Exam Computability Theory

January 10, 2012, 14.00–17.00

This exam consists of 5 exercises; see also the back side
Advice: first do those exercises you can do straight away; then start thinking about the others.
SUCCES!

Exercise 1:
Prove that the following functions are primitive recursive:

a) $\log(x)$, defined as: $\log(0) = 0$, and for $x > 0$, $\log(x)$ is the unique n such that $2^n \leq x < 2^{n+1}$

b) the function f, where $f(x)$ is the number of divisors of x if $x > 0$, and $f(0) = 0$

Exercise 2:
Prove that there is no partial recursive function F with the following properties:

i) if φ_e is total then $F(e)$ is defined

ii) if φ_e is total and for some m, $\varphi_e(m) = 0$ and $\forall n < m\, \varphi_e(n) \neq 1$ then $F(e) = 0$

iii) if φ_e is total and for some m, $\varphi_e(m) = 1$ and $\forall n < m\, \varphi_e(n) \neq 0$ then $F(e) = 1$

[Hint: apply the recursion theorem]

Exercise 3:
By Tot we denote the set $\{e \mid \varphi_e \text{ is total}\}$. For each of the following sets A, determine whether or not A is many-one reducible to Tot. Justify your answers.

a) $A = \{e \mid \varphi_e \text{ is total and not eventually constant}\}$

b) $A = \mathcal{K}$, the standard set

c) $A = \{e \mid W_e \text{ contains infinitely many powers of } 2\}$

Exercise 4:
For the following sets A, classify A in the arithmetical hierarchy. That is: find a smallest possible n and state that $A \in \Pi_n$, $A \in \Sigma_n$ or $A \in \Delta_n$; and prove that this is so. You are *not* required to prove that your classification is optimal.

a) $A = \begin{array}{l}\{e \mid \text{ there are } x_1 < x_2 < \cdots < x_e \in \mathrm{dom}(\varphi_e) \\ \qquad\qquad \text{such that } \varphi_e(x_1) < \cdots < \varphi_e(x_e)\}\end{array}$

b) $A = \{e \mid \varphi_e \text{ is total and strictly increasing}\}$

c) $A = \begin{array}{l}\{e \mid \text{for every } x,\ \mathrm{dom}(\varphi_e) \cap \{x, x+1\} \\ \qquad\qquad \text{contains exactly one element}\}\end{array}$

Exercise 5:
Let G be the Gödel sentence for PA; we have PA $\nvdash G$ and PA $\nvdash \neg G$. Let Prov(x) be the formula (in the language of PA) which expresses that there is a formal proof in PA of the formula with Gödel number x.

a) Show that there is a sentence ϕ in the language of PA such that

$$\mathrm{PA} \vdash \phi \leftrightarrow \neg\mathrm{Prov}(\overline{\ulcorner G \rightarrow \phi \urcorner})$$

b) Let ϕ be as in a). Show that $\mathrm{PA} + G \nvdash \phi$.