theory of flat functors* form a small category $\mathcal{C}$.

Given a small category $\mathcal{C}$, let $\mathcal{L}_{\mathcal{C}}$ be the language which has:

for every object C of $\mathcal{C}$ a sort C;

for every arrow $f : C \to D$ in $\mathcal{C}$, a function symbol $f : C \to D$.

The geometric theory $\mathrm{Flat}(\mathcal{C})$ has the following sequents:

1) For every commutative triangle

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[dr, "h"'] & D \arrow[d, "g"] \\ & E \end{tikzcd}$$

a sequent $\top \vdash_{x^C} h(x) = g(f(x))$.

2) A sequent

$$\top \vdash \bigvee_{C \in \mathcal{C}_0} \exists x^C (x = x).$$

3) A sequent

$$\top \vdash_{x^C, y^D} \bigvee_{f:E \to C, g:E \to D} \exists z^E (f(z) = x \wedge g(z) = y)$$

4) A sequent

$$f(x) = g(x) \vdash_{x^C} \bigvee_{h:D \to C, fh=gh} \exists y^D (h(y) = x)$$

Exercise 65 Show that for a topos $\mathcal{E}$, a model of $\mathrm{Flat}(\mathcal{C})$ in $\mathcal{E}$ is nothing but a flat functor $\mathcal{C} \to \mathcal{E}$; and hence, that the topos $\widehat{\mathcal{C}}$ classifies models of $\mathrm{Flat}(\mathcal{C})$.

Admittedly, in this example the classifying topos is still a presheaf topos. However, this changes if we extend the theory $\mathrm{Flat}(\mathcal{C})$ according to section 2.3.

Definition 4.13 Let $(\mathcal{C}, J)$ be a site. The theory $\mathrm{FlatCont}(\mathcal{C}, J)$ of flat and J-continuous functors from $\mathcal{C}$, is an extension of the theory $\mathrm{Flat}(\mathcal{C})$ by the following axioms: for every object C of $\mathcal{C}$ and every covering sieve $R \in J(C)$ we have the axiom

$$\top \vdash_{x^C} \bigvee_{f:D \to C, f \in R} \exists y^D (f(y) = x)$$

Theorem 2.16 now implies:

Proposition 4.14 *A model of* $\mathrm{FlatCont}(\mathcal{C}, J)$ *is a topos* $\mathcal{E}$ *is nothing but a flat and* J*-continuous functor from* $\mathcal{C}$ *to* $\mathcal{E}$*. Therefore, the topos* $\mathrm{Sh}(\mathcal{C}, J)$ *classifies models of* $\mathrm{FlatCont}(\mathcal{C}, J)$*.*

And we conclude:

Theorem 4.15 (Classifying Topos Theorem, part I) *Every Grothendieck topos is the classifying topos of some geometric theory.*

The geometric theory which a Grothendieck topos classifies is by no means unique, as the following example shows.

Example 4.16 (MM, §VIII.8) Let Δ be the category of nonempty finite ordinals and order-preserving (i.e., $\leq$-preserving) functions. The presheaf category $\widehat{\Delta}$ is of paramount importance in algebraic topology and higher category theory; it is the *category of simplicial sets*. In the indicated section of their book, MacLane and Moerdijk give a detailed proof of the fact that $\widehat{\Delta}$ classifies the theory of linear orders with distinct top and bottom elements, and order-preserving maps which also preserve top and bottom.

This looks rather different from the category $\mathrm{Flat}(\Delta)$!

If geometric theories T and T' have equivalent classifying toposes, we call them *Morita equivalent*. In a picture strongly advocated by Olivia Caramello, the classifying topos forms a "bridge" between the theories T and T'.

4.3 Syntactic categories

In section 4.2 we have already seen (in the notations $\phi \vdash_{\vec{x}} \psi$ and $[\![\phi]\!]_{\vec{x}}$) that it is useful to consider so-called *formulas in context*: a formula in context is a pair $[\vec{x}.\phi]$ where ϕ is a geometric formula and $\vec{x}$ a finite list of variables which contains all variables which appear freely in ϕ.

Given a geometric theory T and a geometric sequent $\phi \vdash_{\vec{x}} \psi$, we write $T \models (\phi \vdash_{\vec{x}} \psi)$ to mean that $\phi \vdash_{\vec{x}} \psi$ is true in every model of T in every topos.

There is a deduction system for geometric logic, giving a notion $T \vdash (\phi \vdash_{\vec{x}} \psi)$, which is described in **Elephant**, §D1.3. We have a *Completeness Theorem*, which says that the notions $T \models (\phi \vdash_{\vec{x}} \psi)$ and $T \vdash (\phi \vdash_{\vec{x}} \psi)$ are equivalent; this theorem is outside the scope of these lecture notes. We shall only use the $\models$-notion.

We construct for any geometric theory T a so-called *syntactic category* $\mathrm{Syn}(T)$, as follows.

Call two geometric formulas in context $[\vec{x}.\phi]$ and $[\vec{y}.\psi]$ equivalent if $[\vec{y}.\psi]$ is obtained from $[\vec{x}.\phi]$ by a renaming of variables (both free and bound). An object of $\mathrm{Syn}(T)$ is an equivalence class of such formulas in context. We shall just write $[\vec{x}.\phi]$ for its equivalence class.

When discussing arrows from $[\vec{x}.\phi]$ to $\vec{y}.\psi]$ we may, by our convention on equivalence, assume that the contexts $\vec{x}$ and $\vec{y}$ are disjoint.

A morphism $[\vec{x}.\phi] \to [\vec{y}.\psi]$ in $\mathrm{Syn}(T)$ is an equivalence class of formulas in context $[\vec{x}, \vec{y}.\theta]$ which satisfy:

i) $T \models (\theta(\vec{x}, \vec{y}) \vdash_{\vec{x},\vec{y}} \phi(\vec{x}) \wedge \psi(\vec{y}))$.

ii) $T \models (\phi(\vec{x}) \vdash_{\vec{x}} \exists \vec{y} \theta(\vec{x}, \vec{y}))$.

iii) $T \models (\theta(\vec{x}, \vec{y}) \wedge \theta(\vec{x}, \vec{y'}) \vdash_{\vec{x},\vec{y},\vec{y'}} \vec{y} = \vec{y'})$.

where in the last clause, for $\vec{y} = y_1, \ldots, y_n$ and $\vec{y'} = y'_1, \ldots, y'_n$, $\vec{y} = \vec{y'}$ abbreviates the formula $y_1 = y'_1 \wedge \cdots \wedge y_n = y'_n$.

Two such $\theta(\vec{x}, \vec{y})$ and $\theta'(\vec{x}, \vec{y})$ represent the same morphism if they are equivalent modulo T.

Given morphisms $\theta(\vec{x}, \vec{y}) : [\vec{x}.\phi] \to [\vec{y}.\psi]$ and $\xi : [\vec{y}.\psi] \to [\vec{z}.\chi]$, the composition $\xi \circ \theta : [\vec{x}.\phi] \to [\vec{z}.\chi]$ is represented by the formula $\exists \vec{y}(\theta(\vec{x}, \vec{y}) \wedge \xi(\vec{y}, \vec{z}))$. For any object $[\vec{x}.\phi]$, the identity arrow $[\vec{x}.\phi] \to [\vec{y}.\phi]$ (recall our convention about equivalent formulas in context) is the formula $x_1 = y_1 \wedge \cdots \wedge x_n = y_n$.

Exercise 66 Prove that $\mathrm{Syn}(T)$ is a category.

Definition 4.17 A *geometric category* is a regular category in which subobject lattices have arbitrary joins, and these joins are stable under pullback.

Exercise 67 i) Characterize the monomorphisms in the category $\mathrm{Syn}(T)$.

ii) Show that $\mathrm{Syn}(T)$ is a regular category.

iii) Show that $\mathrm{Syn}(T)$ is a geometric category.

The category $\mathrm{Syn}(T)$ has a tautological model of T: for any sort S, $[\![S]\!]$ is the formula in context $[x^S.x = x]$; for any function symbol $f : S_1, \ldots, S_n \to T$, the arrow $[\![f]\!]$ is the formula

$$f(x_1^{S_1}, \ldots, x_n^{S_n}) = y^T$$

and for any relation symbol $R \subseteq S_1, \ldots, S_n$, the subobject $[\![R]\!]$ is represented by the evident monomorphism with domain $R(x_1^{S_1}, \ldots, x_n^{S_n})$.

For every geometric category $\mathcal{C}$, there is a *geometric topology* on $\mathcal{C}$: the covering sieves are those families$\{f_i : D_i \to C\}_{i \in I}$ for which the subobject

$$\bigvee_{i \in I} \mathrm{im}(f_i)$$

is the maximal subobject of C (here $\mathrm{im}(f_i)$ denotes the image of f_i as subobject of C).

Without proof, we state:

Theorem 4.18 *Let T be a geometric theory. For any cocomplete topos $\mathcal{E}$ (or, for any geometric category $\mathcal{E}$), the category of models of T in $\mathcal{E}$ is equivalent to the category of flat and continuous functors from* $\mathrm{Syn}(T)$ *to $\mathcal{E}$.*

Therefore we have:

Theorem 4.19 (Classifying Topos Theorem, part II) *The category* $\mathrm{Sh}(\mathrm{Syn}(T), J)$, *where J is the geometric topology on* $\mathrm{Syn}(T)$, *is a classifying topos for T. Hence every geometric theory has a classifying topos.*

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