Lecture notes on Cosmology (ns-tp430m)

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Part I: An introduction to the Einstein theory of gravitation

Einstein's theory of gravitation is a geometric theory, in the sense that gravitational forces exerted by masses are mediated by a nontrivial structure of space and time. In particular, in the presence of matter, physical distances between bodies change, and time lapses at a different rate. All information about the effects of a matter distribution on space and time are elegantly encoded in a symmetric metric tensor $g_{\mu\nu}$ with a Lorentzian signature, which means that a local Minkowski metric around the observer has the signature (the signs of the diagonal elements), $\mathrm{sign}[g_{\mu\nu}] = (+,-,-,-)$. (A completely equivalent sign convention, which is often used, is $(-,+,+,+)$.) Once given, the metric tensor completely determines the Lorentzian manifold, which in turn provides a representation of gravitational interactions. In fact, the metric tensor $g_{\mu\nu} = g_{\mu\nu}(\vec{x},t)$ provides complete information on how to measure physical distances and time lapses between space-time points. Furthermore, it fully specifies the dynamics of gravitating bodies, and thus it is of a fundamental importance for the Einstein relativistic theory of gravitation. Before we begin discussing the metric tensor, we shall now briefly consider the metric structure of the special theory of relativity.

1 Special theory of relativity

Special relativity is an important special case of general theory of gravitation, where distances are determined by the Minkowski metric, which can be written in terms of an infinitesimal line element as follows

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \,, \qquad \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv \mathrm{diag}(1,-1,-1,-1)\,. \tag{1}$$

Here and throughout these lectures, we are using Einstein's summation convention over the repeated indices, *e.g.* $\eta_{\mu\nu} dx^\mu dx^\nu \equiv \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} \eta_{\mu\nu} dx^\mu dx^\nu$. Here $x^\mu = (ct, x^i)$ $(i = 1,2,3)$ is a four-vector denoting a coordinate position of a point in space and time.

1.1 Lorentz symmetry

The line element ds denotes an invariant distance between two infinitesimally displaced points in space and time, and it is invariant under Lorentz transformations. A Lorentz transformation is any matrix belonging to real orthogonal matrices in 3+1 dimensional space and time. When taken together, they built up the orthogonal group $O(1,3)$, or equivalently, $SL(2,\mathcal{C})$, where $\mathcal{C}$ denotes the set of complex numbers. In general, Lorentz transformations $\Lambda_{\mu\nu}$ are the 4×4 matrices, which leave invariant the scalar product $A\cdot B$ of two four-vectors A^μ and B^ν, where $A\cdot B \equiv \eta_{\mu\nu}A^\mu B^\nu$.

There are two disjoined classes of Lorentz transformations. Proper Lorentz transformations are the transformations which are by continuous deformations connected with the identity transformation $\delta_{\mu\nu} = \mathrm{diag}(1,1,1,1)$, and whose determinant equals to unity, $\det[\Lambda_{\mu\nu}] = 1$. Improper Lorentz transformations are all other transformations. For example, space inversions and time inversions are examples of improper Lorentz transformations. Any combination of improper Lorentz transformations is also an improper transformation. The condition $\det[\Lambda_{\mu\nu}] = -1$ is a sufficient, but not necessary, condition for a transformation to be improper. For example, $\Lambda_{\mu\nu} = -\delta_{\mu\nu}$ (combination of space and time inversions) is an improper Lorentz transformation with $\det[\Lambda_{\mu\nu}] = 1$.

One can show that the Lorentz group has six generators. Three generators generate rotations, the other three generate boosts. To study the representations of the proper Lorentz group, the following *Ansatz* is useful,

$$\Lambda = \mathrm{e}^{-\vec{\omega}\cdot\vec{S}-\vec{\zeta}\cdot\vec{K}}\,, \tag{2}$$

where $\vec{\omega}$ and $\vec{\zeta}$ are the 3-vectors of rotations and boosts, respectively. The generators of rotations $\vec{S} = (S^i)$ and boosts $\vec{K} = (K^i)$ $(i = 1,2,3)$ satisfy the following commutation relations of the Lorentz algebra,

$$\begin{aligned} \left[S^i, S^j\right] &= \epsilon^{ijl} S^l \\ \left[S^i, K^j\right] &= \epsilon^{ijl} K^l \\ \left[K^i, K^j\right] &= -\epsilon^{ijl} S^l\,, \end{aligned} \tag{3}$$

which constitutes both the algebra of $SL(2,\mathcal{C})$ and of $O(1,3)$. Recall that the commutator of two matrices A and B is defined as $[A,B] = AB - BA$. From Eqs. (3) we see that the order by which we perform two consecutive rotations or boosts matters, since two consequtive rotations or boosts do not in general commute. The symbol ϵ^{ijl} is the totally antisymmetric tensor in 3 space dimensions, defined by $\epsilon^{123} = 1$, $\epsilon^{321} = -1$. The cyclic (even) permutations do not change the value of ϵ^{ijl}. Since ϵ^{ijl} is totally antisymmetric, $\epsilon^{ijl} = 0$ when any pair of indices i, j or l is identical.

The Poincaré group is an inhomogeneous extension of the Lorentz group, and it is obtained by adding space and time translations to the Lorentz group. The Poincaré group has therefore ten generators in total. The four additional generators are associated with space and time translations.

Lorentz transformations can be thought of as linear coordinate transformations,

$$x^\mu \to \tilde{x}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu}\, x^\nu \equiv L^\mu_{\ \nu}\, x^\nu \tag{4}$$

such that $L^\mu_{\ \nu} = \partial\tilde{x}^\mu/\partial x^\nu$. With this we see that the infinitesimal line element ds^2 in Eq. (1) is invariant under a Lorentz transformation, provided

$$\eta_{\mu\nu}\Lambda^\mu_{\ \alpha}\Lambda^\nu_{\ \beta} = \eta_{\alpha\beta}\,. \tag{5}$$

Equivalently $\Lambda_{\mu\alpha}\Lambda^{\alpha\nu} = \eta_\mu^{\ \nu} \equiv \delta_\mu^{\ \nu}$. Thus $\Lambda^{\alpha\nu}$ is the inverse of $\Lambda_{\mu\alpha}$, and it is obtained from $\Lambda_{\mu\alpha}$ simply by raising its indices with the Lorentz metric tensor, $\Lambda^{\alpha\nu} = \eta^{\alpha\beta}\eta^{\nu\mu}\Lambda_{\beta\mu}$. This then implies that $\Lambda^{-1} = \Lambda^T$, which proves the statement that Lorentz transformations belong to the group of orthogonal matrices $O(1,3)$, where 1 and 3 refer to the Lorentzian signature, $(+,-,-,-)$.

1.2 Causal Structure

Next we consider the causal structure of relativistic mechanics, which describes motion of particles whose laws obey Lorentz symmetry.

Two space time points are said to be *light-like* separated if the line element vanishes, $ds^2 = 0$. Geometrically, a space-time can be divided into the regions within the past and future *light-cones*

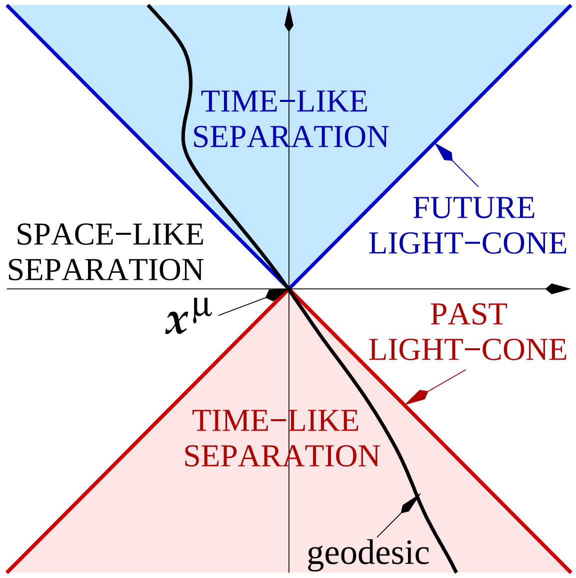

Figure 1: The past and future light-cones in Minkowski space-time separate time-like from space-like distances. Time is on the vertical axis, and space (radial distance) on the horizontal axis. Each point on the diagram corresponds to a two-dimensional sphere S^2 of the spatial section of space-time.

(time-like separations), and the region outside the light-cones (space-like separations). A point on a light cone is light-like separated. This is illustrated on the space-time diagram in Figure 1. Time-like separated points are in general causally connected. They can be connected by a *geodesic*, which is any curve that represents a motion of a point particle, $x^i = x^i(t)$, which is a solution of the equation of motion. The geodesics of massless particles are a collection of light-like separated space-time points.

A point $x^\mu = (ct, x^i)$ lies on the past light-cone of $x_0^\mu = (ct_0, x_0^i)$ if

$$c(t - t_0) = -\|\vec{x} - \vec{x}_0\| \,. \tag{6}$$

Similarly, a point on the future light-cone is determined by

$$c(t - t_0) = \|\vec{x} - \vec{x}_0\| \,. \tag{7}$$

Finally, two points are time-like or space-like separated when

$$c|t - t_0| < \|\vec{x} - \vec{x}_0\| \quad \text{and} \quad c|t - t_0| > \|\vec{x} - \vec{x}_0\| \,, \tag{8}$$

respectively. For any two points on a geodesic, $c|t - t_0| \leq \|\vec{x} - \vec{x}_0\|$. The equality can hold only for massless particles, which reflects the fact that only massless particles (*e.g.* photons) can travel with the speed of light in vacuum.

2 Metric tensor in general relativity

According to general relativity, a space-time reduces to a (locally) Minkowski space-time in the absence of matter, or when matter is sufficiently remote, such that its effects on the metric tensor are unmeasurably small. In addition it is required that no cosmological term is present.

2.1 Newtonian metric tensor

In presence of matter (or when matter is not very distant) physical distances between points in general change. For example, an approximately static distribution of matter, if it is concentrated in a finite region of space $\mathcal{D}$ such that it can be replaced by an equivalent mass $M = \int_{\mathcal{D}} d^3x\rho(\vec{x})$ concentrated at a point $\vec{x}_0 = M^{-1}\int_{\mathcal{D}} d^3x\vec{x}\rho(\vec{x})$ (which we can choose to be at the origin, $\vec{x}_0 = \vec{0}$), sources outside the region $\mathcal{D}$ the following Newton potential at a point $\vec{x}$,

$$\phi_N(\vec{x}) = -G_N\frac{M}{r} \tag{9}$$

where

$$G_N = 6.673(10)\times 10^{-11}\ \frac{\mathrm{m}^3}{\mathrm{kg\,s}^2} \tag{10}$$

is the Newton constant, and $r \equiv \|\vec{x}\,\|$. According to Einstein's theory of gravitation, the physical distances of objects in the gravitational field of this mass distribution are described by the line element,

$$ds^2 = c^2\Big(1+\frac{2\phi_N}{c^2}\Big)dt^2 - \frac{dr^2}{1+2\phi_N/c^2} - r^2 d\Omega^2\,, \tag{11}$$

where $d\Omega^2 = d\theta^2 + \sin^2(\theta)d\varphi^2$ denotes the volume element of the two-dimensional sphere (the sphere in three spatial dimensions), and $\theta \in [0,\pi]$ and $\varphi \in [0, 2\pi]$ are the two angles covering fully the sphere. The general relativistic form of the line element (1) is

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu\,. \tag{12}$$

By comparing (11) and (12) we easily find the metric tensor of a static mass distribution expressed in spherical coordinates (r, θ, φ),

$$g_{\mu\nu} = \begin{pmatrix} 1+2\phi_N/c^2 & 0 & 0 & 0 \\ 0 & -(1+2\phi_N/c^2)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2(\theta) \end{pmatrix}. \tag{13}$$

An important consequence of the change in the physical distance between space-time points is *light bending* around massive bodies, and experimentally confirmed by two expeditions lead by Eddington and Dyson during the solar eclipse on March 29 in 1919. According to the corpuscular theory of light, the Newton theory predicts by a factor two smaller bending angle.

A second important consequence is that photons in a gravitational field gain energy, and hence their frequency is increased according to $\Delta\nu/\nu = -\phi_N/c^2$. This effect was experimentally measured in 1960 by Pound and Rebka. Equivalently, this can be thought of as time dilatation: time passes slower in a system where gravitational fields are stronger, such that the relative time dilatation equals, $\Delta t/t = -\phi_N/c^2$.

2.2 Einstein's equivalence principle

The Einstein's *equivalence principle* states that an observer cannot perform a local experiment, based on which he or she would be able to conclude whether he or she is placed in an accelerating or a gravitating system. The origins of the equivalence principle can be traced back to the equivalence of the inertial and gravitational mass,

$$m_i = m_g\,, \qquad m_i \frac{d^2\vec{x}}{dt^2} = m_g \vec{g} \tag{14}$$

observed first by Galileo, where $\vec{g} = -\nabla\phi_N$ denotes the gravitational field (the gravitational force per unit mass). In a constant gravitational field all bodies are accelerated at an identical rate. More formally, the *weak equivalence principle* ascertains that in any gravitational field, a freely falling observer will not experience any gravitational effects. On the other hand, the *strong equivalence principle* ascertains that all physical laws take the same form in freely falling frames. We will use this to derive the geodesic equation in the next section. Both the weak and strong equivalence principle have been scrutinized by experiments, since any deviation from the equivalence principle would imply that an alternative theory of gravity is a more accurate description of reality than Einstein's theory. So far no violation of the equivalence principle has been observed.

The equivalence principle provides an elegant resolution of the *twin paradox* of special relativity, according to which the twin which travels to a star would find his twin brother more aged, when he returns to the Earth. Indeed, according to the equivalence principle, the time is equally contracted in an accelerated system, as it is in a gravitational field, and the twin in an accelerated ship ages at a slower rate.

2.3 General covariance

From Eqs. (11–13) it follows that the line element is invariant under a *general coordinate transformation* (diffeomorphism),

$$x^\mu \to \tilde{x}^\mu(x)\,, \tag{15}$$

provided ds^2 is invariant, $ds^2 = d\tilde{s}^2$. Now an infinitesimal coordinate transformation

$$d\tilde{x}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\, dx^\alpha\,, \tag{16}$$

and the line element invariance imply that the coordinate transformation (15) induces the following coordinate transformation of the metric tensor,

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu}\, g_{\alpha\beta}(x)\,, \tag{17}$$

while the inverse of the metric tensor transforms as,

$$\tilde{g}^{\mu\nu}(\tilde{x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial \tilde{x}^\nu}{\partial x^\beta}\, g^{\alpha\beta}(x)\,. \tag{18}$$

In general relativity one introduces the notion of covariant vectors A_μ and contravariant vectors A^ν, which are related as $A_\mu = g_{\mu\nu}A^\nu$ [1]. The metric tensor is thus used to lower vector indices. Conversely, the inverse metric tensor, $g^{\mu\nu}$ is used for raise vector indices, $A^\mu = g^{\mu\nu}A_\nu$. The inverse metric tensor $g^{\mu\nu}$ is defined by

$$g^{\mu\rho}g_{\rho\nu} = \delta^\mu_\nu \tag{19}$$

[1] More rigorously, a vector field can be defined in terms of one forms as, $A = A_\mu dx^\mu$. From the transformation law of the one form (16) and the requirement $A = \tilde{A}$, the transformation law for the components of a vector field immediately follows, $\tilde{A}_\mu = (\partial x^\alpha/\partial\tilde{x}^\mu)A_\alpha$.

where $\delta^\mu_{\ \nu} = \mathrm{diag}(1,1,1,1)$ denotes the Kronecker delta. Note that the indices of a tensor are also lowered and raised by the metric tensor and its inverse. For example, we have $T^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}T_{\alpha\beta}$. The metric tensor is a special tensor in that its indices are also raised and lowered by the metric tensor, *cf.* Eq. (19). In particular we have $g^\mu_{\ \nu} = \delta^\mu_{\ \nu}$.

All metrics related by a general coordinate transformation (15) are physically equivalent, and any apparent differences in metrics should not be ascribed to physical effects. In analogy to gauge theories, the effects induced by coordinate transformations (diffeomorphisms) are sometimes called gauge artifacts. Accordingly, any physical observable in a metric theory of gravitation should be invariant under general coordinate transformations (15). This principle of *general covariance*, and the requirement that in the weak field nonrelativistic limit one ought to reproduce the Newton theory, were the main guiding principles that lead Einstein to the discovery of the general theory of relativity.

Just like in special relativity, the causal structure of space-time is determined by light-cones shown in figure 1. The main difference is that the light-cones of general relativity are not represented by straight lines (6–7), but instead they are deformed by the nontrivial structure of the metric tensor.

3 Geodesic equation

Let us now consider a freely falling observer O, who erects a special relativistic coordinate system in its neighbourhood, such that particles move along trajectories $\xi^\mu = \xi^\mu(\tau) = (\xi^0, \xi^i)$ specified by a non-accelerated motion,

$$\frac{d^2\xi^\mu}{ds^2} = 0\,, \tag{20}$$

where the line element $ds = cd\tau$ is proportional to a time variable, such that $ds^2 \equiv c^2 d\tau^2 = \eta_{\mu\nu}d\xi^\mu d\xi^\nu$. Now assume that the motion of O changes in such a way that it can be described by a coordinate transformation,

$$d\xi^\mu = \frac{\partial \xi^\mu}{\partial x^\alpha}\,dx^\alpha\,, \qquad x^\mu = (ct, x^0)\,. \tag{21}$$

This and Eq. (20) then imply that the observer will perceive an accelerated motion of particles governed by the *geodesic equation*,

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}(x)\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0\,, \tag{22}$$

where the new line element is given by Eq. (12), and

$$g_{\mu\nu}(x) \ = \ \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}\eta_{\alpha\beta} \qquad \text{and} \qquad \Gamma^\mu_{\alpha\beta} = \frac{\partial x^\mu}{\partial \xi^\nu}\,\frac{\partial^2\xi^\nu}{\partial x^\alpha\partial x^\beta} \tag{23}$$

denote the metric tensor and the (affine) Levi-Cività connection, respectively. The form for the Levi-Cività connection, also known as the Christoffel symbol, can be inferred by imposing that a covariant derivative (which is defined below in Eq. (33)) of the metric tensor vanishes,

$$\nabla_\lambda g_{\mu\nu} = 0 \rightarrow \frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \Gamma^\alpha_{\lambda\mu}g_{\alpha\nu} + \Gamma^\alpha_{\lambda\nu}g_{\alpha\mu}\,. \tag{24}$$

This relation defines the unique metric compatible connection, also known as the Levi-Cività connection. Now by making use of an appropriate combination of the derivatives of this type, one finds for the Levi-Cività connection,

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}\Big(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}\Big)\,, \tag{25}$$

where $\partial_\alpha g_{\nu\beta} \equiv \partial g_{\nu\beta}/\partial x^\alpha$, etc.

An important consequence of the geodesic equation is that trajectories of particles (including massive particles, as well as massless photons) moving in gravitational fields sourced by a distribution of masses, exhibit an accelerated motion.

More formally, the geodesic equation expresses the conservation of the *covariant derivative* of a velocity 4-vector,

$$\frac{\mathcal{D}u^\mu}{d\tau} \equiv \frac{du^\mu}{d\tau} + \Gamma^{\mu}_{\alpha\beta}(x) u^\alpha u^\beta = 0\,, \tag{26}$$

where $u^\mu = dx^\mu/d\tau$, $d\tau = ds/c$. More generally, a 4-vector field X^μ is covariantly conserved if the covariant derivative $\mathcal{D} = u^\alpha \nabla_\alpha$ with respect to some time parameter τ vanishes, $DX^\mu/d\tau \equiv dX^\mu/d\tau + \Gamma^{\mu}_{\alpha\beta}(x) X^\alpha u^\beta = 0$. One important example of such a vector field is the velocity field u^α, which is covariantly conserved in the absence of external forces, as indicated in Eq. (22). In this spirit, the general relativistic generalisation of Newton's law can be written as

$$m\frac{\mathcal{D}u^\mu}{d\tau} = F^\mu_{\rm ext}\,, \tag{27}$$

where here $F^\mu_{\rm ext}$ denotes a sum over external forces, excluding gravity, and m is particle's mass. For example, for the electromagnetic field, $F^\mu_{\rm ext}$ is the generalized Lorentz force, $F^\mu_{\rm ext} \to F^\mu_{\rm Lorentz} = qF^\mu{}_\rho u^\rho$, where $F^\mu{}_\rho = \nabla_\rho A^\mu - \nabla^\mu A_\rho$, $u^\rho = dx^\rho/d\tau$, ∇^μ the covariant derivative, and q denotes the electric charge.

Let us now define the covariant (vector) derivative, ∇^μ. If for a scalar function, $f = f(x)$, $\nabla^\mu f$ transform as a vector under a general coordinate transformation, then ∇^μ is the covariant derivative. One can easily show that the structure of the covariant derivative acting on a scalar is trivial, $\nabla^\mu f = \partial^\mu f$, by simply showing that $\partial^\mu f$ transforms as a vector under general coordinate transformations. For a vector field A^ν, $\nabla^\mu A^\nu$ ought to transform as a two-indexed tensor, and similarly for tensor fields. To be more concrete, note first that a derivative of a vector field transforms as,

$$\tilde{A}_{\mu,\nu} = \frac{\partial x^\alpha}{\partial\tilde{x}^\mu}\frac{\partial x^\beta}{\partial\tilde{x}^\nu} A_{\alpha,\beta} + \frac{\partial^2 x^\alpha}{\partial\tilde{x}^\mu\partial\tilde{x}^\nu} A_\alpha\,, \tag{28}$$

and from (23) it follows that the connection transforms noncovariantly,

$$\tilde{\Gamma}^\rho_{\mu\nu} = \Gamma^\gamma_{\alpha\beta}\frac{\partial x^\alpha}{\partial\tilde{x}^\mu}\frac{\partial x^\beta}{\partial\tilde{x}^\nu}\frac{\partial\tilde{x}^\rho}{\partial x^\gamma} + \frac{\partial^2 x^\alpha}{\partial\tilde{x}^\mu\partial\tilde{x}^\nu}\frac{\partial\tilde{x}^\rho}{\partial x^\alpha} \tag{29}$$

Taking these two transformation laws together, we easily find,

$$\tilde{\Gamma}^\rho_{\mu\nu}\tilde{A}_\rho = \Gamma^\gamma_{\alpha\beta}A_\gamma\frac{\partial x^\alpha}{\partial\tilde{x}^\mu}\frac{\partial x^\beta}{\partial\tilde{x}^\nu} + \frac{\partial^2 x^\alpha}{\partial\tilde{x}^\mu\partial\tilde{x}^\nu}A_\alpha \tag{30}$$

Upon subtracting this from (28), we find

$$\tilde{A}_{\mu,\nu} - \tilde{\Gamma}^\rho_{\mu\nu}\tilde{A}_\rho = \Big(A_{\alpha,\beta} - \Gamma^\gamma_{\alpha\beta}A_\gamma\Big)\frac{\partial x^\alpha}{\partial\tilde{x}^\mu}\frac{\partial x^\beta}{\partial\tilde{x}^\nu}\,. \tag{31}$$

We have thus reached the conclusion that the following quantity transforms as a tensor,

$$\nabla_\mu A_\nu \equiv A_{\nu;\mu} = A_{\nu,\mu} - \Gamma^\rho_{\mu\nu} A_\rho \,, \tag{32}$$

which defines the *covariant derivative* of a vector field. Similarly, the covariant derivative of a two-indexed tensor reads

$$\nabla_\rho T_{\mu\nu} \equiv T_{\mu\nu;\rho} = T_{\mu\nu,\rho} - \Gamma^\alpha_{\mu\rho} T_{\alpha\nu} - \Gamma^\alpha_{\nu\rho} T_{\mu\alpha} \,, \tag{33}$$

where we used a rather standard notation, according to which a semicolon (;) denotes a covariant derivative ($_{;\beta} \equiv \nabla_\beta$), and a colon (,) denotes an ordinary derivative ($_{,\beta} \equiv \partial_\beta$).

4 Geodesic deviation

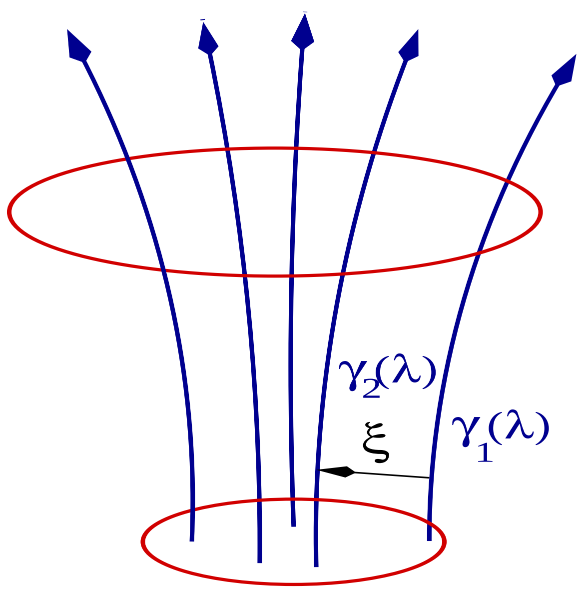

Figure 2: The neighboring geodesics $\gamma_1(s)$ and $\gamma_2(s)$ used in the derivation of the equation of geodesic deviation. The physical distance between the geodesics is denoted by the vector field $\xi^\mu(s)$.

We shall now show how one obtains an equation which controls the rate of change of the physical separation of neighboring geodesics of test particles. To that aim consider two geodesic paths $\gamma_1(\lambda)$ and $\gamma_2(\lambda)$ traced by nearby test particles, with the coordinate vectors, $x^\mu(\lambda)$ and $x^\mu(\lambda) + \xi^\mu(\lambda)$, as shown in figure 2,

$$\begin{aligned} \frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}(x) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} &= 0 \\ \frac{d^2(x^\mu + \xi^\mu)}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}(x+\xi) \frac{d(x^\alpha + \xi^\alpha)}{d\lambda} \frac{d(x^\beta + \xi^\beta)}{d\lambda} &= 0 \,. \end{aligned} \tag{34}$$

Upon subtracting these two equations, we get to first order the physical distance ξ^α between the two geodesics, which is assumed to be a small parameter,

$$\frac{d^2 \xi^\mu}{d\lambda^2} + (\partial_\nu \Gamma^\mu_{\alpha\beta}) \xi^\nu \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} + 2\Gamma^\mu_{\alpha\beta} \frac{d\xi^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0 \,. \tag{35}$$

On the other hand, after some effort a second covariant derivative of the vector ξ^μ can be written as,

$$\begin{aligned}\frac{\mathcal{D}^2\xi^\mu}{d\lambda^2} &= \frac{d^2\xi^\mu}{d\lambda^2} + (\partial_\nu\Gamma^\mu_{\alpha\beta})\xi^\nu u^\alpha u^\beta + 2\Gamma^\mu_{\alpha\beta}\frac{d\xi^\alpha}{d\lambda}u^\beta \\ &+ (\partial_\nu\Gamma^\mu_{\alpha\beta})\xi^\alpha u^\beta u^\nu - (\partial_\alpha\Gamma^\mu_{\beta\nu})\xi^\alpha u^\beta u^\nu - \Gamma^\mu_{\alpha\beta}\xi^\alpha\Gamma^\beta_{\rho\sigma}u^\rho u^\sigma + \Gamma^\mu_{\alpha\beta}\Gamma^\beta_{\rho\sigma}\xi^\rho u^\sigma u^\alpha\,, \end{aligned} \tag{36}$$

where $u^\mu = dx^\mu/d\lambda$. By comparing this with Eq. (35) we immediately see that (35) can be written in a covariant form,

$$\frac{\mathcal{D}^2\xi^\mu}{d\lambda^2} = \xi^\alpha u^\beta u^\gamma\left[\partial_\gamma\Gamma^\mu_{\alpha\beta} - \partial_\alpha\Gamma^\mu_{\beta\gamma} + \Gamma^\mu_{\gamma\nu}\Gamma^\nu_{\alpha\beta} - \Gamma^\mu_{\alpha\nu}\Gamma^\nu_{\beta\gamma}\right]. \tag{37}$$

The expression in the square brackets defines the *Riemann curvature tensor* $\mathcal{R}^\mu{}_{\beta\gamma\alpha}$. With that in mind, the equation of *geodesic deviation* can be recast to the simple form,

$$\frac{\mathcal{D}^2\xi^\mu}{d\lambda^2} = \mathcal{R}^\mu{}_{\alpha\beta\gamma}u^\alpha u^\beta\xi^\gamma\,, \tag{38}$$

with the Riemann curvature tensor

$$\mathcal{R}^\mu{}_{\alpha\beta\gamma} = \partial_\beta\Gamma^\mu_{\alpha\gamma} - \partial_\gamma\Gamma^\mu_{\alpha\beta} + \Gamma^\mu_{\nu\beta}\Gamma^\nu_{\gamma\alpha} - \Gamma^\mu_{\nu\gamma}\Gamma^\nu_{\beta\alpha}\,. \tag{39}$$

Equation (38) may be used as the definition of the Riemann curvature tensor. Alternatively, it may be defined in terms of the double covariant derivative acting on a covariant vector field $A_\alpha = g_{\alpha\nu}A^\nu$ as follows. The covariant derivative ∇_β acts on A_α as indicated in Eq. (32). A second covariant derivative acts then on $A_{\mu;\nu}$ as on a two-indexed covariant tensor field $B_{\mu\nu}$ (*cf.* Eq. (148)),

$$B_{\alpha\beta;\gamma} = B_{\alpha\beta,\gamma} - \Gamma^\mu_{\alpha\gamma}B_{\mu\beta} - \Gamma^\mu_{\beta\gamma}B_{\alpha\mu}\,. \tag{40}$$

The difference of two double covariant derivatives then defines the Riemann curvature tensor,

$$[\nabla_\gamma, \nabla_\beta]A_\alpha \equiv A_{\alpha;\beta;\gamma} - A_{\alpha;\gamma;\beta} = \mathcal{R}^\mu{}_{\alpha\beta\gamma}A_\mu\,. \tag{41}$$

It is not hard to show that this definition results in the expression for the Riemann curvature tensor, which is identical to Eq. (39). By studying the symmetries of the Riemann curvature tensor, one can show that $\mathcal{R}^\mu{}_{\alpha\beta\gamma}$ has 20 independent components (in $3+1$ dimensional space-time).

The equation of geodesic deviation (38) controls the congruence of nearby geodesics. In a flat space-time, the curvature tensor vanishes, and hence $\mathcal{D}^2\xi^\mu/d\lambda^2 = d^2\xi^\mu/d\lambda^2 = 0$. This is just saying that two initially parallel geodesics remain parallel at all times. In curved space-times however, the Riemann tensor is nonvanishing, and as a consequence a freely moving observer sees a relative acceleration of nearby freely moving test particles. One manifestation of this is the *tidal effect* (sometimes referred to as the 'tidal force') of distant masses, which for example, explains the tides on the Earth as a consequence of the difference in the attractive gravitational force of the Moon at different places on the Earth.

5 The Einstein field equation

The Einstein field equation cannot be derived. It can be obtained by postulating the principle of general covariance, by requiring that in the weak field nonrelativistic limit one recovers the Newton theory of gravitation, and by requiring that the equation of motion contains at most two time derivatives. For simplicity, we shall first state the Einstein equation, and then show that it reduces to the Newton theory.

The *Einstein field equation* for the classical theory of gravitation is

$$G_{\mu\nu} - \frac{\Lambda}{c^2} g_{\mu\nu} = \frac{8\pi G_N}{c^4} T_{\mu\nu} \,, \tag{42}$$

where $G_{\mu\nu}$ denotes the Einstein curvature tensor, $T_{\mu\nu}$ is the stress-energy-momentum tensor (or in short the stress-energy tensor) of all gravitating matter, $G_N = 6.673(10) \times 10^{-11}$ m^3/kg/s^2 is the Newton constant, $c = 299\,792\,458$ km/s is the speed of light in vacuum, and Λ denotes the cosmological term. Sometimes Λ is considered as a part of the stress-energy tensor. The corresponding stress-energy tensor is then, $T_{\Lambda\,\mu\nu} = [c^2/(8\pi G_N)]\Lambda g_{\mu\nu}$.

To generalise the conservation law for the stress-energy tensor to the relativistic theory of gravitation one defines the *covariant conservation law* of the stress-energy tensor,

$$\nabla^\mu T_{\mu\nu} = 0 \,. \tag{43}$$

Any known form of matter builds up a stress-energy tensor that is covariantly conserved. Hence, the consistency of the Einstein field equation (42) implies the following *Bianchi identity* for the Einstein curvature tensor,

$$\nabla^\mu G_{\mu\nu} = 0 \,. \tag{44}$$

One can show that this condition defines uniquely (up to an overall multiplicative constant proportional to $g_{\mu\nu}$) the Einstein curvature tensor in terms of the Ricci curvature tensor and the Ricci curvature scalar,

$$G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R} \,, \qquad \mathcal{R} = g^{\alpha\beta}\mathcal{R}_{\alpha\beta} \,. \tag{45}$$

where the Ricci tensor is defined in terms of a contraction of the Riemann curvature tensor (39) as follows,

$$\mathcal{R}_{\mu\nu} = \mathcal{R}^{\alpha}{}_{\mu\alpha\nu} \,. \tag{46}$$

In the absence of asymmetric stresses, the stress-energy tensor is symmetric in its two indices, $T_{\mu\nu} = T_{\nu\mu}$, which leaves a priory ten independent component functions. The Einstein curvature tensor is in this case also symmetric, $G_{\mu\nu} = G_{\nu\mu}$, such that it as well contains at most ten independent components. The Bianchi identity (44) and the covariant stress-energy conservation (43) further restrict the number of independent functions to six. This then implies that the metric tensor is completely specified by six independent functions (six degrees of freedom). It turns out that only two degrees correspond to the propagating degrees of freedom (gravitons), while the other four are excited only when sourced by matter, and do not propagate in the radiation zone, that is far from the matter distribution. Two out of these four are the gravitational potentials (the (spatial) Newton potential and the 'time-like' potential), while the other two are vector-like, and are typically sourced by a matter distribution with a nonvanishing vorticity.

The Einstein field equation (42) and Eq. (45) define how matter curves space-time, which is expressed though a nontrivial dependence of the metric tensor on space and time, $g_{\mu\nu} = g_{\mu\nu}(x)$, such that it cannot be removed by an arbitrary coordinate transformation. Conversely, $g_{\mu\nu}(x)$ specifies motion of matter, such that the Einstein equation (42) is a self-contained equation that described dynamics of matter fields *via* a geometric theory. Observe that the Riemann curvature tensor (39) contains terms, which are of the form a single derivative acting on the Levi-Cività connection, and terms which are quadratic forms in the connection. The Levi-Cività connection (25) is in turn expressed in terms of single derivatives acting on the metric tensor. This then implies that the Einstein field equation (42) contains derivatives of the metric tensor up to second order in space and time, and it that sense it resembles the Maxwell theory of electromagnetism and the Klein-Gordon equation for scalar matter. (The dynamics of fermions is specified by the Dirac equation, which at a first sight contains first order derivatives only. Nevertheless, one can show that the matrix structure of the Dirac equation implies that, provided spin is conserved, fermions also obey a second order differential equation.) The principal difference between the dynamics of matter fields and the dynamic of gravitational field are the nonlinear terms, contained in the quadratic forms in the Levi-Cività connection, which makes the Einstein theory of gravitation a much more complex theory than its matter counterparts. Fortunately, these terms are dynamically relevant only in strong gravitational fields.

A second important difference is that in the theory of gravitation the dynamical field is the metric tensor, which is a two indexed symmetric tensor field, while in the matter sector there are vector fields (photons, gluons, weak bosons), spinor fields (fermions) and scalar fields (the recently discovered higgs particle). As a consequence, upon quantisation, one finds that the propagating degrees in gravitation are the spin-two massless gravitons (which propagate at the speed of light in vacuum), while in the matter sector one finds that the propagating modes in gauge theories are the spin-one vector particles, the propagating modes in fermionic theories are the spin one-half fermions, and finally the propagating modes in the scalar sector are the spin-zero scalars.

The strength of the coupling between the gravitational and matter fields is governed by the coupling constant, $8\pi G_N/c^4 \sim 2\times 10^{-43}$ $\mathrm{s}^2/\mathrm{kg}/\mathrm{m}$. This is an extremely small constant, such that only in the presence of matter under extreme conditions (large energy densities), the matter effects on space and time can be strong. Such extreme conditions are found, for example, in galactic centers, many of which are believed to host black holes. However, since gravitation is a long range interaction (recall that the Newton gravitational potential decreases with distance as $1/r$), the effects of a mass distributed throughout space are cumulative, and a relatively dilute matter distribution, if spread over large regions, can have a large effect on the structure of space and time. This is precisely the case with the Universe, where one finds a relatively low matter density (the matter density in the Universe is on average about 10^{-29} g/cm^3), which is to a good approximation homogeneous and isotropic, when averaged over large volumes. Indeed, today we have experimental means for testing the large scale structure of space-time. This has been used extensively to test the Einstein theory of gravitation on cosmological scales, as well as to study the evolution of the Universe.

5.1 The Hilbert-Einstein action

The Einstein field equation (42) can be derived by varying the Hilbert-Einstein action

$$\begin{aligned} S_{HE} &= -\int d^4x\sqrt{-g}\,\frac{c^4}{16\pi G_N}\Big(\mathcal{R}+2\frac{\Lambda}{c^2}\Big) && (47)\\ S &= S_{HE}+S_{\rm matter} \\ S_{\rm matter} &= \int d^4x\sqrt{-g}\mathcal{L}_{\rm matter}\,, && (48)\end{aligned}$$

where $g=\det[g_{\mu\nu}]$ is the determinant of the metric tensor, $\mathcal{R}$ is the Ricci curvature scalar, Λ is the cosmological term, and $\sqrt{-g}\mathcal{L}_{\rm matter}$ is the matter field Lagrangian. Note that the Hilbert-Einstein action (47) is the most general action which transforms as a scalar under general coordinate transformations, and which contains terms up to second order in derivatives of the metric tensor. There are in principle two unspecified constants in the action (47), which are not determined by the symmetry (general covariance). One constant is the dimensionless constant multiplying $(c^4/G_N)\mathcal{R}$, and it can be determined by requiring that the Einstein theory of gravitation reduces to the Newton theory in the weak field limit, and we show how to do that in the next section. The second constant is proportional to the cosmological term Λ, and it can be determined by considering the dynamics of gravitating bodies on very large (cosmological) scales. This illustrates how powerful the principle of general covariance can be when constructing the gravitational action.

For example, for a real scalar field $\phi = \phi(x)$ we have,

$$\sqrt{-g}\mathcal{L}_{\rm matter} = \sqrt{-g}\Big(\frac{1}{2}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi)\Big)\,, \tag{49}$$

where $V(\phi)$ denotes the scalar field potential, such that $d^2V/d\phi^2 \equiv V'' = m_\phi^2 c^2/\hbar^2$ defines the scalar field mass-squared, and $V'''' \equiv \lambda_\phi$ defines the scalar field quartic self-coupling.

In order to calculate the variation δS of the action (47), we first observe that (see Problem 1.3)

$$\delta g = g g^{\mu\nu}\,\delta g_{\mu\nu} = -g g_{\mu\nu}\,\delta g^{\mu\nu}\,, \tag{50}$$

which immediately implies

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\,\delta g^{\mu\nu}\,. \tag{51}$$

Recalling that $\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$ yields the following intermediate result for the variation of the Hilbert-Einstein action,

$$\delta S_{HE} = \int d^4x\sqrt{-g}\Big(-\frac{c^4}{16\pi G_N}\delta g^{\mu\nu}\Big(\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} - \frac{\Lambda}{c^2}g_{\mu\nu}\Big) + \frac{c^4}{16\pi G_N}g^{\mu\nu}\delta\mathcal{R}_{\mu\nu}\Big)\,. \tag{52}$$

The variation of the Ricci tensor $\delta\mathcal{R}_{\mu\nu}$ can be easily found by transforming to a local Minkowski frame, in which $g_{\mu\nu} \to \eta_{\mu\nu} + O(\partial_\alpha g_{\mu\nu})$, such that we have

$$\delta\mathcal{R}_{\mu\nu} = \delta\mathcal{R}^\alpha_{\mu\alpha\nu} \simeq \delta\partial_\alpha\Gamma^\alpha_{\mu\nu} - \delta\partial_\nu\Gamma^\alpha_{\mu\alpha}\,. \tag{53}$$

This then implies

$$g^{\mu\nu}\delta\mathcal{R}_{\mu\nu} \simeq \partial_\alpha\Big(g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu}\Big) - \partial_\nu\Big(g^{\mu\nu}\delta\Gamma^\alpha_{\mu\alpha}\Big)\,, \tag{54}$$

where we inserted $g^{\mu\nu}$ inside the derivatives, which is legitimate in the local Minkowski frame. Since the left-hand-side of Eq. (54) is a scalar, the right-hand-side must also be a scalar, which implies that the covariant form of Eq. (54) must read,

$$g^{\mu\nu}\delta\mathcal{R}_{\mu\nu} = \nabla_\alpha\Big(g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^\beta_{\mu\beta}\Big)\,. \tag{55}$$

This has the form of a covariant divergence, such that upon integration over an invariant measure in Eq. (52), the variation of the Ricci curvature tensor does not contribute to the Einstein field equation,

$$\int d^4x\sqrt{-g}g^{\mu\nu}\delta\mathcal{R}_{\mu\nu} = \int d^4x\sqrt{-g}\nabla_\alpha\Big(g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^\beta_{\mu\beta}\Big) = 0\,. \tag{56}$$

The last equality follows from the simple observation that the covariant divergence of the contravariant vector appearing in (56) can be also written as

$$\nabla\cdot A \equiv \nabla_\alpha A^\alpha = \partial_\alpha A^\alpha + \Gamma^\alpha_{\beta\alpha}A^\beta = \frac{1}{\sqrt{-g}}\partial_\alpha\Big(\sqrt{-g}A^\alpha\Big)\,, \qquad \Gamma^\alpha_{\beta\alpha} = \frac{1}{\sqrt{-g}}\partial_\beta\sqrt{-g}\,. \tag{57}$$

It then follows from the Gauss's integral theorem that the integral (56) can be replaced by a closed surface integral, whereby the surface S_α is arbitrary. If we take the surface sufficiently far from any masses, such that variation the metric tensor can be chosen to vanish everywhere on the surface, $\delta g^{\mu\nu}|_{S_\alpha} = 0$, then the integral in Eq. (56) vanishes.

Next we observe that varying the matter field action (48) yields,

$$\delta S_{\rm matter} = \int d^4x\sqrt{-g}\delta g^{\mu\nu}\,\frac{1}{2}T_{\mu\nu}\,, \tag{58}$$

where we defined the *stress-energy tensor* in terms of the variation of the matter action with respect to the metric tensor as follows,

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_{\rm matter}}{\delta g^{\mu\nu}}\,. \tag{59}$$

For example, for the scalar field matter (49) one finds,

$$T_{\mu\nu} = (\partial_\mu\phi)(\partial_\nu\phi) - g_{\mu\nu}\mathcal{L}_{\rm matter}\,. \tag{60}$$

By taking account of the intermediate results (52) and (56), we arrive at the following form for the variation of the action (47–48),

$$\delta S = \int d^4x\sqrt{-g}\delta g^{\mu\nu}\Big[-\frac{c^4}{16\pi G_N}\Big(G_{\mu\nu} - g_{\mu\nu}\frac{\Lambda}{c^2}\Big) + \frac{1}{2}T_{\mu\nu}\Big]\,. \tag{61}$$

Now requiring that δS vanishes for an arbitrary variation of the metric tensor $\delta g^{\mu\nu}$ yields the Einstein field equation (42).

6 Weak field limit

We shall now study the question of correspondence between the Einstein and Newton theory of gravitation, which is realised when gravitational fields are weak.

According to the equation of geodesic deviation (38), two freely moving test particles in a curved space time move along trajectories that appear to experience different acceleration. As a consequence, the respective geodesics that are initially set to be parallel eventually deviate from being parallel. This effect can be ascribed to the tidal fields of distant masses, or to the gravitational field of a smoothly distributed matter in the vicinity.

For freely falling test particles one may choose a locally Minkowski coordinate frame, with respect to which the observer does not move. In this frame $dx^\mu/d\lambda = u^\mu = \delta^\mu_{\ 0}$, and the spatial components of the equation of geodesic deviation (38) simplify to

$$\frac{\mathcal{D}^2\xi^i}{d\lambda^2} = -\mathcal{R}^i_{\ 0j0}\xi^j\,, \tag{62}$$

where we used $u^\mu = \delta^\mu_{\ 0}$, and the antisymmetry property of $\mathcal{R}^i_{\ \mu\nu\gamma}$ under the exchange of the last two indices.

Let us now compare the expression (62) with the corresponding expression one obtains in Newton's theory, which we desire to correspond to the weak field limit of the Einstein theory. The acceleration in the Newton theory along the geodesics $\gamma_1(\lambda)$ and $\gamma_2(\lambda)$ (see figure 2) is given by,

$$\frac{d^2x^i}{dt^2} = \partial^i\phi_N|_{\gamma_1}\,, \qquad \frac{d^2(x^i+\xi^i)}{dt^2} = \partial^i\phi_N|_{\gamma_2}\,, \tag{63}$$

where ϕ_N denotes the Newton potential. Upon subtracting equations (63), and working to linear order in ξ^i, we get,

$$\frac{d^2\xi^i}{dt^2} = \xi^j \partial_j \partial^i \phi_N \,, \tag{64}$$

where we used $\partial^i\phi_N|_{\gamma_2} = \partial^i\phi_N|_{\gamma_1} + \xi^j\partial_j\partial^i\phi_N|_{\gamma_1}$, and $\partial^j = -\partial_j$. The weak field correspondence of this equation with Eq. (62) then requires

$$\mathcal{R}^i{}_{0j0} \longleftrightarrow -\partial^i\partial_j \frac{\phi_N}{c^2} \,, \tag{65}$$

where we identified the parameter λ with time t multiplied by the speed of light, $\lambda \equiv ct$, and we approximated the covariant derivative in Eq. (62) by an ordinary derivative, $\mathcal{D}^2\xi^i/d\lambda^2 \to c^{-2}d^2\xi^i/dt^2$.

Taking the trace of (65) then yields,

$$\mathcal{R}^i{}_{0i0} = \mathcal{R}_{00} \longleftrightarrow \partial_i^2\frac{\phi_N}{c^2} = \frac{4\pi G_N}{c^2}\rho_N \,, \tag{66}$$

where (for later convenience) we added to the right hand side the usual source of the Newtonian Poisson equation. Here ρ_N denotes the density of matter, which differs by a factor of c^2 from the energy density appearing in the stress-energy tensor, $\rho = \rho_N c^2$.

In order to make the desired connection with the Einstein theory, we now take the trace of the Einstein field equation (42) by multiplying it by $g^{\mu\nu}$,

$$\mathcal{R} = -\frac{8\pi G_N}{c^4}T - 4\frac{\Lambda}{c^2} \,, \qquad T \equiv g^{\mu\nu}T_{\mu\nu} \,, \tag{67}$$

upon which Eq. (42) can be recast to the form,

$$\mathcal{R}_{\mu\nu} = \frac{8\pi G_N}{c^4}\Big(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\Big) - g_{\mu\nu}\frac{\Lambda}{c^2} \,. \tag{68}$$

We now assume that the stress-energy tensor of matter can be well approximated by the ideal fluid form,

$$T_{\mu\nu} = (\rho + \mathcal{P})\frac{u_\mu u_\nu}{c^2} - g_{\mu\nu}\mathcal{P} \,, \tag{69}$$

where ρ denotes the energy density and $\mathcal{P}$ the pressure of the fluid. Recall that the observer is in the freely falling frame, in which now $dx_\mu/dt \equiv u_\mu = c\delta_\mu^{\,0}$, $g_{\mu\nu} \simeq \eta_{\mu\nu}$, implying that $T_{00} = \rho$, $T_{0i} = 0$, $T_{ij} = \mathcal{P}\delta_{ij}$, $T = \rho - 3\mathcal{P}$. From this we find that Eq. (68) can be rewritten as

$$\begin{aligned} \mathcal{R}_{00} &= \frac{4\pi G_N}{c^4}\Big(\rho + 3\mathcal{P}\Big) - \frac{\Lambda}{c^2} \\ \mathcal{R}_{ij} &= \frac{4\pi G_N}{c^4}\delta_{ij}\Big(\rho - \mathcal{P}\Big) + \delta_{ij}\frac{\Lambda}{c^2} \,. \end{aligned} \tag{70}$$

With this we can rewrite the correspondence relation (66) between the Einstein and Newton theory in the form

$$\mathcal{R}_{00} = \frac{4\pi G_N}{c^4}\Big(\rho + 3\mathcal{P}\Big) - \frac{\Lambda}{c^2} \quad\longleftrightarrow\quad \frac{1}{c^2}\partial_i^2\phi_N = \frac{4\pi G_N}{c^2}\rho_N \,. \tag{71}$$

We see that the Newton limit is reproduced only when $\rho = \rho_N c^2$, $\Lambda = 0$ and $\mathcal{P} = 0$. This is justified for dust, representing an extremely nonrelativistic matter, for which the pressure contribution is negligible when compared to that of the energy density, $\mathcal{P} \ll \rho$. This is certainly not a good approximation for relativistic fluids (such as neutrinos or photons). In particular for a photon fluid we have $\mathcal{P} = (1/3)\rho$, such that the true relativistic source of the Newtonian potential is

$$\rho_{\text{active}} \equiv \rho + 3\mathcal{P} \,. \tag{72}$$

ρ_{active}/c^2 is sometimes referred to as the active gravitational mass density and, as we will see, it is of a fundamental importance for cosmology, since at early epochs the Universe was predominantly made up of relativistic matter.

The Λ-term does not have a Newtonian equivalent, although sometimes an energy density is associated to Λ, which is of the form, $\rho_\Lambda = [c^2/(8\pi G_N)]\Lambda$, and whose equation of state reads, $\mathcal{P}_\Lambda = w_\Lambda \rho_\Lambda$, with $w_\Lambda = -1$. The Λ-term is very small in the Universe, and it becomes dynamically relevant only on very large (cosmological) scales.

So far we have established a link between the Einstein and Newton theory of gravitation, by establishing a correspondence between certain components of the Riemann curvature tensor and spatial derivatives of the Newton potential. We shall now show how to construct the metric tensor in the weak field limit.

Since the coupling between matter and gravitation is weak (recall that it is governed by $8\pi G_N/c^4 \sim 2\times 10^{-43}\ \mathrm{s^2 kg^{-1} m^{-1}}$), it is often a very good approximation to linearise around the flat Minkowski space-time, in particular when one is asking questions about the evolution of local structures (galaxies, clusters of galaxies, etc.),

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,, \qquad \eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1) \,, \tag{73}$$

where $h_{\mu\nu}$ represents a deviation from the flat metric, $\eta_{\mu\nu}$.

Working to linear order in $h_{\mu\nu}$ we easily find the Levi-Cività connection (25),

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\eta^{\alpha\beta}\Big(\partial_\mu h_{\beta\nu} + \partial_\nu h_{\mu\beta} - \partial_\beta h_{\mu\nu}\Big) \,, \tag{74}$$

and for the Ricci tensor $\mathcal{R}_{\mu\nu} = \mathcal{R}^\alpha{}_{\mu\alpha\nu}$ (39) and scalar $\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$,

$$\begin{aligned}
\mathcal{R}_{\mu\nu} &= \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} \\
&= \frac{1}{2}\eta^{\alpha\beta}\Big(\partial_\alpha\partial_\mu h_{\beta\nu} + \partial_\nu\partial_\beta h_{\mu\alpha}\Big) - \frac{1}{2}\Big(\Box h_{\mu\nu} + \partial_\nu\partial_\mu h\Big) \\
\mathcal{R} &= \eta^{\alpha\beta}\eta^{\mu\nu}\partial_\alpha\partial_\mu h_{\beta\nu} - \Box h \,, \qquad \Box \equiv \eta^{\alpha\beta}\partial_\alpha\partial_\beta \,, \quad h \equiv \mathrm{Tr}[h_{\mu\nu}] = \eta^{\nu\mu}h_{\mu\nu} \,,
\end{aligned} \tag{75}$$

plus higher order terms.

Since we are working to first order in $h_{\mu\nu}$, we can raise and lower indices with $\eta_{\mu\nu}$. It is quite straightforward to check that the Bianchi identity is automatically satisfied by the Ricci tensor (75),

$$\nabla^\mu G_{\mu\nu} = \partial^\mu\Big(\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu}\Big) = 0 \,. \tag{76}$$

The metric tensor $h_{\mu\nu}$ is symmetric in its indices, and thus it has in general 10 component functions, but not all of them are independent. Four of the functions can be constrained by imposing the invariance under the general linear coordinate transformations,

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu(x) \,. \tag{77}$$

The metric tensor transforms as

$$h_{\mu\nu} \to \tilde{h}_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \,. \tag{78}$$

One can easily show that, provided $\Box \xi^\mu = 0$, the metric tensor is invariant under the coordinate transformation (77) provided it satisfies the following gauge condition (analogous to Lorentz gauge in electrodynamics, $\partial^\mu A_\mu = 0$),

$$\partial^\mu h_{\nu\mu} - \frac{1}{2}\partial_\nu h = 0 \,. \tag{79}$$

In this gauge the Ricci tensor and scalar (75) simplify to,

$$\mathcal{R}_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu} \,, \qquad \mathcal{R} = -\frac{1}{2}\Box h \,. \tag{80}$$

Assuming that the stress-energy tensor takes on the ideal fluid form (69), we find that the Einstein equations in the fluid rest frame (*cf.* Eqs. (68–70)), in the weak field limit, and in gauge (79), reduce to the following simple form,

$$\Box h_{00} = -\frac{8\pi G_N}{c^4}\Big(\rho + 3\mathcal{P}\Big) + 2\frac{\Lambda}{c^2} \tag{81}$$

$$\Box h_{0i} = 0 \tag{82}$$

$$\Box h_{ij} = -\frac{8\pi G_N}{c^4}\delta_{ij}\Big(\rho - \mathcal{P}\Big) - 2\delta_{ij}\frac{\Lambda}{c^2} \,. \tag{83}$$

Note that (83) describes the equation for gravitation waves in the weak field limit in presence of matter sources, and we comment on its significance below.

In order to complete the analysis, we now take the nonrelativistic limit, in which $\Box \to -\partial_i^2$, and the pressure $\mathcal{P} \to 0$. In addition we assume $\Lambda \to 0$. We then find

$$\begin{aligned} \partial_l^2 h_{00} &= \frac{8\pi G_N}{c^4}\rho \\ \partial_l^2 h_{0i} &= 0 \\ \partial_l^2 h_{ij} &= \frac{8\pi G_N}{c^4}\delta_{ij}\rho \,, \end{aligned} \tag{84}$$

from which we conclude

$$h_{00} = \frac{2\phi_N}{c^2} \,, \qquad h_{ij} = \delta_{ij}\frac{2\phi_N}{c^2} \,, \qquad h_{0i} = 0 \,, \tag{85}$$

such that in the weak field nonrelativistic limit the line element takes the form,

$$ds^2_{\rm Newton} = \Big(1 + \frac{2\phi_N}{c^2}\Big)c^2dt^2 - \Big(1 - \frac{2\phi_N}{c^2}\Big)\delta_{ij}dx^i dx^j \,. \tag{86}$$

When the potential is a spherically symmetric distribution of matter, $\phi_N = \phi_N(r)$, than (to linear order in the potential) the line element (86) simplifies to

$$ds^2_{\rm Newton} = \Big(1 + \frac{2\phi_N}{c^2}\Big)c^2dt^2 - \Big(1 - \frac{2\phi_N}{c^2}\Big)dr^2 - r^2\Big(d\theta^2 + \sin^2(\theta)d\varphi^2\Big) \,. \tag{87}$$

Note that Eqs. (86–87) represent the weak field limit of the line element (11), in which $\phi_N \ll c^2$, where ϕ_N denotes the Newton potential (see also Problem 1.4).

A careful reader has certainly noticed that we have not yet discussed gravitational waves, which are also a part of the weak field analysis. Gravitational waves correspond to homogeneous (relativistic) solutions of equation (83), such that they can be thought of as a linear superposition of plane waves which propagate with the speed of light in vacuum. In the light of the discussion in section 1.2, gravitational waves propagate on the light-cone, and hence are consistent with causality. In the physical gauge, in which $h_{0\nu} = 0$ and h_{ij} is traceless and transverse ($h_i^{\ i} = 0$, $\partial_j h_{ij} = 0$), the two physical degrees of freedom correspond to the two mutually orthogonal deformations of space, such that a circle placed orthogonally to the wave propagation is deformed to an ellipse. Quantum mechanically, these two degrees of freedom correspond to the two states of the massless spin-two graviton. These projections of spin on the direction of motion are known as helicities, and can be either plus two or minus two (in units of $\hbar$). We postpone a more detailed analysis of gravitational waves to Part IV, in which we discuss the production of gravitational waves in an inflationary epoch of the early Universe, during which the Universe expands in an accelerated fashion.

7 Tests of general relativity

General relativity has been tested on many grounds, and up to this moment no deviations have been found from the theory's predictions. Here we mention several tests, and consider in some detail time dilatation, reshift and light deflection by the gravitational field. The latter, for example, induces lensing of cosmic microwave background radiation. An interested reader may consult, for example, Norbert Straumann, *General relativity and relativistic astrophysics* (Springer-Verlag, 1984), and Clifford M. Will, "The Confrontation between general relativity and experiment," Living Rev. Rel. **9** (2006) 3 [arXiv:gr-qc/0510072].

Tests of the Einstein theory of gravitation include:

(1) *Advance of the perihelion of a planet.* A disagreement of advance of the perihelion (the point of closest approach of a planet to the Sun) of Mercury with the Newton theory prediction was known since a long time ago. In 1859 Le Verrier suggested that the anomaly could be explained by an unobserved planet Vulcan in an orbit close to the Sun. A general relativistic calculation was performed in 1916, and explained the Mercury anomaly with an accuracy of better than 1%. The prediction of the Einstein theory for advance of the Mercury perihelion is $\Delta\varphi_{\rm Einstein} = 42.98''$ per century, while the observed value is $43''$ (the agreement implied by the Lunar ranging measurements is about 0.3%).

(2) *Gravitational bending of light* was first measured during the total solar eclipse in 1919 to an accuracy of about 10% by two scientific teams lead by Dyson and Eddington. The most accurate modern measurements are based on about 2 million quasar and galaxy observations by VLBI (1999) over the whole sky, and yield a confirmation of the Einstein theory to an accuracy of about 0.02%. An alternative confirmation has been reached by the satellite Hyparcos at the level of 0.1%.

(3) *Gravitational redshift of light* was first observed on an Earth experiment in 1960 by Pound and Rebka. The measurement, which was based on the Mössbauer effect, and was subsequently improved by Pound and Snider in 1965 to an accuracy of about 1%. The accuracy was further improved to 2×10^{-4} in a rocket experiment by Versot and Levine in 1976.

(4) *Time delay in gravitational field.* This effect has been observed by radar echoes (radar-ranging off the retro-reflector placed in 1969 by Appolo 11 on the Moon), as well as by passive reflections of radar signals off Mercury and Venus, when they were on the opposite side of the Sun. The active reflection off the Viking satellite mirror resulted in an 0.1% test of the Einstein theory prediction.

(5) *Gravitational lensing* has first been observed in 1979. The first Einstein ring was observed in 1988 on MG 1131 + 0456 in the constellation Leo by Jacqueline Hewitt. Nowadays, gravitational lensing is routinely observed in deep sky images (*e.g.* by the Hubble space telescope and, starting in 2020, by the Large Synoptic Survey Telescope, LSST, whose construction began in 2012).

(6) *Geodetic precession* is generated by the warped nature of space-time, and it arises when one massive body rotates in the gravitational field of another body. The general relativistic theory of gravitation predicts a precession of about $2''$ per year of the Earth rotation axis due to the gravitational interaction between the Earth and the Moon, and it has been confirmed by the Lunar laser ranging experiments to an accuracy of about 0.7%. The Stanford-NASA gyroscope satellite experiment (Gravity Probe B), `http://einstein.stanford.edu/`, `http://www.gravityprobeb.com/`, launched in April 2004, has the designed accuracy goal of 5×10^{-5}. The final results (C. W. F. Everitt et al, Phys. Rev. Lett. **106**, 221101 (2011)) give for the geodetic precession 6.602 ± 0.018 arcsec/year, which agrees with the predicted value of 6.606 arcsec/year.

(7) *Gravitomagnetic precession (Lense-Thirring effect).* The effect was independently predicted by G. E. Pugh (1959) and by Leonard I. Schiff (1960). Space-time is warped due to a non-vanishing angular momentum of the Earth. The warping causes a precession of the axis of rotating bodies in the Earth orbit (the spin-orbit coupling in the theory of gravitation). The effect is tiny (about 2.1×10^{-2} arcsec/year for the Earth) and the detection has been recently claimed by Ciufolini et al, based on the precession of the two LAGEOS satellites. The result represents a 20% accurate confirmation of the prediction of general relativity. Even though Gravity Probe B has been designed to measure gravitomagnetic precession to an accuracy of about 2%, the actual result had an accuracy of 'only' 20%. The result reported by C. W. F. Everitt et al (Phys. Rev. Lett. **106**, 221101 (2011)) is $37.2 \pm 7.2 \times 10^{-3}$ arcsec/year, and it agrees well with the value predicted by general relativity, 39.2×10^{-3} arcsec/year. Since the Gravity Probe B result is more reliable than the LAGEOS satellite results, it is considered yet another novel test of general relativity.

(8) *Gravitational radiation* from binary pulsar systems. In 1974 Joseph Taylor and Russell Hulse (Nobel Prize 1993) discovered a binary pulsar system, consisting of a neutron star orbiting around as-of-yet unseen companion, which they named PSR 1913+16. The pulsar period is 59 ms, the orbital period about 7.75 hours, and the eccentricity 0.617. In fact, the pulsing period P_p and its slow-down rate dP_p/dt are known to a very high accuracy,

$$P_p = 59.029997929613(7)\ \mathrm{ms}\,, \qquad \frac{dP_p}{dt} = 8.62713(8)\times 10^{-18}\,, \tag{88}$$

while the orbital period decreases with the rate

$$\frac{dP_b}{dt} = -2.422(6)\times 10^{-12}\,, \tag{89}$$

which is caused predominantly by gravitational radiation, present in a significant amount only in strong gravitational fields. When the effect of galactic rotation is subtracted (89), a comparison with the general relativistic prediction yields,

$$\frac{(dP_b/dt)_{\rm GR}}{(dP_b/dt)_{\rm observed}} = 1.0023 \pm 0.005\,, \tag{90}$$

which tests general relativity to an accuracy of about 0.5%. This is so far the only indirect confirmation for the existence of gravitational radiation, and at present one of a few tests of general relativity in strong gravitational fields.

(9) *Black holes.* The center of our galaxy (Milky Way) harbours a black hole placed in the constellation Sagitarrius A^* (Sag A^*), which is about 26000 light years away from us, and whose mass is about $4.1\times10^6\ M_\odot$, $M_\odot$ being the mass of the Sun, which is about 8.2×10^{36} kg (The Max Planck Institute for Extraterrestrial Physics in Garching, Germany, and the UCLA Galactic Center Group, Los Angeles). [2] Our black hole will soon become active, and between 2014 and 2018 it will accrete a significant amount of matter. A feast is soon to come! Finally, there are also strong indications that other galaxies (active galactic nuclei) and quasars harbor massive black holes in their centra, whose mass could reach a value as high as $10^{10}\ M_\odot$. The largest black hole so far observed is believed to be in the galaxy NGC 4889. Its mass is estimated to be in the range from 6 billion to 37 billion solar masses.

(10) *Direct detection of gravitational radiation.*

In 2014, the advanced LIGO (LIGO stands for the Laser Interferometer Gravitational-Wave Observatory) will start taking data, and direct detection from a merger of two compact stars is expected within a year. Compact objects are neutron stars and black holes. If successful, that will be the first direct detection of gravitational waves. Moreover, apart from cosmological tests, these will represent first tests of the strong gravity regime (that is beyond linear regime) of general relativity. The LIGO is physically at two locations: first is the LIGO Livingston Observatory in Livingston, Louisiana, and second is the LIGO Hanford Observatory, near Richland, Washington. For more information see `http://ligo.org/`. More to the future, the EU plans the underground Einstein Telescope (`http://www.et-gw.eu/`, `http://en.wikipedia.org/wiki/Einstein_Telescope`) and the satellite mission LISA (`http://sci.es` `http://lisa.nasa.gov/`).

(11) *Cosmological tests of general relativity.*

Cosmology has been used to test general relativity in the strong regime through gravitational redshift of photons. Moreover, motion of galaxies and clusters of galaxies on large scales tell us that *dark matter* and *dark energy* must be added in order to make general relativity consistent with observations. It seems that most of the observations of dark matter can be well explained by modeling it with a cold (nonrelativistic) fluid (or 'dust'), whose equation of state is $w_{\rm dm} = \mathcal{P}_{\rm dm}/\rho_{\rm dm} = 0$. On the other hand, observations of dark energy can be eplained by a cosmological constant, whose equation of state is $w_{\rm de} \equiv w_\Lambda = \mathcal{P}_\Lambda/\rho_\Lambda = -1$. Alternatively, it is possible to modify general relativity on large scales, such that its effects

[2]For a video experience see `http://en.wikipedia.org/wiki/Supermassive_black_hole`. The video can be seen at `http://en.wikipedia.org/wiki/File:A_Black_Hole%E2%80%99s_Dinner_is_Fast_Approaching_-_Part_2.ogv`.

mimick those of dark matter and dark energy. Studies of modified gravity are an active area of research, and the final judgement (on whether the right thing is to modify gravity or to add dark matter and dark energy to general relativity) has not as yet been made.

8 Gravitational time dilatation and gravitational redshift

Perhaps the simplest way of understanding time dilatation and gravitational redshift is to consider two observers O_1 and O_2 placed in a stationary gravitational field well approximated by the Newton potential. We assume further that the observers do not move with respect to the center of mass, such that the gravitational field they observe appears to be static. The reader should keep in mind the spherically symmetric Schwarzschild metric and the corresponding line element, as given in Eqs. (11–13), which for convenience we quote again,

$$ds^2 = c^2\Big(1+\frac{2\phi_N}{c^2}\Big)dt^2 - \frac{dr^2}{1+2\phi_N/c^2} - r^2\Big(d\theta^2+\sin^2(\theta)d\varphi^2\Big)\,, \qquad \phi_N = -G_N\frac{M}{r}\,. \tag{91}$$

Let us now consider a high frequency light ray passing by two observers, O_1 and O_2, and let us assume that the observers measure a time lapse between two subsequent light crests, which we denote by δt_1 and δt_2, respectively. The time lapses δt_1 and δt_2 can then be easily related by noting that an observer at an asymptotic infinity, where the metric is Minkowski flat, would measure a time lapse $\delta\tau$ between two subsequent wave crests of the same wave, such that the following simple relation holds,

$$\delta\tau = \sqrt{g_{00}(r_1)}\delta t_1 = \sqrt{g_{00}(r_2)}\delta t_2\,. \tag{92}$$

From this we immediately conclude that the two time lapses are related as

$$\frac{\delta t_1}{\delta t_2} = \sqrt{\frac{g_{00}(r_2)}{g_{00}(r_1)}}\,, \tag{93}$$

which in a weak gravitational field reduces to

$$\frac{\delta t_1}{\delta t_2} \simeq 1 + \frac{\phi_N(r_2)-\phi_N(r_1)}{c^2}\,. \tag{94}$$

This implies that the observer, which is placed deeper in the potential well, such that its potential is more negative, measures a longer lapse between the wave crests. This effect is known as the *gravitational time dilatation*.

As an example, let us assume that the first observer $O_1 \equiv O_\odot$ is located on the surface of the Sun, where $\phi_N(r_1) \equiv \phi_\odot \simeq -2.12\times10^{-6}c^2$, and O_2 is on the surface of the Earth, where the potential is much smaller, and can be to a good approximation neglected, $\phi_{\rm Earth}\simeq 0$. Eq. (94) then implies

$$\begin{aligned}\delta t_{\rm Earth} &\simeq \Big(1+\frac{\phi_\odot}{c^2}\Big)\delta t_\odot \\ &= \delta t_\odot - 2.12\times10^{-6}\delta t_\odot\,, \end{aligned} \tag{95}$$

such that the time lapse between the two crests measured on the Earth is shorter than what would be measured on the surface of the Sun. Thus, on the Earth surface time lapses faster than on the surface of the Sun (time contraction), and conversely, $\delta t_\odot = \delta t_{\rm Earth} + 2.12\times 10^{-6}\delta t_{\rm Earth}$. [3]

The question we now address is, what is the frequency of light measured by the observers O_1 and O_2. The phase of the wave crests/throughs in a diagonal metric can be represented by the simple formula

$$\Phi(\vec{x}_A;\vec{x}_B) = \frac{1}{\hbar}\int_{\vec{x}_A}^{\vec{x}_B} g_{\mu\nu}(x)p^\mu dx^\nu = \frac{1}{\hbar}\int_{t_A}^{t_B} g_{00}(x')\frac{E}{c}dt' - \frac{1}{\hbar}\int_{\vec{x}_A}^{\vec{x}_B} g_{ii}(x')\vec{p}\cdot d\vec{x}'\,, \tag{97}$$

where $p^\mu = (E/c, p^i)$ denotes the 4-vector of energy and momentum, and $\hbar = h/(2\pi) = 1.054\times 10^{-34}$ Js is the reduced Planck constant, $h = 6.6262\times10^{-34}$ Js. Since the (static) observers O_1 and O_2 measure a short time interval between two wave crests, to leading order in δt Eq. (97) simplifies to,

$$\delta\Phi(\vec{x}_A;\vec{x}_B) = \pi = g_{00}(r_1)\frac{E_1}{\hbar c}\delta t_1 = g_{00}(r_2)\frac{E_2}{\hbar c}\delta t_2\,. \tag{98}$$

By making use of Eqs. (92–93) and of $E = \hbar\nu$, we easily find

$$\frac{E_1}{E_2} = \frac{\nu_1}{\nu_2} = \sqrt{\frac{g_{00}(r_2)}{g_{00}(r_1)}} \simeq 1 + \frac{1}{c^2}\Big(\phi_N(r_2) - \phi_N(r_1)\Big)\,, \tag{99}$$

such that energy redshifts as photons climb out of a gravitational potential. This phenomenon is known as the *gravitational redshift of light*, and is has been first observed on the Earth in 1960 by Pound and Rebka. When applied to the expanding Universe, the gravitational reshift is responsible for the redshift of photons from distant sources, and from the cosmic microwave background radiation.

A simple interpretation of this result is obtained by noting that photons are just a special case of particles, whose relativistic energy is given by the Einstein relation, $E = \sqrt{p^2c^2 + m^2c^4}$, with zero mass $m = 0$ and a momentum $p = \hbar\nu/c$, where ν denotes the frequency of the photon. When placed in a gravitational potential, $\phi_N = \phi_N(\vec{x})$, the energy of particles with $m \neq 0$ is modified as,

$$E = \sqrt{p^2c^2 + m^2c^4} + m\phi_N\,. \tag{100}$$

Since the gravitational field is conservative, the energy of particles moving in a gravitational field must be conserved, which implies

$$E(\vec{x}_1, \vec{p}_1) = E(\vec{x}_2, \vec{p}_2)\,. \tag{101}$$

[3] The following consideration is erroneous. I encourage the reader to find a flaw in the following reasoning. The equivalence principle implies that time dilatation in an accelerated system is the same as time dilatation observed in a system placed in a gravitational field of an equal magnitude. A quantitative estimate of time dilatation can be then easily found by identifying acceleration with gravitational field. We thus have, $\vec{a} = \vec{g} = -\partial_{\vec{x}}\phi_N$, or equivalently, $\phi_N(x) = \int_{\vec{x}}^\infty \vec{a}(\vec{x})\cdot d\vec{x}$. This then implies the following simple expression for time dilatation in an accelerated system, measured with respect to an inertial system (both systems are assumed to be placed in a vanishingly small gravitational field),

$$\frac{\Delta t_{\rm dilatation}(\vec{x}_1,\vec{x}_2)}{t_{\rm inertial}} = \frac{1}{c^2}\int_{\vec{x}_1}^{\vec{x}_2}\vec{a}(\vec{x})\cdot d\vec{x}\,, \tag{96}$$

where the system accelerates from point $\vec{x}_1$ to point $\vec{x}_2$, and $t_{\rm inertial}$ denotes time lapse in the inertial (nonaccelerated) system. This expression represents an estimate of the aging difference of the two twins, in the twin paradox of special relativity, which however gives an incorrect answer. The correct answer can be obtained by a standard use of the ligh-cone diagram. I encourage the reader to find the flaw in the reasoning in this footnote.

For nonrelativistic particles, this reduces simply to the conservation of the kinetic plus potential energy. This expression is however meaningless for photons, since their mass is zero. A heuristic derivation for the photons can be nonetheless obtained by replacing $m\phi_N$ by $(E/c^2)\phi_N$ in the expression (100), which means that photons behave as if they had a gravitational mass equal to p/c. This then implies,

$$\hbar\nu_1\Big(1+\frac{\phi_N(\vec{x}_1)}{c^2}\Big) = \hbar\nu_2\Big(1+\frac{\phi_N(\vec{x}_2)}{c^2}\Big)\,, \tag{102}$$

which agrees with (99).

9 Light deflection

A gravitationally induced light deflection was first measured during the Solar eclipse on March 29, 1919 by two expeditions, organized by Frank Dyson and Arthur Eddington, respectively. The observations took place in the Brazilian city of Sobral (Dyson), and in the Portuguese island of Principe off the West coast of Africa (Eddington). The observers compared positions of stars at night with the respective positions during the eclipse, and found for the light deflection angle (in arc seconds),

$$\alpha = (1.98\pm 0.16)'' \quad (\mathrm{Sobral, Dyson})\,, \qquad \alpha = (1.61\pm 0.40)'' \quad (\mathrm{Principe, Eddington})\,, \tag{103}$$

in agreement with the prediction of the Einstein theory, $\alpha = 1.75'' R_\odot/d$, where $R_\odot$ denotes the Sun radius, and d the closest distance of the photon to the Sun center.

The Newton's corpuscular theory of light predicts a bending angle which is by a factor two smaller. This can be understood as follows. In the corpuscular theory of light the origin of the effect is in light bending, and would correspond to the bending of light rays with respect to absolute straight lines, defined for example by rigid rods. But there are no absolute straight lines in the Einstein theory. The space-time of general relativity is curved around massive bodies, resulting in an additional effect identical in size to the bending angle of light corpuscles in the Newton theory, explaining thus the general relativistic result.

Let us start our analysis with the general relativistic action for a point particle,

$$S = mc\int ds = \int L dt\,, \qquad L = mc\sqrt{g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}\,. \tag{104}$$

The corresponding canonical 4-momentum $p^\mu = (E/c, p^i)$ is,

$$p_\nu = \frac{\partial L}{\partial(dx^\nu/dt)} = \frac{dt}{ds}mcg_{\nu\mu}\frac{dx^\mu}{dt}\,, \qquad \frac{ds}{dt} = \sqrt{g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}\,. \tag{105}$$

The corresponding Euler-Lagrange equation is then,

$$\frac{dp_\nu}{dt} = \frac{1}{2}(\partial_\nu g_{\alpha\beta})p^\alpha\frac{dx^\beta}{dt} \tag{106}$$

where $p^\mu = mcdx^\mu/ds$. Note that this is just a convenient rewriting of the geodesic equation (36), with the identification, $p^\mu = mu^\mu$. In general a massive particle must propagate on the mass shell, which implies,

$$g_{\mu\nu}p^\mu p^\nu = m^2c^2 \Longleftrightarrow g_{\mu\nu}u^\mu u^\nu = c^2\,. \tag{107}$$

For the photons however, $m \to 0$, and this simplifies to

$$g_{\mu\nu} p^\mu p^\nu = 0 \,, \tag{108}$$

which establishes the photon dispersion relation, $p^0 = p^0(p^i, x^\mu)$.

We are interested in light propagation in the presence of a quasistationary mass distribution which produces weak gravitational fields (weak lensing), hence the line element can be to a good accuracy modeled by the following Newtonian diagonal form,

$$ds^2 \simeq \Big(1 + 2\frac{\phi_N}{c^2}\Big) c^2 dt^2 - \Big(1 - 2\frac{\phi_N}{c^2}\Big)\delta_{ij} dx^i dx^j \,, \tag{109}$$

where $\phi_N = \phi_N(\vec{x})$ is the Newton potential of the quasistationary mass distribution.

First we note that in a quasistationary Newtonian metric, the Euler-Lagrange equation for light (106) implies the conservation law of the canonical energy p_0/c,

$$\frac{dp_0}{dt} = \frac{d}{dt}\Big((1 + 2\phi_N/c^2)p^0\Big) = 0 \,, \tag{110}$$

while for the spatial momentum we get,

$$\begin{aligned} \frac{dp_i}{dt} &= -\frac{d}{dt}\Big((1 - 2\phi_N/c^2)p^i\Big) \\ &= \frac{c}{2}(\partial_i g_{\alpha\beta})\frac{p^\alpha p^\beta}{p^0} \,. \end{aligned} \tag{111}$$

Upon dividing this by the conserved quantity, $-(1 + 2\phi_N/c^2)(p^0/c) = \text{const.}$, we find

$$\frac{d}{dt}\Big((1 - 4\phi_N/c^2)\frac{d\vec{x}}{dt}\Big) = -2\nabla\phi_N \,, \tag{112}$$

where we took account of $dx^\alpha/dt = cp^\alpha/p^0$ and $(dx^i/dt)^2 = c^2$. This equation describes the lensing of light in a weak quasistationary gravitational field in general relativity. Note that for relativistic bodies, the gravitational field (the force per unit mass), $-2\nabla\phi_N$, appears to be by a factor two larger than what one would expect from the naïve Newtonian limit.

In a simple case when light bending is small, we can take a light ray to move from a source S in the y direction, $v_y = c$, and we can integrate (112) once to obtain,

$$\frac{d\vec{x}(y)}{dt} = -\frac{2}{c}\int_{y_S}^{y} dy' \nabla' \phi_N \,. \tag{113}$$

where y_S denote the source position, and we made use of $dt = dy/c$. The light bending angle $\alpha_x = \alpha_x(y_S, y_O)$ in x direction accumulated between the source at $\vec{x}_S$ and the observer at $\vec{x}_O$ is then $(\alpha_x = v_x/v_y = v_x/c)$,

$$\alpha_x = -\frac{2}{c^2}\int_{y_S}^{y_O} dy \partial_x \phi_N \,. \tag{114}$$

This is the main result of this section.

We shall now apply formula (114) to the simple case of a point like mass distribution of a mass M located at the origin, in which case, $\phi_N = -G_N M/r$. Eq. (114) then yields

$$\begin{aligned} \alpha_x &= -\frac{2G_N M x}{c^2}\int_{y_S}^{y_O} \frac{dy}{(x^2 + y^2)^{3/2}} \\ &\simeq -\frac{4G_N M}{c^2 d} \,, \end{aligned} \tag{115}$$

where $x = d$ represents the closest distance of the light ray to the mass M. When applied to the Sun, whose mass and the radius are given by $M_\odot = 2 \times 10^{30}$ kg and $R_\odot = 7 \times 10^8$ m, and taking $y_S \to -\infty$, $y_O \to +\infty$, we get the famous Einstein's result (for the Sun),

$$\alpha_\odot(d) = 1.75'' \frac{R_\odot}{d} \,. \tag{116}$$

It is of interest to note that bending angles of a similar magnitude are produced by typical elliptical and spiral galaxies (see Problem 1.5).

10 Coupling of matter fields to gravitation

According to the principle of general covariance, matter fields couple to gravitation such that the corresponding matter action is generally covariant. We now discuss how to construct generally covariant actions for the relevant matter fields (scalars, gauge fields, fermions).

10.1 Scalar fields

We start with the simplest case of a real scalar field with a canonical kinetic term, and whose potential is given by $V = V(\phi)$. If we restrict ourselves to terms containing up to second order derivatives, the covariant scalar action is then given by,

$$\begin{aligned} S_\phi &= \int d^4x \sqrt{-g}\mathcal{L}_\phi \\ \sqrt{-g}\mathcal{L}_\phi &= \frac{1}{2}\sqrt{-g}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \sqrt{-g}V(\phi) - \frac{1}{2}\sqrt{-g}\xi\phi^2\mathcal{R} \,. \end{aligned} \tag{117}$$

The Euler-Lagrange equation of motion is obtained by varying S_ϕ with respect to ϕ,

$$\frac{1}{\sqrt{-g}}\partial_\mu\Big(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\Big) + \frac{dV(\phi)}{d\phi} + \xi\mathcal{R}\phi = 0 \,. \tag{118}$$

Note that the derivative in this equation becomes the covariant derivative, $\nabla_\mu = (-g)^{-1/2}\partial_\mu(-g)^{1/2}$, when it acts on the vector field, $\nabla^\mu\phi = \partial^\mu\phi$, such that the derivative operator in (118) is nothing but the d'Alembertian operator $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ as it acts on a scalar field.

The scalar potential V often contains a constant, quadratic and quartic term only,

$$V(\phi) = V_0 + \frac{1}{2}\frac{m_\phi^2 c^2}{\hbar^2}\phi^2 + \frac{\lambda_\phi}{4!}\phi^4 \,. \tag{119}$$

Note that V_0 is redundant in the sense that it can be combined with the cosmological term of the Hilbert-Einstein action (47),

$$-\frac{c^2}{8\pi G_N}\Lambda_0 + V_0 \to -\frac{c^2}{8\pi G_N}\Lambda \,, \tag{120}$$

and has no independent physical meaning. This also explains why some authors like to put the cosmological term as part of the matter action. Phase transitions in field theory are often represented by a potential of the type (119) with $V_0 > 0$ and $m_\phi^2 < 0$, showing that phase transitions in the early Universe (for example, electroweak phase transition and quantum-chromodynamic (QCD) phase transition) are intricately related to the problem of vacuum energy in the theory of gravitation (the cosmological constant problem). This question has gained in importance by the recent (2012) discovery of the higgs particle by the ATLAS and CMS collaborations at the LHC experiment at CERN, Geneva.

The last term in the scalar Lagrangian (117) represents the coupling of scalar fields to the Ricci scalar of gravitation, and it is generally covariant. Up to this moment there are no strong observational constraints on the size or sign of ξ. An exception are composite scalar fields which are made up of some more fundamental fields that are not Lorentz scalars. For example, chiral condensates of quantum-chromodynamics (QCD) can be thought of as effective scalar fields made up of fundamental fermion fields, whose spinor structure forbids covariant coupling to the Ricci curvature scalar, implying that in this case we expect $\xi = 0$. A similar conclusion is reached if composites are made of gauge fields, *e.g.* we expect that the composite field $g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}$ does not couple to gravitation, $\xi = 0$.

10.2 Abelian gauge fields

Next, we consider Abelian gauge fields. A generalisation to nonabelian gauge fields is straightforward. The covariant field strength can be defined in terms of gauge field as follows,

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{121}$$

where the last equality follows quite trivially from the antisymmetry in the definition of the field strength tensor. The Abelian gauge field action is then simply,

$$\begin{aligned} S_{\rm gauge} &= \int d^4x\sqrt{-g}\mathcal{L}_{\rm gauge} \\ \sqrt{-g}\mathcal{L}_{\rm gauge} &= -\frac{1}{4}\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} \,. \end{aligned} \tag{122}$$

The equation of motion for the gauge field A_μ is obtained by varying (122) with respect to A_μ, and it reads,

$$\partial_\mu\Big(\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma}\Big) = 0 \,. \tag{123}$$

An important property of gauge fields is that in conformally flat space-times, whose metric can be written in the conformally flat form,

$$g_{\mu\nu} = a(x)^2\eta_{\mu\nu} \,, \qquad \eta_{\mu\nu} = {\rm diag}(1,-1,-1,-1) \,, \tag{124}$$

Eq. (123) reduces to the simple Maxwell equation,

$$\eta^{\mu\rho}\partial_\mu F_{\rho\sigma} = 0 \,. \tag{125}$$

In deriving this we made use of the metric tensor inverse $g^{\mu\nu} = a(x)^{-2}\eta^{\mu\nu}$, $\eta^{\mu\nu} = {\rm diag}(1,-1,-1,-1)$, and of $\sqrt{-g} = a^4$. We have just proved that gauge fields couple conformally to gravitation. An important consequence of this fact is that in conformal space-times (124), examples of which are de Sitter and power law inflationary space-times, as-well-as Friedmann-Lemaître-Robertson-Walker (FLRW) space-times, at the classical level, gauge fields do not couple to gravitation (they feel no gravitational pull). One says that in conformal space times gauge fields live in *conformal vacuum*, given by the appropriately normalised solution of (125). This lead to the popular belief that there cannot be much photon production through photon coupling to gravitation from the early universe epochs, which is in fact incorrect.

Eq. (118) implies that scalar fields do not in general couple conformally to gravitation. Indeed, inserting the conformally flat metric (124) into the scalar equation (118) results in

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi + 2\eta^{\mu\nu}\frac{\partial_\mu a}{a}\partial_\nu\phi + a^2\frac{dV(\phi)}{d\phi} + a^2\xi\mathcal{R}\phi = 0\,, \tag{126}$$

which shows that scalar fields do in general feel gravitational force in conformal space-times. If space-time is in addition homogeneous, in which case the scale factor $a = a(\eta)$ is a function of conformal time η only (defined as $dt = ad\eta$), because of the time derivative acting on ϕ, the second term in (126) looks like a damping term. The damping coefficient, $2\mathcal{H} = 2aH$, is given in terms of the Hubble parameter H, which is defined as $H(t) = (1/a)da/dt$, and therefore it is often called Hubble damping. Hubble damping has important consequences for cosmology, since it is in the crux of the mechanism for production of cosmological perturbations during inflationary epoch, which in turn seed structures of the Universe. We shall come back to this question when we discuss scalar and tensor cosmological perturbations.

10.3 Frame fields and fermionic fields

Due to the spinorial structure of fermion fields ψ, fermions transform nontrivially under general coordinate transformations, $\psi \to S\psi$, where S denotes a matrix in spinor space. For that reason, getting the covariant form of the Dirac equation requires special care. This is easiest done by making use of the frame field (also known as the *vierbein* or *tetrad*) formalism. The frame field can be defined in terms of the metric tensor as follows,

$$g_{\mu\nu}(x) = e_\mu^{\ a}(x)e_\nu^{\ b}(x)\eta_{ab}\,, \qquad \eta_{ab} = \mathrm{diag}(1,-1,-1,-1)\,, \quad (a,b = 0,1,2,3)\,, \tag{127}$$

and hence can be thought of as the transformation of the metric tensor to a locally flat coordinate system (known as the tangent space) with the metric tensor η_{ab}. The set of tangent spaces at all points of the space-time is known as the tangent bundle.

The following generalisation of the anticommutation relation for the Dirac matrices is very natural (since it is generally covariant)

$$\{\gamma_\mu, \gamma_\nu\} \equiv \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}\,. \tag{128}$$

This implies that the Dirac matrices acquire space-time dependence, which can be easily disentangled by making use of the *vierbein*,

$$\gamma_\mu(x) = e_\mu^{\ a}(x)\gamma_a\,, \tag{129}$$

where γ_a are the Dirac matrices of the corresponding flat (tangent) space erected at point x^μ, and they obey the standard flat space-time anticommutation relation,

$$\{\gamma_a, \gamma_c\} = 2\eta_{ac}\,, \qquad \eta_{ac} = \mathrm{diag}(1,-1,-1,-1)\,. \tag{130}$$

In order to obtain the covariant formulation of the Dirac equation, it is necessary to introduce a *spin connection* Γ_ν, which are 4×4 matrices in spinor space. The spin connection (a better name would be the spinor connection) is used to define the covariant derivative acting on Dirac spinors,

$$\nabla_\mu\psi \equiv \partial_\mu\psi - \Gamma_\mu\psi\,. \tag{131}$$

More precisely, the spin connection Γ_μ and the Levi-Cività connection, $\Gamma^\mu_{\alpha\beta}$ can be used to define the covariant derivative acting on any object, whose transformation properties under general coordinate transformations are known. For example, γ_μ has both one vector and two spinor indices, and hence the covariant derivative acting on γ_μ is given by

$$\nabla_\mu \gamma_\nu \equiv \partial_\mu \gamma_\nu - \Gamma^\alpha_{\mu\nu}\gamma_\alpha - \Gamma_\mu \gamma_\nu + \gamma_\nu \Gamma_\mu = 0\,. \tag{132}$$

The covariant derivative of the Dirac matrices must vanish, since there exists a coordinate transformation which transforms γ_μ to a locally space-time independent form, and in which $\nabla^\nu \to \partial^\nu$, implying (132). Given that γ_μ can be constructed from the *vierbeins*, Eqs. (129) and (132) determine the spin connection Γ_μ up to an additive multiple of the unit matrix.

We can now write the generally covariant form of the fermionic action,

$$S_{\rm fermion} = \int d^4x \sqrt{-g}\mathcal{L}_{\rm fermion} \tag{133}$$

$$\sqrt{-g}\mathcal{L}_{\rm fermion} = \sqrt{-g}\bar{\psi} i\gamma^\mu \nabla_\mu \psi - \sqrt{-g}\frac{m_\psi c}{\hbar}\bar{\psi}\psi\,, \tag{134}$$

where $\bar{\psi} = \psi^\dagger \gamma^0(x)$, $\nabla_\mu \psi = (\partial_\mu - \Gamma_\mu)\psi$, and m_ψ denotes the fermion mass. Note that the Lagrangian (134) is hermitean, as it should be. Upon varying the action (133–134) with respect to $\bar{\psi}$, we easily get the fermion equation of motion for curved space-times,

$$i\hbar\gamma^\mu(\partial_\mu - \Gamma_\mu)\psi - m_\psi c\psi = 0\,. \tag{135}$$

For example, for the conformally flat metric (124), the vierbeins are simply,

$$e^c_\mu(x) = \delta^c_\mu a(x)\,, \qquad e^\mu_c(x) = \delta^\mu_c a(x)^{-1}\,, \qquad (\mu = 0,1,2,3;\ c = 0,1,2,3)\,, \tag{136}$$

such that

$$\gamma^\mu = a^{-1}\delta^\mu_c \gamma^c\,, \tag{137}$$

where γ^c are the flat space Dirac matrices. After some algebra, one finds that following the Dirac equation (see Problem 1.7) holds,

$$\Big(\hbar\gamma^a\partial_a + iam_\psi\Big)\psi_{\rm cf} = 0\,, \qquad \psi_{\rm cf} = a^{3/2}\psi\,, \tag{138}$$

where here $\gamma^a = (\gamma^0, \gamma^i)$ denote the flat space-time Dirac matrices. From Eq. (138) we conclude that massless fermions couple conformally to gravitation, in the sense that the conformally rescaled massless fermion field $\psi_{\rm cf} \equiv a^{3/2}\,\psi$ does not couple to gravitation at the classical level.

As it is indicated in Eq. (138), the presence of a mass term breaks conformal coupling of fermions to gravitation. This effect has been used to motivate the study of fermion pair production in rapidly expanding space-times of the early Universe (inflationary epoch and early radiation era). The production is negligible today, since the rate of fermion pair production is determined by the rate of change of the effective mass term am, which is in turn proportional to the Hubble expansion rate today, which is tiny.

Finally, we recall that matter couples to gravitation through the stress-energy tensor, which can be calculated for any matter field $\phi_{\rm matter} \in \{\phi, \psi, A_\mu\}$, etc., by varying the appropriate matter action $S_{\rm matter} \in \{S_\phi, S_{\rm fermion}, S_{\rm gauge}\}$ with respect to the metric tensor (see Eq. (59)),

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{S_{\rm matter}}{\delta g^{\mu\nu}}\,. \tag{139}$$

11 Alternative theories of gravitation*

The Einstein theory of gravitation (or general relativity) has up to now passed all experimental tests. In the future nevertheless, we may witness emergence of a more accurate theory of gravitation, which reduces to the Einstein theory in a certain limit. Moreover, there are some observations on large (galactic and cosmological) scales, whose explanation may as well be found by extending the Einstein theory. One motivation to extend the Einstein theory is to provide alternative explanation for the missing matter problem of the Universe, which is standardly explained by adding the appropriate amount of nonbaryonic matter (*dark matter*), which apart from gravitational interaction interacts very weakly with visible matter. This explanation is now questioned by the work of Douglas Clowe et al. ("A Direct Empirical Proof of the Existence of Dark Matter," Ap. J. Lett. **648** (2006) L109L113 [arXiv:astro-ph/0608407]) where evidence is presented that the dark and visible matter of the merging *Bullet Cluster* 1E0657-558 are widely separated, see also `http://en.wikipedia.org/wiki/Bullet_Cluster`. This observation alone presents a strong support in favour of theories of dark matter, and it is very hard to explain within extended theories of gravitation that provide an alternative explanation for dark matter. Howver, Bullet Cluster seems to undergo a high-velocity merger (the relative velocity of the two clusters is around 4500 km/s), evident from the spatial distribution of the hot, X-ray emitting gas, which is difficult to account for within the standard cosmology with cold dark matter and Gaussian initial seeds for galaxy formation formed during an inflationary epoch.

Furthermore, the Universe appears spatially flat on cosmological scales, even though the amount of visible and dark matter makes up only about one-third of what is required to explain the observed flatness. It is possible to get a flat universe by adding a cosmological term of the right magnitude. The true explanation may as well be more subtle, and may arise from an alternative theory of gravitation, or from an exotic matter component which does not cluster, and which interacts with other matter only gravitationally, or very weakly.

Finally, there are good theoretical reasons, based on which one may argue that the Einstein theory cannot be the complete theory of gravitation. Namely, when the Einstein theory is canonically quantised, one obtains a perturbatively nonrenormalisable quantum field theory. This suggests that the true theory of quantum gravitation is more complex than the canonically quantised Einstein theory.

Here we briefly review just a couple of simple extensions of the Einstein theory. One should keep in mind that none of these examples solves the problem of perturbative nonrenormalisability of the Einstein theory.

A very simple extension of the Einstein theory is the Jordan-Fierz-Brans-Dicke theory (JFBD), (Jordan, 1949; Fierz, 1956; Brans and Dicke, 1961), in which the Newton constant is a function of a gravitational scalar field Φ, which in turn couples to the trace of the matter stress-energy tensor, and thus Φ can vary in space and time. The action of the theory has the form (in the Jordan-Fierz physical frame)

$$S_{\rm JFBD} = -\frac{c^4}{16\pi G_N^*}\int d^4x\sqrt{-g}\left[\Phi\mathcal{R} + \frac{\omega}{\Phi}g^{\mu\nu}(\partial_\mu\Phi)(\partial_\nu\Phi)\right] + S_{\rm matter}[\psi_{\rm matter}, g_{\mu\nu}]\,, \tag{140}$$

where $\psi_{\rm matter}$ denotes matter fields, ω is a dimensionless constant, Φ is a dimensionless gravitational scalar field, and G_N^* is the bare Newton constant, such that G_N^*/Φ reduces to the Newton constant G_N when $\Phi = $ const. The equations of motion are obtained by varying the action (140),

$$G_{\mu\nu} = \tfrac{8\pi G_N^*}{c^4}\tfrac{1}{\Phi}T_{\mu\nu} + \tfrac{\omega}{\Phi^2}\Big((\partial_\mu\Phi)(\partial_\nu\Phi) - \tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}(\partial_\alpha\Phi)(\partial_\beta\Phi)\Big) + \tfrac{\omega}{\Phi}\Big(\nabla_\mu\nabla_\nu\Phi - g_{\mu\nu}(g^{\alpha\beta}\nabla_\alpha\nabla_\beta\Phi)\Big)$$

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta\Phi \equiv \frac{1}{\sqrt{-g}}\partial_\alpha\Big(\sqrt{-g}g^{\alpha\beta}\partial_\beta\Phi\Big) = \frac{8\pi G_N^*}{c^4}\frac{1}{2\omega+3}T\,, \qquad (141)$$

where $T_{\mu\nu} = 2(-g)^{-1/2}\delta S_{\rm matter}/\delta g^{\mu\nu}$ is the matter stress-energy tensor, and $T = g^{\mu\nu}T_{\mu\nu}$ is its trace. From equations (141) we see that the JFBD theory reduces to the Einstein theory in the limit when $\omega \to \infty$. Since the measured Newton constant is $G_N = G_N^*/\Phi$, a variation in space or time in matter density will induce a variation of G_N. Up to this moment, no space-time variations of the Newton constant have been observed. Solar system observations place a lower bound of $\omega > 600$. On the other hand, the VLBI experiments place a stricter bound, $\omega > 3500$. Further tests of the JFBD theory are based on the observed constancy of the Newton constant. Indeed, since $G_N^{-1}dG_N/dt = \Phi^{-1}d\Phi/dt$, a lower limit on $G_N^{-1}dG_N/dt$ implies an upper limit on the time variation of Φ. In the JFBD theory one would naturally expect a variation of the order of the Hubble parameter today, $G_N^{-1}dG_N/dt \sim H_0 = 0.74 \pm 0.03 \times 10^{-10}$/year. The observed upper bound is significantly smaller, $G_N^{-1}dG_N/dt \leq 5 \times 10^{-12}$/year (Lunar ranging, Viking radar reflection). Based on this bound, one cannot yet rule out the JFBD theory, since the observations which lead to the limit are performed locally (in the Solar system and now). Since the Solar system is virialised, one does not sense that the Universe is expanding.

The Jordan-Fierz-Brans-Dicke theory belongs to a more general class of theories, which are known as the *scalar-tensor* theories (STe) of gravitation (Bergmann 1968, Nordtvedt 1970, Wagoner 1970), and whose action is obtained by generalising the action (140), such that ω becomes a function of the scalar field, $\omega = \omega(\Phi)$, and one adds a potential to Φ, $V = V(\Phi)$. In general in STe theories the cosmological term can be absorbed in V.

Another viable extension of the Einstein theory are the scalar-tensor-vector theories of gravitation (SVeTe). One example of SVeTe has recently been proposed by Beckenstein (2004) as a generally covariant version of the nonrelativistic MOND theory of gravitation (Milgrom), which was at once believed to provide an explanation for the missing matter problem on the galactic scales by changing the Newton law for very small accelerations. Another possible extension of the Einstein theory of gravitation is an old idea considered by Einstein. In this theory the metric tensor is extended to include the antisymmetric components, $g_{\mu\nu} \to \hat{g}_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu}$, where $B_{\mu\nu} = -B_{\nu\mu}$. While it has been shown that such theories are generically unstable, and hence not viable, stability may be restored by adding a mass term. Consequently, in such a theory the Newton force law gets modified on scales given by the inverse mass of the antisymmetric B-field. If the mass is appropriately chosen, such a theory may become a viable candidate for the explanation of the missing mass problem in galaxies, as well as for the enhanced photon lensing observed in galactic clusters (Moffat, 2004). Other ideas pursued by modern researchers include massive gravity theories and bi-metric theories of gravity. For a recent review on the subject, we suggest "Modified Gravity and Cosmology" by Timothy Clifton, Pedro G. Ferreira, Antonio Padilla and Constantinos Skordis (Phys.Rept. **513** (2012) 1-189, DOI: 10.1016/j.physrep.2012.01.001 [e-Print: arXiv:1106.2476 [astro-ph.CO]]).

Finally, we mention theories, whose action includes higher order curvature invariants, examples of which are, $\mathcal{R}^2$, $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$, $\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$, etc. Since the equations of motions in these theories contain higher order derivatives, they contain more than two independent solutions. Quite generically, some of these solutions are unstable. An extra effort is needed to construct a higher order derivative theory in which unstable modes are absent. This may be done only in very special cases, but most of these cases can be mapped onto a scalar-tensor theory, and thus they contain no new physics.

Problems

1.1. Maxwell's theory of electromagnetism. *(5 points)*
The special relativistic form of the Maxwell action coupled to matter can be written as,

$$\begin{aligned} \mathcal{S}_{\text{Maxwell+matter}} &= \int d^4x \Big(\mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{Maxwell}} \Big) \\ \mathcal{L}_{\text{Maxwell}} &= -\frac{1}{4}\eta^{\mu\alpha}\eta^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} \\ \mathcal{L}_{\text{matter}} &= -\eta^{\mu\nu}A_\mu j_\nu \,. \end{aligned} \tag{142}$$

where $j^\nu = (c\rho, \vec{j}\,)$ represents a charged matter current density, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the antisymmetric field-strength tensor. The F_{0i} components harbour the electric field strength, $E^i = F_{0i}$, while the F_{ij} components host the magnetic field strength, $B^i = -(1/2)\epsilon^{ijl}F_{jl}$; ϵ^{ijl} ($\epsilon^{123} = 1$) is a fully antisymmetric symbol in the indices $i, j, l = 1, 2, 3$.

By making use of the action principle, derive the following two (inhomogeneous) Maxwell's equations,

$$\nabla \cdot \vec{E} = j^0 \equiv c\rho \,, \qquad \nabla \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \vec{j} \tag{143}$$

How would you obtain the homogeneous Maxwell's equations,

$$\nabla \cdot \vec{B} = 0 \,, \qquad \nabla \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0\,? \tag{144}$$

1.2. The geodesic equation. *(5 points)*
By making use of the action principle, derive the geodesic equation for the 4-velocity of a point particle, $u^\mu = dx^\mu/d\tau$ from the following general relativistic action for a point particle,

$$S_{\text{point particle}} = m\int ds = mc\int d\tau \Big(g_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\Big)^{1/2} . \tag{145}$$

In proving this you may use $\nabla_\alpha g_{\mu\nu} = 0$ (see Problem 3d below). What is the special relativistic limit of the action (145)?

1.3. General covariance and tensors. *(10 points)*

(a) *(3 points)*
Show that

$$\int d^4x \sqrt{-g} \tag{146}$$

represents a generally covariant measure. The symbol $g = \det[g_{\mu\nu}]$ denotes the determinant of the metric tensor. You may find the following definition of the determinant of a 2-indexed tensor $t_{\mu\nu}$ useful,

$$\epsilon_{\mu\nu\rho\sigma}\det[t_{\mu\nu}] = \epsilon_{\alpha\beta\gamma\delta}t_{\mu\alpha}t_{\nu\beta}t_{\rho\gamma}t_{\sigma\delta} \,, \tag{147}$$

where $\epsilon_{\mu\nu\rho\sigma}$ represents a totally antisymmetric Levi-Cività δ-symbol in 3+1 dimensions, such that $\epsilon_{0123} = 1$, and it is antisymmetric under exchange of any two indices. The ϵ-symbol vanishes whenever any two indices are identical.

Show that $\sqrt{-g}\epsilon_{\mu\nu\rho\sigma}$ transforms as the components of a four-indexed covariant tensor, while $\epsilon_{\mu\nu\rho\sigma}/\sqrt{-g}$ transforms as the components of a contravariant tensor.

(b) *(2 points)*

Show that the covariant derivative of a two indexed contravariant tensor field $T^{\mu\nu}$ reads (*cf.* Eq. (40)),

$$\nabla_\alpha T^{\mu\nu} \equiv T^{\mu\nu}{}_{;\alpha} = T^{\mu\nu}{}_{,\alpha} + \Gamma^\mu_{\rho\alpha} T^{\rho\nu} + \Gamma^\nu_{\rho\alpha} T^{\mu\rho} \,. \tag{148}$$

(c) *(2 points)*

Show that the covariant derivative of the metric tensor vanishes,

$$\nabla_\alpha g_{\mu\nu} = 0 \,. \tag{149}$$

(d) *(3 points)*

Show that the Einstein curvature tensor $G_{\mu\nu} = \mathcal{R}_{\mu\nu} - (1/2)\mathcal{R} g_{\mu\nu}$ satisfies the following Bianchi identity

$$\nabla^\nu G_{\mu\nu} = 0 \,. \tag{150}$$

Hint: Show first the following cyclic derivative property for the Riemann curvature tensor,

$$\nabla_\gamma \mathcal{R}_{\mu\nu\alpha\beta} + \nabla_\alpha \mathcal{R}_{\mu\nu\beta\gamma} + \nabla_\beta \mathcal{R}_{\mu\nu\gamma\alpha} = 0 \,, \tag{151}$$

then covariantize it, and finally contract the appropriate indices.

1.4. The Schwarzschild metric. *(13 points)*

Consider the Schwarzschild line element (Karl Schwarzschild, 1916), which defines the metric of a static distribution of mass, which can be approximated by a mass M located at the origin $\vec{x} = \vec{0}$,

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu = (1 + 2\phi_N/c^2) c^2 dt^2 - \frac{dr^2}{1 + 2\phi_N/c^2} - r^2 \Big(d\theta^2 + \sin^2(\theta)\, d\varphi^2 \Big) \,, \tag{152}$$

where $\phi_N = -G_N M/r$ denotes Newton's potential, and $G_N = 6.673(10) \times 10^{-11}$ m^3/kg/s^2 Newton's constant. (For simplicity we set $c = 1$.)

(a) *(6 points)*

This solution can be derived by starting with the general spherically symmetric line element,

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2 \Big(d\theta^2 + \sin^2(\theta)\, d\varphi^2 \Big) \,, \tag{153}$$

where by spherical symmetry, $\nu = \nu(r, \tau)$ and $\lambda = \lambda(r, \tau)$ are functions of the radial distance $r = \sqrt{x^2 + y^2 + z^2}$ and time τ. By calculating the Levi-Cività connection,

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \Big(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta} \Big) \,. \tag{154}$$

and the Riemann and Ricci curvature tensor,

$$\mathcal{R}^\mu{}_{\alpha\beta\gamma} = \partial_\beta \Gamma^\mu_{\alpha\gamma} - \partial_\gamma \Gamma^\mu_{\alpha\beta} + \Gamma^\mu_{\sigma\beta} \Gamma^\sigma_{\gamma\alpha} - \Gamma^\mu_{\sigma\gamma} \Gamma^\sigma_{\beta\alpha} \,, \qquad \mathcal{R}_{\alpha\beta} = \mathcal{R}^\mu{}_{\alpha\mu\beta} \,, \tag{155}$$

show that the elements of the Einstein tensor

$$G_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} - \frac{1}{2} \mathcal{R} g_{\mu\nu} \,, \tag{156}$$

(here $\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$ denotes the Ricci scalar curvature) can be expressed in terms of the functions ν and λ defined in (153) as follows,

$$
\begin{aligned}
G^0_{\ 0} &= \mathrm{e}^{-\lambda}\Big(\frac{1}{r}\frac{d\lambda}{dr} - \frac{1}{r^2}\Big) + \frac{1}{r^2} \\
G^1_{\ 1} &= -\mathrm{e}^{-\lambda}\Big(\frac{1}{r}\frac{d\nu}{dr} + \frac{1}{r^2}\Big) + \frac{1}{r^2} \\
G^1_{\ 0} &= -\mathrm{e}^{-\lambda}\frac{1}{r}\frac{d\lambda}{d\tau} \\
G^0_{\ 1} &= \mathrm{e}^{-\nu}\frac{1}{r}\frac{d\lambda}{d\tau}
\end{aligned}
\tag{157}
$$

and

$$
G^2_{\ 2} = G^3_{\ 3} = -\frac{1}{2}\mathrm{e}^{-\lambda}\Big(\frac{1}{2}\frac{d\nu}{dr}\frac{d\lambda}{dr} + \frac{1}{r}\frac{d\lambda}{dr} - \frac{1}{r}\frac{d\nu}{dr} - \frac{1}{2}\Big(\frac{d\nu}{dr}\Big)^2 - \frac{d^2\nu}{dr^2}\Big) + \frac{1}{2}\mathrm{e}^{-\nu}\Big(\frac{d^2\lambda}{d\tau^2} + \frac{1}{2}\Big(\frac{d\lambda}{d\tau}\Big)^2 - \frac{1}{2}\frac{d\lambda}{d\tau}\frac{d\nu}{d\tau}\Big)\,,
\tag{158}
$$

and other elements of $G^\mu_{\ \nu}$ vanish.

(b) *(3 points)*

Upon imposing the sourceless (vacuum) Einstein equation, stating that the Einstein curvature tensor vanishes in the vacuum,

$$
G^\mu_{\ \nu} = 0\,,
\tag{159}
$$

and with a help of the Bianchi identity,

$$
\nabla^\nu G^\mu_{\ \nu} = 0\,,
\tag{160}
$$

show that Eqs. (157) imply Eqs. (158), such that only the following three equations are independent,

$$
\begin{aligned}
\mathrm{e}^{-\lambda}\Big(\frac{1}{r}\frac{d\lambda}{dr} - \frac{1}{r^2}\Big) + \frac{1}{r^2} &= 0 \\
\mathrm{e}^{-\lambda}\Big(\frac{1}{r}\frac{d\nu}{dr} + \frac{1}{r^2}\Big) - \frac{1}{r^2} &= 0 \\
\frac{d\lambda}{d\tau} &= 0\,.
\end{aligned}
\tag{161}
$$

(c) *(2 points)*

Next show that the general solution of these equations has the form,

$$
\begin{aligned}
r\mathrm{e}^{-\lambda} &= r + \text{constant} \\
\lambda + \nu &= h(\tau)\,,
\end{aligned}
\tag{162}
$$

where $h(\tau)$ denotes a general function of time. This solution is valid everywhere in space, except at the origin, $r = 0$. By the time reparametrization,

$$
t = t(\tau) = \int^\tau \mathrm{e}^{h(\tau')/2} d\tau'
\tag{163}
$$

this then reduces to the Schwarzschild solution (152).

(d) *(2 points)*

Discuss the physical meaning of the special points $r = 0$ and $r = 2G_N M/c^2$.

(e*) *(extra 2 points)*

What is the physical reason that there are no dynamical solutions?

NB: If you find it too difficult to solve the time dependent problem, make a stationary ansatz from the beginning, $\lambda = \lambda(r)$, $\nu = \nu(r)$. For this you will earn up to 10 points, if you solve (a)-(d) correctly.

1.5. Light deflection of a thermal sphere. *(7 points)*

Newtonian spherically symmetric gravitating systems of many particles satisfy the Poisson equation for the gravitational Newton potential ϕ_N,

$$\nabla^2 \phi_N = 4\pi G_N \rho_N \,, \tag{164}$$

where ρ_N denotes the mass density, which in a spherically symmetric system is a function of the distance r from the center of mass, $\phi_N = \phi_N(r)$. In an equilibrated system, the distribution of particles can be approximated by the thermal distribution function $f = f(r, v)$, which is a function of $v = |\vec{v}|$ and $r = |\vec{r}|$ only,

$$f = \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} \exp\Big(- \frac{v^2/2 + \phi_N}{\sigma^2} \Big) \,, \tag{165}$$

where v is particle's velocity, $\phi_N = \phi_N(r)$ Newton's gravitational potential, $\sigma^2 = \langle \vec{v}^2 \rangle / 3 \equiv k_B T/m$ and $\rho = \rho_1 \mathrm{e}^{-\phi_N/\sigma^2}$ the density of particles, and

$$\rho(r) = \int d^3 v f \,. \tag{166}$$

(a) *(2 points)*

Show that ϕ_N satisfies the equation of motion,

$$\frac{d^2}{dr^2}\phi_N + \frac{2}{r}\frac{d}{dr}\phi_N = 4\pi G_N \rho_1 \exp\Big(- \frac{\phi_N(r)}{\sigma^2} \Big) \,. \tag{167}$$

(b) *(2 points)*

Show that one analytic solution of this equation can be found, which is known as the *thermal sphere*. It reads

$$\begin{aligned} \rho(r) &= \frac{\sigma^2}{2\pi G_N r^2} \\ \phi_N(r) &= -\sigma^2 \ln \Big(\frac{\sigma^2}{2\pi G_N \rho_1 r^2} \Big) \end{aligned} \tag{168}$$

Next show that the mass inside a radius r reads,

$$M(r) = \frac{2 k_B T}{m G_N} r \,. \tag{169}$$

Discuss the limits $r \to 0$ and $r \to \infty$.

(c) *(3 points)*

Calculate the deflection angle of light in the presence of a mass distribution of a thermal sphere, by making use of the formula,

$$\vec{\alpha}(d) = -\frac{2}{c^2}\int d\ell\, \nabla_\perp \phi_N(\vec{x}) \quad \Longrightarrow \quad \alpha(d) = -\frac{2}{c^2}\int dy \partial_x \phi_N(x) \tag{170}$$

where $\nabla_\perp$ is the gradient operator in the lens plane, whose two components are transversal (perpendicular) to the photon path, ℓ is the distance along the light geodesic, and d is the shortest distance from the center of mass ($\vec{x} = 0$) to the geodesic.

Assume that mass distribution of an elliptical galaxy can be well approximated by a thermal sphere, with a typical dispersion of a velocity component $\sigma = 100$ km/s. Calculate the light deflection angle originating at a distant point source (quasar or galaxy).

1.6. The Friedmann equations. *(10+5* points)*
Consider the spatially flat metric (here we set $c = 1$),

$$g_{\mu\nu} = \mathrm{diag}(1, -a^2, -a^2, -a^2)\,. \tag{171}$$

(a) *(2 points)*

Calculate the corresponding Levi-Cività connection,

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}\Big(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}\Big)\,. \tag{172}$$

(b) *(3 points)*

Calculate the Riemann curvature tensor and the Ricci tensor, by making use of the expressions,

$$\mathcal{R}^\mu{}_{\alpha\beta\gamma} = \partial_\beta \Gamma^\mu_{\alpha\gamma} - \partial_\gamma \Gamma^\mu_{\alpha\beta} + \Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\gamma\alpha} - \Gamma^\mu_{\sigma\gamma}\Gamma^\sigma_{\beta\alpha}\,, \qquad \mathcal{R}_{\alpha\beta} = \mathcal{R}^\mu{}_{\alpha\mu\beta}\,, \tag{173}$$

and show that the Ricci tensor has the form,

$$\mathcal{R}_{00} = -3\frac{\ddot{a}}{a}\,, \qquad \mathcal{R}_{ij} = -\Big(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2}\Big) g_{ij}\,, \tag{174}$$

while the Ricci scalar reads,

$$\mathcal{R} \equiv g^{\mu\nu}\mathcal{R}_{\mu\nu} = -6\Big(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\Big)\,. \tag{175}$$

(c) *(2 points)*

By making use of the Einstein equation ($c = 1$)

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}\,, \tag{176}$$

where

$$G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R} \tag{177}$$

denotes the Einstein curvature tensor, Λ denotes the cosmological term, and the stress-energy tensor of an ideal fluid equals in the fluid rest frame, in which $u^\mu = (1, \vec{0}\,)$,

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu - g_{\mu\nu} p \,, \tag{178}$$

derive the Friedmann (Friedmann-Lemaître-Robertson-Walker, `FLRW`) equations,

$$\begin{aligned} H^2 \equiv \frac{\dot{a}^2}{a^2} &= \frac{8\pi G_N}{3}\rho + \frac{\Lambda}{3} \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G_N}{3}(\rho + 3p) + \frac{\Lambda}{3} \,. \end{aligned} \tag{179}$$

(d) *(3 points)*

Show that the covariant stress-energy conservation implies,

$$\dot{\rho} + 3H(p + \rho) = 0 \,. \tag{180}$$

Show that this is not an independent constraint, and that it can be derived from (179).

Discuss the solutions of equations (179–180) for the cases (1) $\rho = p = 0$, $\Lambda = \Lambda_0 = \text{const.}$, (2) $p = w\rho \propto 1/a^4$, $\Lambda = 0$ (what is the value of w in this case?), and (3) $\rho \propto 1/a^3$, $\Lambda = 0$ (what is the value of w in this case?).

(e) *(5* points)*

Generalize your treatment to a space-time with curved space sections, whose line element in spherical coordinates reads,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2(t)\frac{dr^2}{1 - kr^2} - a^2(t) r^2 \Big(d\theta^2 + \sin^2(\theta)\, d\varphi^2\Big) \,. \tag{181}$$

When $k = +1$ ($k = -1$) this metric describes an expanding universe with positively (negatively) curved spatial sections. For $k = 0$, the metric (181) reduces to the spatially flat metric (171). Derive the corresponding Friedmann equations.

1.7. *Fermions in curved space-times.* (*5 points)*
Consider the following covariant Dirac action for fermions in curved space times,

$$S_{\text{fermion}} = \int d^4x \sqrt{-g} \Big(\bar{\psi} i\gamma^\mu \nabla_\mu \psi + m_\psi \bar{\psi}\psi\Big) \,, \tag{182}$$

where $\bar{\psi} = \psi^\dagger \gamma^0(x)$ and m_ψ denotes the fermion mass. The covariant derivative acting on a fermion field is given in terms of the spin connection Γ_μ as,

$$\nabla_\mu \psi = (\partial_\mu - \Gamma_\mu)\psi \,, \tag{183}$$

which is in turn defined by

$$\nabla_\mu \gamma_\nu \equiv \partial_\mu \gamma_\nu - \Gamma^\alpha_{\mu\nu}\gamma_\alpha - \Gamma_\mu \gamma_\nu + \gamma_\nu \Gamma_\mu = 0 \,. \tag{184}$$

(a) *(3 points)*

Calculate the elements of the spin connection Γ_μ and of the Levi-Cività connection in homogeneous conformal space-times, whose metric is of the following conformally flat form,

$$g_{\mu\nu} = a(\eta)^2\eta_{\mu\nu} \equiv e^a_\mu(x)e^b_\nu(x)\eta_{ab}\,, \qquad \eta_{ab} = \mathrm{diag}(1,-1,-1,-1)\,, \quad (a,b=0,1,2,3)\,, \quad (185)$$

where $a = a(\eta)$ denotes the scale factor, which is a function of conformal time η (*e.g.* in de Sitter space-time, $a = -1/(H_I\eta)$ $(\eta < 0)$, where H_I denotes the Hubble parameter of de Sitter space). Show first that in conformal space-times the vierbein has the form

$$e^c_\mu(x) = \delta^c_\mu a(x)\,, \qquad e^\mu_c(x) = \delta^\mu_c a(x)^{-1}\,, \qquad (\mu = 0,1,2,3;\ c = 0,1,2,3)\,. \quad (186)$$

such that the Dirac matrices are,

$$\gamma^\mu(x) = e^\mu_{\ a}\gamma^a = a(t)^{-1}\delta^\mu_{\ a}\gamma^a\,. \quad (187)$$

(b) *(2 points)*

By making use of (182) and (183), show that the equation of motion for fermions in homogeneous conformal space-times can be written as

$$\Big(\gamma^0\partial_0 + \gamma^i\partial_i - iam_\psi\Big)\psi_c = 0\,, \qquad \psi_c = a^{3/2}\psi\,, \quad (188)$$

where here γ^0 and γ^i denote the flat space Dirac matrices. Comment on the physical implications of this result.

1.8. *The Jordan-Fierz-Brans-Dicke (JFBD) theory of gravitation. (3 + 5* points)*

(a) *(3 points)*

The equation of motion for the scalar gravitational field Φ in the JFBD theory reads,

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta\Phi \;\equiv\; \frac{1}{\sqrt{-g}}\partial_\alpha\Big(\sqrt{-g}g^{\alpha\beta}\partial_\beta\Phi\Big) = \frac{8\pi G_N^*}{c^4}\frac{1}{2\omega+3}T\,, \quad (189)$$

where G_N^* denotes the bare Newton constant, ω is a dimensionless constant, $T = g^{\mu\nu}T_{\mu\nu}$ is the trace of the matter stress-energy tensor, $T_{\mu\nu} = 2(-g)^{-1/2}\delta S_{\rm matter}/\delta g^{\mu\nu}$. Assume that within a FLRW universe, the dominant matter component is scaling as nonrelativistic particles, and estimate the variation of the Newton 'constant' with time, $\dot G_N/G_N$, where $G_N = G_N^*/\Phi$.

(b*) *(3* points)*

In the Jordan-Fierz (physical) frame, in which the JFBD action reads,

$$S_{\rm JFBD} = -\frac{c^4}{16\pi G_N^*}\int d^4x\sqrt{-g}\left[\Phi\mathcal{R} + \frac{\omega}{\Phi}g^{\mu\nu}(\partial_\mu\Phi)(\partial_\nu\Phi)\right] + S_{\rm matter}[\psi_{\rm matter}, g_{\mu\nu}]\,, \quad (190)$$

where $\psi_{\rm matter}$ denotes the matter fields. Show that the JFBD action in the Einstein (conformal) frame, which is defined by the following conformal rescaling of the metric tensor,

$$g_{\mu\nu} = A^2(\varphi)g^E_{\mu\nu}\,, \qquad A^2(\varphi) = \Phi^{-1}\,, \qquad \alpha(\varphi)^2 \equiv \left(\frac{d\ln(A)}{d\varphi}\right)^2 = \frac{1}{2\omega+3}\,, \quad (191)$$

reads

$$S_{\rm E} = -\frac{c^4}{16\pi G_N^*}\int d^4x\sqrt{-g^E}\left[\mathcal{R}^E + 2g^{E\mu\nu}(\partial_\mu\varphi)(\partial_\nu\varphi)\right] + S_{\rm matter}[\psi_{\rm matter}, A^2(\varphi)g^E_{\mu\nu}]\,, \quad (192)$$

with $g^E = \det[g^E_{\mu\nu}]$.

(c*) *(2* points)*

What are the equations of motion for $g^E_{\mu\nu}$ and φ in the Einstein frame?