Exam Advanced Topics in Logic A

November 10, 2005, 14.00–17.00

THIS EXAM CONSISTS OF 4 PROBLEMS

Advice: first do those problems you can do right away; then, start thinking about the others. Good luck!

Problem 1:

a) Prove that the function $F(x) = x\underbrace{!\cdots!}_{x}$ (x with x factorials) is primitive recursive.

b) The same for the function $G(x) = \underbrace{F(F(\cdots(F}_{F(x)}(x))\cdots))$ ($F(x)$ iterations of F), where F is the function of part a).

[Hint: in both cases, first define a suitable function of *two* arguments]

Problem 2:
Show that for every total recursive function F there is a total recursive function G with the properties:

i) For every x, $F(x) \leq G(x)$;

ii) the set $\{j(n, G(n)) \mid n \in \mathbb{N}\}$ is primitive recursive.

[Hint: use the Kleene T-predicate]

Problem 3:
For a theory T in the language of PA we say that T is *recursive* if the set $\{\ulcorner\phi\urcorner \mid T \vdash \phi\}$ is recursive.

a) Show that if T is consistent and recursive, there is a consistent, recursive extension T' of T such that T' is complete.
[Hint: use a recursive enumeration of all codes of sentences in the language]

b) Conclude from a) that the theory PA is not recursive.

Problem 4:
Recall that $\Box\phi$ is short for $\exists x\overline{\mathrm{Prf}}(x, \overline{\ulcorner\phi\urcorner})$. The following three properties hold:

D1	$\mathrm{PA} \vdash \phi \;\Rightarrow\; \mathrm{PA} \vdash \Box\phi$
D2	$\mathrm{PA} \vdash (\Box(\phi \to \psi) \wedge \Box\phi) \to \Box\psi$
D3	$\mathrm{PA} \vdash \Box\phi \to \Box\Box\phi$

By the Diagonalization Lemma, there is a sentence ϕ in the language of PA, such that

$$\mathrm{PA} \vdash \phi \leftrightarrow \Box(\phi \to \neg\Box\phi)$$

a) Using only D1 and D2, show that ϕ is independent of PA, that is, neither ϕ nor $\neg\phi$ are theorems of PA.

b) Again using only D1 and D2, argue that the sentence ϕ is false in the standard model.

c) Using D1–D3, prove that $\mathrm{PA} \vdash \phi \leftrightarrow \Box\bot$