Exam Computability Theory
January 10, 2012, 14.00–17.00

This exam consists of 5 exercises; see also the back side
Advice: first do those exercises you can do straight away; then start thinking about the others.
Succes!

Exercise 1:
Prove that the following functions are primitive recursive:

a) \( \log(x) \), defined as: \( \log(0) = 0 \), and for \( x > 0 \), \( \log(x) \) is the unique \( n \) such that \( 2^n \leq x < 2^{n+1} \)

b) the function \( f \), where \( f(x) \) is the number of divisors of \( x \) if \( x > 0 \), and \( f(0) = 0 \)

Exercise 2:
Prove that there is no partial recursive function \( F \) with the following properties:

i) if \( \varphi_e \) is total then \( F(e) \) is defined

ii) if \( \varphi_e \) is total and for some \( m \), \( \varphi_e(m) = 0 \) and \( \forall n < m \varphi_e(n) \neq 1 \) then \( F(e) = 0 \)

iii) if \( \varphi_e \) is total and for some \( m \), \( \varphi_e(m) = 1 \) and \( \forall n < m \varphi_e(n) \neq 0 \) then \( F(e) = 1 \)

[Hint: apply the recursion theorem]

Exercise 3:
By \( \text{Tot} \) we denote the set \( \{e | \varphi_e \text{ is total}\} \). For each of the following sets \( A \), determine whether or not \( A \) is many-one reducible to \( \text{Tot} \). Justify your answers.

a) \( A = \{e | \varphi_e \text{ is total and not eventually constant}\} \)

b) \( A = K \), the standard set

c) \( A = \{e | W_e \text{ contains infinitely many powers of 2}\} \)
Exercise 4:
For the following sets $A$, classify $A$ in the arithmetical hierarchy. That is: find a smallest possible $n$ and state that $A \in \Pi_n$, $A \in \Sigma_n$ or $A \in \Delta_n$; and prove that this is so. You are not required to prove that your classification is optimal.

a) $A = \{ e | \text{there are } x_1 < x_2 < \cdots < x_e \in \text{dom}(\varphi_e) \text{ such that } \varphi_e(x_1) < \cdots < \varphi_e(x_e) \}$

b) $A = \{ e | \varphi_e \text{ is total and strictly increasing} \}$

c) $A = \{ e | \text{for every } x, \text{ dom}(\varphi_e) \cap \{ x, x + 1 \} \text{ contains exactly one element} \}$

Exercise 5:
Let $G$ be the Gödel sentence for PA; we have $PA \not\vdash G$ and $PA \not\vdash \neg G$. Let $Prov(x)$ be the formula (in the language of PA) which expresses that there is a formal proof in PA of the formula with Gödel number $x$.

a) Show that there is a sentence $\phi$ in the language of PA such that

$$PA \vdash \phi \iff \neg Prov(\overline{\neg G \rightarrow \phi})$$

b) Let $\phi$ be as in a). Show that $PA + G \not\vdash \phi$. 