GENERAL RELATIVITY

Tutorial problem set 13, 09.12.2016.

■ **PROBLEM 33** De Sitter-Schwarzschild solution.

This problem is similar to the first problem from last week's problem set. You will solve the Einstein's equation in vacuum ($T_{\mu\nu} = 0$) for the spherically symmetric metric, only now we will assume a non-vanishing cosmological constant $\Lambda > 0$. Start with the following ansatz for the line element,

$$ds^2 = -f(t,r)dt^2 + w(t,r)dr^2 + r^2 d\Omega^2 \ . \tag{33.1}$$

(a) Write down all the non-trivial components of the Einstein equation.

The non-vanishing components of the Riemann tensor for line element (33.1) are

$$R^0{}_{101} = \frac{1}{4f^2 w}\Big[2fw\partial_t^2 w - f(\partial_t w)^2 - w\partial_t f\partial_t w + f\partial_r f\partial_r w - 2wf\partial_r^2 f + w(\partial_r f)^2\Big] \ , \tag{33.2}$$

$$R^0{}_{202} = -\frac{r}{2fw}\partial_r f \ , \qquad R^0{}_{303} = -\frac{r}{2fw}\sin^2\vartheta\,\partial_r f \ , \tag{33.3}$$

$$R^0{}_{212} = -\frac{r}{2fw}\partial_t w \ , \qquad R^0{}_{313} = -\frac{r}{2fw}\sin^2\vartheta\,\partial_t w \ , \tag{33.4}$$

$$R^1{}_{212} = \frac{r}{2w^2}\partial_r w \ , \qquad R^1{}_{313} = \frac{r}{2w^2}\sin^2\vartheta\,\partial_r w \ , \qquad R^2{}_{323} = \frac{w-1}{w}\sin^2\vartheta \ , \tag{33.5}$$

and the non-vanishing components of the Ricci tensor are R_{00}, R_{11}, R_{22}, R_{33}, and $R_{01} = R_{10}$. You do not need to calculate the Ricci scalar explicitly (from the Riemann tensor), why?
Hint: examine the trace of the Einstein equation.

(b) Argue from the tr component of the Einstein equation that w is independent of time, $w(t,r) = w(r)$.

(c) By taking an appropriate linear combination of the remaining non-trivial components of the Einstein equation show that f must be of the form $f(t,r) = F(t)A(r)$, and $A(r) = 1/w(r)$. Further, argue that $F(t)$ can be subsumed into the redefinition of time, and needs not be considered.

(d) Finally, show that Einstein equation implies

$$A(r) = 1 - \frac{R_S}{r} - \frac{\Lambda r^2}{3} \ . \tag{33.6}$$

By requiring the standard Schwarzschild limit when $\Lambda \to 0$ fix the constant R_S.

(e) In these coordinates the horizons are located at the radii defined by $A(r) = 0$. One can determine their location by solving this condition, which is not so straightforward to solve. It is a cubic equation which can be written in a compact form as

$$x^3 - y^2(x-1) = 0 \ , \tag{33.7}$$

where $x = r/R_S$, $y = R_H/R_S$, and $R_H = \sqrt{3/\Lambda}$. There is a regime for y where this condition has no solutions $r > 0$, and a regime for y where it has two solutions $r > 0$. The second regime corresponds to two horizons (a black hole one and a cosmological one). In between these two regimes, there is a critical value $y = y_{cr}$ for which the horizons overlap exactly at $x = x_{cr}$. It means that (33.7) has a double root $r > 0$, and it can be factorized as

$$0 = (x - x_{cr})^2 (x - \alpha) \ , \tag{33.8}$$

where α is some constant smaller than zero, and therefore we are not interested in it. By equating (33.8) with (33.7) for $y = y_{cr}$ it is possible to determine what y_{cr} and x_{cr} are. From these values find the condition that M and Λ have to satisfy in order for the two horizons to exist.

How would you address the question of what the space-time looks like for $y < y_{cr}$?

■ PROBLEM 34 Gravitational redshift in Schwarzschild space-time.

Consider two observers in Schwarzschild space-time, at fixed spatial coordinates $(r_1, \vartheta_0, \varphi_0)$ and $(r_2, \vartheta_0, \varphi_0)$ outside of the Schwarzschild radius. Suppose that observer 1 (let's call him Marko) emits a light signal radially towards observer 2 (let's call him Bruno).

(a) How are the proper times of of these observers related to the coordinate time t?

Label the proper time interval (as measured by Marko) between the emission of of two light crests by $\Delta\tau_1$. What is the the relation between $\Delta\tau_1$, and Δt, where Δt is the interval of coordinate time between the emission of two light crests (you can view Δt as the time elapsed between the emission of two crests as measured by an observer at $r \to \infty$).

(b) What is the proper time interval $\Delta\tau_2$ (as measured by Bruno) between the two successive crests received? Express your result in terms $\Delta\tau_1$, R_S, r_1 and r_2.

(c) Compute the ratio of the two frequencies ν_1 (emitted by Marko), and ν_2 (received by Bruno), and show that in the limit $r_1, r_2 \gg R_S$ one has

$$\frac{\nu_2}{\nu_1} = 1 + \phi_1 - \phi_2 \ , \tag{34.1}$$

where ϕ_1 and ϕ_2 are gravitational potentials at respective coordinates.