Exam Category Theory and Topos Theory
May 26, 2014; 10:00–13:00

THIS EXAM CONSISTS OF FIVE PROBLEMS
Advice: first do those problems you can do right away; then, start thinking about the others.
Good luck!

Exercise 1.

a) Prove that every faithful functor reflects monos and epis.

b) Give a counterexample to show that a) fails for isomorphisms.

c) Prove that every full and faithful functor reflects isomorphisms.

Solution: a) Let F faithful and f an arrow such that $F(f)$ is mono. If $fg = fh$ then $F(f)F(g) = F(f)F(h)$ so $F(g) = F(h)$ since $F(f)$ is mono; hence $g = h$ because F is faithful. Therefore f is mono, and F reflects monos. The argument for epis is dual.
b) Consider the functor from the category $\{0 < 1\}$ to the category $\{0 \simeq 1\}$; it is faithful but does not reflect isos.
c) Let F full and faithful and f such that $F(f)$ has an inverse, say k. Since F is full, there is g such that $F(g) = k$. Then $F(fg) = F(f)F(g) = F(f)k = \mathrm{id} = F(\mathrm{id})$ so since F is faithful, $fg = \mathrm{id}$. Similarly, $gf = \mathrm{id}$ and g is aninverse for f. So F reflects isomorphisms.

Exercise 2. For each of the functors given below, determine whether it preserves all limits, and whether it preserves all colimits. Give a short argument in each case.

a) The functor $\mathrm{Pos} \to \mathrm{Pos}$ which sends a poset P to its opposite P^{op} (so, $x \leq y$ in P^{op} iff $y \leq x$ in P).

b) The functor $\mathrm{Set} \to \mathrm{Set}$ given by $X \mapsto A + X$, where A is a fixed nonempty set and $+$ denotes disjoint sum.

c) The inclusion functor from $\mathrm{Pos}_\perp$ to Pos; the objects of $\mathrm{Pos}_\perp$ are posets with a bottom element, and its arrows are order-preserving maps which preserve the bottom element.

Solution:a) The functor $(\cdot)^{\mathrm{op}}$ is an isomorphism of categories (it is its own inverse), so it preserves all limits and colimits.

b) This functor preserves neither the terminal object nor the initial object, so it does not preserve all limits or all colimits.
c) $\mathrm{Pos}_\perp$ is the category of algebras for a monad on Pos; the monad which adds a new bottom element to a poset. And the inclusion $\mathrm{Pos}_\perp \to \mathrm{Pos}$ is the forgetful functor. This functor preserves all limits. It does not preserve all colimits; for example it does not preserve the initial object.

Exercise 3. An object M of a category $\mathcal{C}$ is called *injective* if for any diagram

$$\begin{array}{ccc} A & & \\ {\scriptstyle m}\downarrow & \searrow^{f} & \\ B & & M \end{array}$$

with m mono, there is an arrow $g : B \to M$ satisfying $gm = f$.

a) Let $\mathcal{C}, \mathcal{D}$ be categories and $\mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$ functors with $F \dashv G$. Prove: if F preserves monos, then G preserves injective objects.

b) Formulate the dual statement to part a).

Solution: a) Let M be injective in $\mathcal{C}$. In order to prove that $G(M)$ is injective in $\mathcal{D}$, consider a diagram

$$\begin{array}{ccc} A & & \\ {\scriptstyle m}\downarrow & \searrow^{f} & \\ B & & G(M) \end{array}$$

in $\mathcal{D}$ with m mono. This transposes over the adjunction $F \dashv G$ to a diagram

$$\begin{array}{ccc} F(A) & & \\ {\scriptstyle F(m)}\downarrow & \searrow^{\tilde{f}} & \\ F(B) & & M \end{array}$$

for which, by injectivity of M (and the fact that $F(m)$ is mono), there is an arrow $F(B) \to M$ making the triangle commute. The transpose of this arrow: $B \to G(M)$ then does it for the original diagram.
b) Call an object P *projective* if, given a diagram

$$\begin{array}{ccc} P & & A \\ & \searrow & \downarrow{\scriptstyle e} \\ & & B \end{array}$$

with e epi, there is an arrow $P \to A$ making the triangle commute. Then the following holds: if $\mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$ are functors with $F \dashv G$ and G preserves epis, then F preserves projectives.

Exercise 4. In this exercise we work again with the notion of 'injective object' defined in Exercise 3. Let $\mathcal{E}$ be a topos with subobject classifier $1 \xrightarrow{t} \Omega$.

a) Prove that Ω is injective.

b) Prove that every object of the form Ω^X is injective.

c) Conclude that for every object X there is a monomorphism from X into an injective object (One says: "$\mathcal{E}$ has enough injectives").

Solution: a) Consider a diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & \Omega \\ {\scriptstyle m}\downarrow & & \\ B & & \end{array}$$

with m mono. Let $A' \xrightarrow{n} A$ be a mono which represents the subobject of A classified by ϕ. Let $\chi : B \to \Omega$ be the classifying map of the subobject $A' \xrightarrow{mn} B$. We claim that $\chi m = \phi$. Indeed, we have a diagram of pullbacks

$$\begin{array}{ccccc} A' & \xrightarrow{\text{id}} & A' & \longrightarrow & 1 \\ {\scriptstyle n}\downarrow & & {\scriptstyle mn}\downarrow & & \downarrow{\scriptstyle t} \\ A & \xrightarrow[m]{} & B & \xrightarrow[\chi]{} & \Omega \end{array}$$

which show that the subobject of A classified by χm is n, which is also the subobject classified by ϕ. So $\chi m = \phi$.
b) A topos is cartesian closed so we have $(-) \times X \dashv (-)^X$. So the result follows from Exercise 3a) if we know that the functor $(-) \times X$ preserves monos. Indeed, because

$$\begin{array}{ccc} A \times X & \xrightarrow{m \times \text{id}} & B \times X \\ {\scriptstyle \pi_A}\downarrow & & \downarrow{\scriptstyle \pi_B} \\ A & \xrightarrow[m]{} & B \end{array}$$

is a pullback, m mono implies $m \times \mathrm{id}$ mono.
c) Let $\delta : X \times X \to \Omega$ classify the subobject $X \to X \times X$, and let $\tilde{\delta} : X \to \Omega^X$ be its exponential transpose. We claim that $\tilde{\delta}$ is mono. Suppose $g, h : Y \to X$ satisfy $\tilde{\delta} g = \tilde{\delta} h$. Transposing, we get that $\delta(g \times \mathrm{id}_X) = \delta(h \times \mathrm{id}_X) : Y \times X \to \Omega$.

Now $\delta(g \times \mathrm{id}_X)$ classifies the subobject represented by $\langle \mathrm{id}_Y, g \rangle : Y \to Y \times X$, i.e. the graph of g. And similarly, $\delta(h \times \mathrm{id}_X)$ classifies the graph of h. If these graphs coincide, then g must be equal to h, as you can ckeck for yourself.

Exercise 5. Let $\mathcal{C}$ be a small category; we work in the category $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ of presheaves on $\mathcal{C}$. Let P be such a presheaf. We define a presheaf $\tilde{P}$ as follows: for an object C of $\mathcal{C}$, $\tilde{P}(C)$ consists of those subobjects α of $y_C \times P$ which satisfy the following condition: for all arrows $f : D \to C$, the set

$$\{y \in P(D) \,|\, (f, y) \in \alpha(D)\}$$

has at most one element.

a) Complete the definition of $\tilde{P}$ as a presheaf.

b) Show that there is a monic map $\eta_P : P \to \tilde{P}$ with the following property: for every diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & P \\ {\scriptstyle m}\big\downarrow & & \\ B & & \end{array}$$

with m mono, there is a unique map $\tilde{g} : B \to \tilde{P}$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & P \\ {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle \eta_P} \\ B & \xrightarrow[\tilde{g}]{} & \tilde{P} \end{array}$$

is a pullback square. The arrow $P \stackrel{\eta_P}{\to} \tilde{P}$ is called a "partial map classifier" for P.

c) Show that the assignment $P \mapsto \tilde{P}$ is part of a functor $\tilde{(\cdot)}$ in such a way that the maps η_P form a natural transformation from the identity functor to $\tilde{(\cdot)}$, and all naturality squares for η are pullbacks.

Solution: a) If $f : D \to C$ is an arrow in $\mathcal{C}$ and $\alpha \in \tilde{P}(C)$ so α is a subobject of $y_C \times P$, let $\tilde{P}(f)(\alpha)$ be the subobject of $y_D \times P$ given by the pullback diagram

$$\begin{array}{ccc} \tilde{P}(f)(\alpha) & \longrightarrow & y_D \times P \\ \downarrow & & \downarrow{\scriptstyle y_f \times \mathrm{id}_P} \\ \alpha & \longrightarrow & y_C \times P \end{array}$$

So, $\tilde{P}(f)(\alpha)(E)$ is the set

$$\{(g, y) \mid g : E \to D, y \in P(E), (fg, y) \in \alpha(E)\}$$

from which it is clear that if $\alpha \in \tilde{P}(C)$ then $\tilde{P}(f)(\alpha) \in \tilde{P}(D)$. Moreover, this is clearly a contravariant functor $\mathcal{C} \to \mathrm{Set}$.

b) Define η_P as follows: for each object C of $\mathcal{C}$ and $y \in P(C)$, $(\eta_P)_C(y)$ is the subobject α of $y_C \times P$ given by

$$\alpha(D) \ = \ \{(f, z) \in y_C(D) \times P(D) \mid z = P(f)(y)\}$$

Convince yourself that this is natural, well defined (i.e., $(\eta_P)_C(y) \in \tilde{P}(C)$), and monic.

We have to show that $\tilde{g}$ is determined uniquely by the requirement that

$$\begin{array}{ccc} A & \xrightarrow{g} & P \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \eta_P} \\ B & \xrightarrow[\tilde{g}]{} & \tilde{P} \end{array}$$

is a pullback square. First of all, the requirement means that the square must commute, so if $y \in B(C)$ and $y = m_C(x)$ for some (necessarily unique) $x \in A(C)$, then $\tilde{g}_C(y)$ must be equal to $(\eta_P)_C(g(x))$.

Moreover, for $y \in B(C)$ and $f : D \to C$, we must have that $\tilde{g}_C(y)(D)$ contains an element precisely when $A(f)(y)$ is in the image of the map m_D. This together means that we have no choice but to put

$$\tilde{g}_C(b)(D) \ = \ \{(f, g_D(x)) \mid x \in A(D), m_D(x) = B(f)(y)\}$$

c) Given $\phi : P \to Q$ we define $\tilde{\phi} : \tilde{P} \to \tilde{Q}$ by demanding that

$$\begin{array}{ccc} P & \xrightarrow{\phi} & Q \\ {\scriptstyle \eta_P}\downarrow & & \downarrow{\scriptstyle \eta_Q} \\ \tilde{P} & \xrightarrow[\tilde{\phi}]{} & \tilde{Q} \end{array}$$

be a pullback. Since η_P is mono, such $\tilde{\phi}$ exists uniquely by part b). And the uniqueness implies that the assignment $\phi \mapsto \tilde{\phi}$ is functorial.