Retake Exam Gödel's Incompleteness Theorems

June 25, 2015, 10.00–13.00
With Solutions

THIS EXAM CONSISTS OF 4 PROBLEMS; SEE ALSO BACK SIDE
Advice: first do those problems you can do right away; then, start thinking about the others.
Please write your name, student number and e-mail address clearly *on the sheets you hand in*
Good luck!

Exercise 1. Let $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ be the two functions which are uniquely determined by the equation

$$n = 2^{f(n)}(2g(n) + 1) - 1$$

for all $n \in \mathbb{N}$.
Show that the functions f and g are primitive recursive.

Solution: The function $n \mapsto 2^n$ is primitive recursive, as well as the relation $x|y$. Also, the primitive recursive functions are closed under bounded minimisation. So the function f, which can be defined as

$$f(n) \;=\; \mu z \leq n.2^{z+1} \not| (n+1)$$

is primitive recursive. Then the function g, which can be defined as

$$g(n) \;=\; \mu z \leq n.zf(n) = n + 1$$

is also primitive recursive.

Exercise 2.

a) Let ϕ be an $\mathcal{L}_{\text{PA}}$-sentence which is true in all *nonstandard* models of PA. Prove that PA $\vdash \phi$.

b) Let $\mathcal{M}$ be a nonstandard model of PA, and let ϕ be an $\mathcal{L}_{\text{PA}}(\mathcal{M})$-sentence (so a sentence with constants from the model $\mathcal{M}$) which is true in every proper end-extension of $\mathcal{M}$. Prove that $\mathcal{M} \models \phi$.

Solution: This exercise is basically about the concept of *elementary extension.*

a) We only need to show that $\mathcal{N} \models \phi$, where $\mathcal{N}$ denotes the standard model. For then, we know that every model of PA satisfies ϕ, whence $\mathrm{PA} \vdash \phi$ by the Completeness Theorem for first-order logic.

By considering the $\mathcal{L}_{\mathrm{PA}} \cup \{c\}$-theory $\{\phi \,|\, \mathcal{N} \models \phi\} \cup \{c > \overline{n} \,|\, n \in \mathbb{N}\}$, which is consistent by the Compactness Theorem, we see that $\mathcal{N}$ has a proper elementary extension, which satisfies ϕ because it is a nonstandard model. By elementariness, $\mathcal{N} \models \phi$, as desired.

b) Here we use the McDowell-Specker Theorem, which says that $\mathcal{M}$ has a proper elementary end-extension. This extension satisfies ϕ by assumption; hence by elementariness, $\mathcal{M} \models \phi$.

Exercise 3. Recall that the notation $\Box\phi$ stands for $\exists x \overline{\mathrm{Prf}}(x, \overline{\ulcorner \phi \urcorner})$ and that for $\Box$ the following three "derivability conditions" hold:

- D1 $\mathrm{PA} \vdash \phi$ implies $\mathrm{PA} \vdash \Box\phi$
- D2 $\mathrm{PA} \vdash \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$
- D3 $\mathrm{PA} \vdash \Box\phi \to \Box\Box\phi$

You may use without proof, that conditions D1 and D2 imply
$\mathrm{PA} \vdash \Box(\phi \wedge \psi) \leftrightarrow \Box\phi \wedge \Box\psi$.
Let G be the Gödel sentence: $\mathrm{PA} \vdash G \leftrightarrow \neg\Box G$.

a) Prove that there is a sentence ϕ satisfying

$$\mathrm{PA} \vdash \phi \leftrightarrow (G \to \neg\Box\phi)$$

b) For ϕ as in a), prove: if $\mathrm{PA} \vdash \phi$ then $\mathrm{PA} \vdash \Box\bot$.

c) For ϕ as in a), prove: if $\mathrm{PA} \vdash \neg\phi$ then $\mathrm{PA} \vdash \bot$.

Solution: a) Apply the Diagonalisation Lemma to the formula $G \to \neg\exists x \overline{\mathrm{Prf}}(x, v)$.

b) Assume $\mathrm{PA} \vdash \phi$. Then $\mathrm{PA} \vdash G \to \neg\Box\phi$ by choice of ϕ, and also $\mathrm{PA} \vdash \Box\phi$ by the assumption and D1. Hence, $\mathrm{PA} \vdash \neg G$. By Gödel's Second Incompleteness Theorem, $\mathrm{PA} \vdash G \leftrightarrow \neg\Box\bot$. So, $\mathrm{PA} \vdash \Box\bot$.

c) Assume $\mathrm{PA} \vdash \neg\phi$. Then $\mathrm{PA} \vdash \Box\neg\phi$ by D1, and $\mathrm{PA} \vdash G \wedge \Box\phi$ by choice of ϕ and logic. Combining $\mathrm{PA} \vdash \Box\phi$ and $\mathrm{PA} \vdash \Box\neg\phi$ we obtain $\mathrm{PA} \vdash \Box\bot$; and combining this with $\mathrm{PA} \vdash G$, so again $\mathrm{PA} \vdash \neg\Box\bot$ by Gödel's Second, we get $\mathrm{PA} \vdash \bot$.

Exercise 4. For this exercise, I remind you of the *partial truth predicates* for PA: there is a Σ_n-formula $\mathrm{Tr}_n(y, s)$ such that for every Σ_n-formula $\phi(v_0)$ with at most the variable v_0 free, we have

$$\mathrm{PA} \vdash \forall s(\mathrm{Tr}_n(\overline{\ulcorner\phi\urcorner}, s) \leftrightarrow \phi[s/v_0])$$

Let a sequence $\phi_0(v_0), \phi_1(v_0), \ldots$ of Σ_n-formulas in at most the free variable v_0 be given, in such a way that the function $k \mapsto \ulcorner\phi_k(v_0)\urcorner$ is recursive. Let $\mathcal{M}$ be a nonstandard model of PA. Suppose that for each n we have

$$\mathcal{M} \models \exists x(\phi_0(x) \wedge \cdots \wedge \phi_n(x))$$

Show that there is an element a of $\mathcal{M}$ such that $\mathcal{M} \models \phi_n(a)$ for all $n \in \mathbb{N}$.

Solution: The function $k \mapsto \ulcorner\phi_k(v_0)\urcorner$ is recursive, so representable in PA by a formula $F(x, y)$. We have:

(1) $\mathrm{PA} \vdash F(\overline{k}, \overline{\ulcorner\phi_k(v_0)\urcorner})$

(2) $\mathrm{PA} \vdash \exists! y F(\overline{k}, y)$

for all k. Also,

(3) $\mathrm{PA} \vdash \forall s(\mathrm{Tr}_n(\overline{\ulcorner\phi_k(v_0)\urcorner}, s) \leftrightarrow \phi_k(s))$

since ϕ_k is assumed to be a Σ_n-formula. Therefore,

(4) $\mathrm{PA} \vdash \forall x(\phi_k(x) \leftrightarrow \exists u(F(\overline{k}, u) \wedge \mathrm{Tr}_n(u, x)))$

Moreover we know that

(5) $\mathrm{PA} \vdash \forall x(x < \overline{m+1} \leftrightarrow x = \overline{0} \vee \cdots \vee x = \overline{m})$

and therefore we can conclude that

(6) in PA, the formula $\phi_0(x) \wedge \cdots \phi_m(x)$ is equivalent to the formula

$$\forall v < \overline{m+1} \exists u(F(v, u) \wedge \mathrm{Tr}_n(u, x))$$

By the assumption that $\mathcal{M} \models \exists x(\phi_0(x) \wedge \cdots \wedge \phi_m(x))$ for every natural number m, we have

(7) $\mathcal{M} \models \exists x \forall v < \overline{m+1} \exists u(F(v,u) \wedge \mathrm{Tr}_n(u,x))$

Applying Overspill, there is a nonstandard element $c \in \mathcal{M}$ such that

(8) $\mathcal{M} \exists x \forall v < c \exists u(F(v,u) \wedge \mathrm{Tr}_n(u,x))$

Let $a \in \mathcal{M}$ be a witness for (8): $\mathcal{M} \models \forall v < c \exists u(F(v,u) \wedge \mathrm{Tr}_n(u,a))$. Then for every standard m we have

(9) $\mathcal{M} \models \mathrm{Tr}_n(\overline{\ulcorner \phi_m(v_0) \urcorner}, a)$

which, by the defining property of the formula Tr_n, means $\mathcal{M} \models \phi_m(a)$. This is what we needed to prove.