Exam Category Theory and Topos Theory
June 18, 2018; 10:00–13:00
With solutions

Exercise 1.Let $\mathcal{C}$ be a locally small category. For an object X of $\mathcal{C}$ we define the *representable* functor $R_X : \mathcal{C} \to \mathrm{Set}$ by

$$R_X(A) \;=\; \mathcal{C}(X, A)$$

(and on arrows by composition)

a) (3 pts) Prove that the functor R_X preserves monomorphisms.

b) (4 pts) Assume that the category $\mathcal{C}$ has all small coproducts. Show that R_X has a left adjoint.

c) (3 pts) Suppose $F : \mathcal{C} \to \mathrm{Set}$ is a functor and $\mu : R_X \Rightarrow F$ a natural transformation. Show that μ is completely determined by the element $\mu_X(\mathrm{id}_X)$ of $F(X)$.

Solution:

a) Suppose $f : A \to B$ is mono in $\mathcal{C}$; we have to prove that $R_X(f) : \mathcal{C}(X, A) \to \mathcal{C}(X, B)$ is injective. So suppose $g_1, g_2 : X \to A$ are elements of $\mathcal{C}(X, A)$ such that $R_X(f)(g_1) = R_X(f)(g_2)$. That means: $fg_1 = fg_2$. Since f is mono, $g_1 = g_2$. So indeed, $R_X(f)$ is injective, that is: mono in Set.

b) For a set B, define $L_X(B)$ to be the coproduct of B many copies of X in $\mathcal{C}$, i.e. $\coprod_{b\in B} X$. We have

$$\mathcal{C}(L_X(B), A) \simeq \mathcal{C}(\coprod_{b\in B} X, A) \simeq \prod_{b\in B} \mathcal{C}(X, A) \simeq \mathrm{Set}(B, \mathcal{C}(X, A))$$

which isomorphisms are all natural; so this establishes the adjunction.

c) This is just the Yoneda lemma.

Exercise 2. Let $\mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$ be an adjunction, with $F \dashv G$. We assume furthermore that $\mathcal{C}$ and $\mathcal{D}$ are regular categories, that the counit ε of the adjunction is split mono, and that the functor G preserves regular epimorphisms.

Let $G(X) \xrightarrow{e} Y \xrightarrow{m} G(X')$ be a diagram in $\mathcal{D}$, with e regular epi and m mono. Show that Y is isomorphic to an object in the image of the functor G.

Solution: The assumption that ε is split mono implies (in fact, is equivalent to) the statement that G is full. To prove this: assume $f : G(X) \to G(X')$ is an arrow in $\mathcal{D}$. Let $r_X : X \to FG(X)$ be a retraction for ε_X. Let $\tilde{f} : FG(X) \to X$ be the transpose of f under the adjunction $F \dashv G$. Note, that $\tilde{f}$ is equal to the composite

$$FG(X) \xrightarrow{F(f)} FG(X') \xrightarrow{\varepsilon_{X'}} X'$$

We consider the arrow $g = \tilde{f} r_X : X \to X'$ in $\mathcal{C}$. The transpose of $G(g)$ is the map

$$FG(X) \xrightarrow{FG(r_X)} FGFG(X) \xrightarrow{FG(\tilde{f})} FG(X') \xrightarrow{\varepsilon_{X'}} X'$$

which, by naturality of ε, is equal to the composite $\tilde{f} r_X \varepsilon_X$, which is equal to $\tilde{f}$. Since $G(g)$ and f have the same transpose, they are equal. We conclude that G is full.

Since G is full, choose $h : X \to X'$ in $\mathcal{C}$ such that $G(h) = me$. Let, by regularity of $\mathcal{C}$, $X \overset{e'}{\to} Z \overset{m'}{\to} X'$ be a regular epi-mono factorization of h. Now G preserves regular epis by assumption and monos because it is a right adjoint (any limit-preserving functor preserves monos); therefore we have a regular epi-mono factorization

$$G(X) \overset{G(e')}{\to} G(Z) \overset{G(m')}{\to} G(X')$$

of $G(h) = me$. By uniqueness of regular epi-mono factorizations in $\mathcal{D}$, we have that $Y \simeq G(Z)$, as desired.

Exercise 3. Let $\mathcal{C}$ be a cartesian closed category and $h : A \to B$ an epimorphism in $\mathcal{C}$. Prove that for every object X of $\mathcal{C}$, the arrow $X^h : X^B \to X^A$ is mono.

Solution: Suppose $a, b : W \to X^B$ is a parallel pair satisfying $X^h a = X^h b$. To prove: $a = b$. We look at the following commutative diagram:

$$W \times A \xrightarrow{a \times \mathrm{id}_A} X^B \times A \xrightarrow{\mathrm{id}_{X^B} \times h} X^B \times B \xrightarrow{\mathrm{ev}} X$$

$$W \times A \xrightarrow{\mathrm{id}_W \times h} W \times B \xrightarrow{a \times \mathrm{id}_B} X^B \times B$$

from which we learn that the transpose of $X^h a$ is the composite

$$\mathrm{ev}(a \times \mathrm{id}_B)(\mathrm{id}_W \times h).$$

Since the transposes of $X^h a$ and $X^h b$ are assumed equal, we see that

$$\mathrm{ev}(a \times \mathrm{id}_B)(\mathrm{id}_W \times h) \;=\; \mathrm{ev}(b \times \mathrm{id}_B)(\mathrm{id}_W \times h)$$

Now the arrow $\mathrm{id}_W \times h$ is epi, because the functor $W \times (-)$, having a right adjoint, preserves epis. Therefore, we get that $\mathrm{ev}(a \times \mathrm{id}_B) = \mathrm{ev}(b \times \mathrm{id}_B)$; that is, the transposes of a and b are equal. It follows that $a = b$, as was to be proved.

A slicker proof is available. Given X, let $G : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ be the functor $X^{(-)}$. Let $\bar{G} : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ be the opposite functor. Since there are natural isomorphisms

$$\mathcal{C}(Y, \bar{G}(W)) \simeq \mathcal{C}(Y, X^W) \simeq \mathcal{C}(W, X^Y) \simeq \mathcal{C}^{\mathrm{op}}(G(Y), W)$$

we see that $G \dashv \bar{G}$. Therefore $\bar{G}$ preserves monos and since h is epi in $\mathcal{C}$ hence mono in $\mathcal{C}^{\mathrm{op}}$, $X^h = \bar{G}(h)$ is mono in $\mathcal{C}$.

Exercise 4. In a poset $(P, \leq)$, a subset $U \subseteq P$ is called *downwards closed* if for every $x \in U$ and $y \leq x$ we have $y \in U$. Let $\mathcal{D}(P)$ be the set of all downwards closed subsets of P, ordered by inclusion.

a) (4 pts) Show that the operation $\mathcal{D}$ has the structure of a monad on Pos, with unit $\eta_P : P \to \mathcal{D}(P)$ which sends $x \in P$ to ${\downarrow}x = \{y \in P \,|\, y \leq x\} \in \mathcal{D}(P)$, and union as multiplication.

b) (4 pts) Suppose $h : \mathcal{D}(P) \to P$ is a $\mathcal{D}$-algebra. Show that h is left adjoint to the unit $\eta_P : P \to \mathcal{D}(P)$, both considered as maps between posets. Conclude that any poset P has at most one $\mathcal{D}$-algebra structure.

c) (2 pts + 1 bonus point) Characterize the posets P which have a $\mathcal{D}$-algebra structure.

Solution:

a) First, we should define $\mathcal{D}$ as a functor. On morphisms $f : P \to Q$, define for a downwards closed subset U of P, its image under $\mathcal{D}(f)$ as the downwards closure of $\{f(x) \,|\, x \in U\}$, i.e. the set

$$\{y \in Q \,|\, \text{for some } x \in U,\, y \leq f(x)\}$$

since simply the pointwise image of U under f fails to be downwards closed in general. One easily checks that with this definition of $\mathcal{D}(f)$, we have a functor. That $\mathcal{D}$ is a monad, is very similar to the proof for the covariant powerset monad; I skip it here.

b) Suppose $h : \mathcal{D}(P) \to P$ is a $\mathcal{D}$-algebra. Since h is order-preserving we see that for $x \in U \in \mathcal{D}(P)$ we have $\downarrow x \subseteq U$, hence $x = h(\downarrow x) \leq h(U)$, so $U \subseteq \downarrow(h(U))$. From this we see that $h(U) \leq x$ implies $\downarrow(h(U)) \subseteq \downarrow x$ so $U \subseteq \downarrow x$; conversely if $U \subseteq \downarrow x$ then $h(U) \leq h(\downarrow x) = x$. We conclude that $h(U) \leq x$ if and only if $U \subseteq \downarrow x = \eta_P(x)$; so h is left adjoint to the unit. We see that up to isomorphism, there can be at most one $\mathcal{D}$-algebra structure on P. But in a poset, isomorphism means equality. So there is at most one algebra structure.

c) By the adjunction shown in part b), we see that $U \subseteq \eta_P(h(U)) = \downarrow h(U)$, so $h(U)$ is an upper bound for U, and it is the least upper bound. Since in a poset, any subset X and its downwards closure have the same upper bounds, we see that a poset P has a $\mathcal{D}$-algebra structure if and only if every subset of P has a least upper bound.

Exercise 5. Recall that in any category, an object M is called *injective* if every diagram

$$\begin{array}{ccc} B & & \\ {\scriptstyle m}\uparrow & & \\ A & \xrightarrow[f]{} & M \end{array}$$

with m mono, can be completed to a commutative diagram

$$\begin{array}{ccc} B & & \\ {\scriptstyle m}\uparrow & \overset{g}{\searrow} & \\ A & \xrightarrow[f]{} & M. \end{array}$$

Recall also that for objects X and Y, X is called a *retract* of Y if there is a diagram $X \xrightarrow{i} Y \xrightarrow{r} X$ such that $ri = \mathrm{id}_X$.

a) (3 pts) Suppose $\mathcal{E}$ is a topos with subobject classifier $1 \xrightarrow{t} \Omega$. Show that Ω is injective.

b) (2 pts) Show that in any cartesian closed category the following holds: if M is injective, then M^X is injective, for any object X.

c) (3 pts) Show that in a topos, every object X admits a mono $X \to \Omega^X$.

d) (2 pts) Prove that in a topos, an object is injective if and only if it is a retract of an object of the form Ω^Y.

Solution:

a) Given a mono $m : A \to B$ and a map $f : A \to \Omega$, let $n : A' \to A$ represent the subobject of A classified by f. Now $mn : A' \to B$ is mono; let $g : B \to \Omega$ classify this mono. In the following diagram:

$$\begin{array}{ccccc} A & \xrightarrow{m} & B & \xrightarrow{g} & \Omega \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle m} & & \\ A & \xrightarrow{\mathrm{id}} & A & & \uparrow{\scriptstyle t} \\ {\scriptstyle n}\uparrow & & \uparrow{\scriptstyle n} & & \\ A' & \xrightarrow[\mathrm{id}]{} & A' & \longrightarrow & 1 \end{array}$$

every square is a pullback. So the whole square is a pullback; hence the arrow gm classifies $n : A' \to A$, which by assumption was classified by f. Therefore $f = gm$, and Ω is injective.

b) Given a mono $m : A \to B$ and a map $f : A \to M^X$, we consider the transpose $\tilde{f} : A \times X \to M$ and the mono $m \times \mathrm{id}_X : A \times X \to B \times X$. By injectivity of M we obtain an arrow $\tilde{g} : B \times X \to M$ making the triangle

$$\begin{array}{ccc} B \times X & & \\ {\scriptstyle m \times \mathrm{id}_X}\uparrow & \searrow^{\tilde{g}} & \\ A \times X & \xrightarrow[\tilde{f}]{} & M \end{array}$$

commute. Taking the transpose of this diagram gives a map $g : B \to M^X$ such that $gm = f$, and M^X is injective.

c) Consider the subobject X of $X \times X$ via the diagonal embedding. Let $d : X \times X \to \Omega$ classify this; and let $\{\cdot\} : X \to \Omega^X$ be the exponential transpose of d. We claim that this map is mono. To see this, consider that for an arrow $f : Y \to X$, the composite $Y \xrightarrow{f} X \xrightarrow{\{\cdot\}} \Omega^X$ transposes to the composite

$$Y \times X \xrightarrow{f \times \mathrm{id}_X} X \times X \xrightarrow{d} \Omega$$

which classifies the graph of f as subobject of $Y \times X$. Therefore if $\{\cdot\}$ coequalizes two maps f and g from Y to X then the graphs of f and g are equal, hence $f = g$.

d) For the "only if" part, suppose M is injective. Considering the mono $\{\cdot\} : M \to \Omega^M$ and the identity $M \to M$ we obtain a map $r : \Omega^M \to M$ which is a retraction for $\{\cdot\}$. So M is a retract of Ω^M.

Conversely, first one proves that every retract of an injective object is injective. Then one applies b) and c) to see that every retract of Ω^Y is injective.