Exam Advanced Topics in Logic A  
November 10, 2005, 14.00–17.00

THIS EXAM CONSISTS OF 4 PROBLEMS  
Advice: first do those problems you can do right away; then, start thinking about the others. Good luck!

Problem 1:

a) Prove that the function $F(x) = x! \cdots !$ (with $x$ factorials) is primitive recursive.

b) The same for the function $G(x) = F(F(\cdots (F(x)) \cdots )) (F(x)$ iterations of $F)$, where $F$ is the function of part a).

[Hint: in both cases, first define a suitable function of two arguments]

Problem 2:
Show that for every total recursive function $F$ there is a total recursive function $G$ with the properties:

i) For every $x$, $F(x) \leq G(x)$;

ii) the set $\{j(n, G(n)) \mid n \in \mathbb{N}\}$ is primitive recursive.

[Hint: use the Kleene T-predicate]

Problem 3:
For a theory $T$ in the language of PA we say that $T$ is recursive if the set $\{\neg \phi \mid T \vdash \phi\}$ is recursive.

a) Show that if $T$ is consistent and recursive, there is a consistent, recursive extension $T'$ of $T$ such that $T'$ is complete.

[Hint: use a recursive enumeration of all codes of sentences in the language]

b) Conclude from a) that the theory PA is not recursive.

Problem 4:
Recall that $\square \phi$ is short for $\exists x \text{Prf}(x, \neg \phi^\frown)$. The following three properties hold:

\begin{align*}
D1 & \quad \text{PA} \vdash \phi \Rightarrow \text{PA} \vdash \square \phi \\
D2 & \quad \text{PA} \vdash (\square (\phi \rightarrow \psi) \land \square \phi) \rightarrow \square \psi \\
D3 & \quad \text{PA} \vdash \square \phi \rightarrow \square \square \phi
\end{align*}

By the Diagonalization Lemma, there is a sentence $\phi$ in the language of PA, such that

$$\text{PA} \vdash \phi \leftrightarrow \square (\phi \rightarrow \neg \square \phi)$$
a) Using only D1 and D2, show that $\phi$ is independent of PA, that is, neither $\phi$ nor $\neg\phi$ are theorems of PA.

b) Again using only D1 and D2, argue that the sentence $\phi$ is false in the standard model.

c) Using D1–D3, prove that PA $\vdash \phi \leftrightarrow \Box \bot$.