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A PEDESTRIAN INTRODUCTION TO NON-EQUILIBRIUM QFT

DRAŽEN GLAVAN AND TOMISLAV PROKOPEC

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I. INTRODUCTION

These notes contain an introduction to non-equilibrium (quantum) physics. For simplicity, to a large extent the relevant non-equilibrium concepts are developed on quantum mechanical systems. How to generalize these results to a quantum field-theoretic setting is indicated when appropriate.

In the four 3-hour lectures the following topics will be covered (* indicates if time permits):

- The density operator and entropy
- Propagators and two-point functions
- The Schwinger-Keldysh or *in-in* formalism
- Resummation techniques
- Applications
- Classical limit*

II. DENSITY OPERATOR

A. Dynamical equations in quantum mechanics

In this section we shall mostly focus on the properties of the density operator for general Gaussian states. General Gaussian states approximate well physical states in many areas of physics, in particular in those situations where the evolution is generated by a quadratic Hamiltonian. Of course, true (cubic, quartic, *etc.*) interactions introduce non-Gaussianities in the system. However, in many applications interactions are weak and hence non-Gaussianities they introduce are small. Consequently, Gaussian states can well approximate evolution of a system if (a) the initial state is approximately Gaussian and if interactions are (in a well defined sense) weak. [1] And finally (and not least importantly) Gaussian states are simple. For these reasons we shall focus our attention to studying general Gaussian states, which are represented by the Gaussian density operator.

A quantum mechanical (ket) state $|\psi, t\rangle$ obeys a Schrödinger equation,

$$\imath\hbar\frac{d}{dt}|\psi, t\rangle = \hat{H}(t)|\psi, t\rangle\,, \tag{1}$$

where $\hat{H}(t)$ denotes a (time dependent, Schrödinger picture) Hamiltonian and $|\psi, t\rangle$ is some state that is an element of the Hilbert space of relevance for the corresponding problem. The Schrödinger equation (1) tells us nothing but that $\hat{T} \equiv \hat{H}/(\imath\hbar)$ generates time translations (just like the momentum operator $\hat{p}/(-\imath\hbar) \to \partial/\partial x$ generates translations in the x-direction) in the sense that for an infinitesimal δt, from Eq. (1) we have

$$|\psi, t + \delta t\rangle = \left[1 - \frac{i\delta t}{\hbar}\hat{H}(t)\right]|\psi, t\rangle\,. \tag{2}$$

Since the Hamiltonian is generally a Hermitean operator, $\hat{H}^\dagger = \hat{H}$, Eq. (1) implies that a bra state $\langle\psi, t| = (|\psi, t\rangle)^\dagger$ evolves according to,

$$-\imath\hbar\frac{d}{dt}\langle\psi, t| = \langle\psi, t|\hat{H}(t)\,. \tag{3}$$

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[1] Of course, there are important examples of strongly interacting physical theories, notable examples being the low-energy limit of QCD in high energy physics and the Hubbard model in condensed matter applications. Even in these cases the Gaussian state approximation may be reasonable, provided one correctly identifies the dynamical degrees of freedom of the strongly interacting theory. *E.g.* in QCD the low energy degrees of freedom are the (composite) mesons and baryons.

A general solution of Eqs. (1–3) can be written in terms of the evolution operator as,

$$|\psi,t\rangle = \hat{U}(t,t_0)|\psi,t_0\rangle\,, \qquad \langle\psi,t| = \langle\psi,t|\hat{U}(t_0,t)\,, \tag{4}$$

where $\hat{U}(t_0,t) = [\hat{U}(t,t_0)]^\dagger$ and $\hat{U}(t,t_0)$ satisfies the following equation of motion,

$$i\hbar\frac{d}{dt}\hat{U}(t,t_0) = \hat{H}(t)\,\hat{U}(t,t_0)\,, \tag{5}$$

whose solution is given in terms of a time-ordered exponential (also known as the Dyson series),

$$\begin{aligned}\hat{U}(t,t_0) &= T\exp\left[-\frac{\imath}{\hbar}\int_{t_0}^{t} dt'\hat{H}(t')\right]\,,\\ &= 1-\frac{\imath}{\hbar}\int_{t_0}^{t} dt'\hat{H}(t') + \Big(-\frac{\imath}{\hbar}\Big)^2\int_{t_0}^{t} dt'\hat{H}(t')\int_{t_0}^{t'} dt''\hat{H}(t'') + \cdots\\ &= 1-\frac{\imath}{\hbar}\int_{t_0}^{t} dt'\hat{H}(t') + \frac{1}{2!}\Big(-\frac{\imath}{\hbar}\Big)^2\int_{t_0}^{t} dt'\int_{t_0}^{t} dt''\,T[\hat{H}(t')\hat{H}(t'')] + \cdots\,.\end{aligned} \tag{6}$$

Notice that in the second line representation of the Dyson series (6) there are no $1/n!$ factors of the usual exponential, and that the integrations are such that the times in the Hamiltonian factors at the left are larger than those at the right. These $1/n!$ factors reappear when the integrations are extended to t, as indicated in the third line of (6). In this case the Hamiltonian factors are time ordered. Of course, in the simple case when the Hamiltonian is time independent, the Hamiltonians at different times commute, rendering time ordering obsolete. In this case the Dyson series reduces to the usual exponential. Similarly, $\hat{U}(t_0,t)$ can be written in terms of an anti-time ordered exponential,

$$\begin{aligned}\hat{U}(t_0,t) &= \hat{U}^\dagger(t,t_0) = \bar{T}\exp\left[\frac{\imath}{\hbar}\int_{t_0}^{t} dt'\hat{H}(t')\right]\,,\\ &= 1+\frac{\imath}{\hbar}\int_{t_0}^{t} dt'\hat{H}(t') + \Big(\frac{\imath}{\hbar}\Big)^2\int_{t_0}^{t} dt'\int_{t_0}^{t'} dt''\hat{H}(t'')\hat{H}(t') + \cdots\,,\end{aligned} \tag{7}$$

where now the Hamiltonian factors that correspond to a later time are pushed to the right. One can easily check by inspection that (4–7) constitute the solutions to (1–3) with the boundary condition $\hat{U}(t_0,t_0) = 1$. Furthermore, the (anti-)time ordering occurring in the above solutions are the principal reason for the Feynman (or the time-ordered) propagator to be the propagator of choice for calculations in quantum field theory such as those of transition amplitudes.

B. The density operator

We can now introduce the density operator for a general pure state $|\psi,t\rangle$ as,

$$\hat{\rho}_{\rm pure}(t) = |\psi,t\rangle\langle\psi,t|\,. \tag{8}$$

From (1–3) it immediately follows that the (Schrödinger picture) density operator $\hat{\rho}_{\rm pure}(t)$ satisfies a *von Neumann-Liouville* equation,

$$\imath\hbar\frac{d}{dt}\hat{\rho}_{\rm pure}(t) = [\hat{H}(t), \hat{\rho}_{\rm pure}(t)]\,, \tag{9}$$

where also the Hamiltonian $\hat{H}(t)$ is in the Schrödinger picture. A mixed state density operator is defined as a superposition (ensemble) of pure state density operators,

$$\hat{\rho}(t) = \sum_i \rho_i |i\rangle\langle i|\,, \quad \Big(\sum_i \rho_i = 1\Big)\,, \tag{10}$$

where $|i\rangle$ is some pure state (such as $|\Psi, t\rangle$) and ρ_i ($0 \leq \rho_i \leq 1$) is the probability that the system is in the state $|i\rangle$. Mixed density operators are useful for the description of the dynamics of (open) quantum statistical systems, in which we have incomplete knowledge about the system. Since the von Neumann-Liouville equation (9) is linear, it is also satisfied for a mixed density operator, such that we have in general,

$$\imath\hbar\frac{d}{dt}\hat{\rho}(t) = [\hat{H}(t), \hat{\rho}(t)]\,. \tag{11}$$

Furthermore, because $\mathrm{Tr}[\hat{\rho}_{\rm pure}] = 1$ (this follows from $\langle\Psi, t|\Psi, t\rangle = 1$), $\sum_i \rho_i = 1$ immediately implies

$$\mathrm{Tr}[\hat{\rho}] = 1\,, \tag{12}$$

whose physical interpretation is simple: (if a measurement is performed) the total probability for the system (*e.g.* a particle) to be found in any state of the Hilbert space must be equal to unity. Note that the normalization (12) is preserved under the evolution (11) (prove it!). Furthermore, for any pure state, $\mathrm{Tr}[\rho^n] = 1$ ($\forall n \in \mathbb{N}$) (show that!), where $\mathbb{N}$ is the set of non-negative integers. In fact one can take the following as the definition of mixed and pure states,

$$\begin{aligned} \texttt{mixed state}: &\quad \mathrm{Tr}[\rho^n] < 1 \quad (\forall n > 1) \\ \texttt{pure state}: &\quad \mathrm{Tr}[\rho^n] = 1 \quad (\forall n > 1)\,. \end{aligned} \tag{13}$$

This can be proven as follows. At any given time there must exist a (complete) basis $|i\rangle$ with respect to which $\hat{\rho}$ is diagonal (of course this basis is generally different at different times). In this basis $\hat{\rho} = \sum_i \rho_i |i\rangle\langle i|$, where $\rho_i = \langle i|\hat{\rho}|i\rangle$ and $0 \leq \rho_i \leq 1$ ($\forall i$). Now, if $\hat{\rho}$ is pure, then only one of the elements ρ_i (say j) does not vanish and $\rho_i = \delta_{ji}$, *i.e.* $\rho_j = 1$ (here for simplicity we neglect the possible complications due to a non-discrete nature of indices i). If, on the other hand, $\hat{\rho}$ is mixed, then $0 \leq \rho_i < 1$ ($\forall i$). This follows from $\sum_i \rho_i = 1$ and from the fact that at

least two elements ρ_i do not vanish. Now, $\rho_i < 1\,(\forall i)$ and $\sum_i \rho_i = 1$, immediately imply that $\mathrm{Tr}[\hat{\rho}^n] = \sum_i \rho_i^n < 1\ (\forall n > 1)$. Since $\mathrm{Tr}[\hat{\rho}^n]$ is independent of the basis, this must be true in general, completing the proof. An analogous consideration shows that, when $0 < n < 1$, then $\mathrm{Tr}[\hat{\rho}^n] > 1$.

The von Neumann-Liouville equation (9) has a different sign (see problem 1) than the *Heisenberg operator equation*, which is the evolution equation for any Hermitean operator,

$$\frac{d}{dt}\hat{\mathcal{O}}_H(t) = \frac{\imath}{\hbar}[\hat{H}(t), \hat{\mathcal{O}}_H(t)] + \Big[\frac{\partial}{\partial t}\hat{\mathcal{O}}_S(t)\Big]_H\,. \tag{14}$$

As opposed to the von Neumann-Liouville equation (9) the Heisenberg equation is written in the Heisenberg picture, which also means that the Hamiltonian is in the Heisenberg picture [2]. The partial derivative in the last term in (14) acts on the operator $\mathcal{O}_S$ in the Schrödinger picture, and thus it vanishes if $\mathcal{O}_S$ is time independent (a time dependence of $\mathcal{O}_S$ can emerge *e.g.* through time dependent backgrounds or couplings to external sources). It is easy to represent the general solutions of Eqs. (11–14) in terms of the evolution operator (6) as,

$$\hat{\rho}(t) = \hat{U}(t, t_0)\hat{\rho}(t_0)\hat{U}(t_0, t)\,, \qquad \hat{\mathcal{O}}_H(t) = \hat{U}(t_0, t)\hat{\mathcal{O}}_S(t)\hat{U}(t, t_0)\,. \tag{15}$$

Both $\hat{U}(t, t_0)$ and $\hat{U}(t_0, t)$ appear in the solutions for $\hat{\rho}(t)$ and $\hat{\mathcal{O}}_H(t)$, such as both time ordering and anti-time ordering is involved in constructing a general evolution of the density operators and of observables (which are represented by Hermitean operators). This will be used in section IV, where we discuss the Schwinger-Keldysh (or in-in) formalism. The evolution of physical observables is, of course, picture independent. The solutions (15) tell us how to obtain this time evolution by working in a specific picture. For example, a physical observable $\mathcal{O}(t)$ is given in the Schrödinger picture by,

$$\mathcal{O}(t) \equiv \langle \hat{\mathcal{O}}_S(t) \rangle = \mathrm{Tr}\left[\hat{\rho}(t)\hat{\mathcal{O}}_S(t)\right]\,, \tag{16}$$

while in the Heisenberg picture the relevant formula is,

$$\mathcal{O}(t) = \mathrm{Tr}\left[\hat{\rho}(t_0)\hat{\mathcal{O}}_H(t)\right]\,. \tag{17}$$

[2] We often switch between the Heisenberg and Schrödinger picture Hamiltonians, without explicitly denoting it. This should not introduce any confusion, because the picture is clear from the context, and the form of the Hamiltonian is picture independent, in the sense that $\hat{H}_S(t) = H(\hat{q}_S, \hat{p}_S; t)$, while $\hat{H}_H(t) = H(\hat{q}_H, \hat{p}_H; t)$, where H is the same function in both pictures. Even though the form of the Hamiltonian does not change, generally they are not the same, *i.e.* $\hat{H}_S(t) \neq \hat{H}_H(t)$.

The results (16) and (17) are obviously identical to each other (because of Eq. (15) and because operators under a trace can be cyclically moved without changing the result).

An important example of a mixed density operator is the thermal density operator, which equals to

$$\hat{\rho}_{\rm th} = \frac{{\rm e}^{-\beta\hat{H}}}{{\rm Tr}\left[{\rm e}^{-\beta\hat{H}}\right]}\,, \tag{18}$$

where $\beta = 1/(k_B T)$, k_B denotes the Stefan-Boltzmann constant and T the temperature. Obviously, $\hat{\rho}_{\rm th}$ is meaningfully defined only for time independent Hamiltonians, and $\hat{\rho}_{\rm th}$ is itself independent of time. This must be so because thermal states are time translation invariant. In many applications however, (18) is taken to be the density operator of a thermal state, even if the Hamiltonian is time dependent. This is justified provided the rate of the interactions Γ that are responsible for thermalisation is much larger than the rate of change of the Hamiltonian, *i.e.* when $\Gamma \gg (d/dt)\ln(H(t))$. This is known as adiabatic limit, and it is often used for approximate calculations in cosmology (thermal state of the plasma in the early Universe setting) and in condensed matter systems (when one adiabatically changes the temperature through a phase transition; the opposite limit is known as quench and the corresponding evolution cannot be approximated by the thermal density operator (18)).

C. Evolution Generated by Quadratic Hamiltonians

Let us consider the evolution of a particle in one spatial dimension generated by the following general quadratic Hamiltonian,

$$\hat{H}(t) = \frac{1}{2}\left[A(t)\hat{Q}^2 + B(t)\hat{P}^2 + C(t)\{\hat{Q},\hat{P}\} + 2D(t)\hat{Q} + 2E(t)\hat{P}(t)\right]\,. \tag{19}$$

Due to the Hermiticity of the Hamiltonian, $A(t), B(t), C(t), D(t)$ and $E(t)$ are real functions of time. Another possible term $\imath F(t)[\hat{q},\hat{p}] = -\hbar F(t)$ constitute a trivial contribution to $\hat{H}$, and hence can be ignored. For example, in the case of a simple harmonic oscillator (SHO), $A = m\omega^2$, $B = 1/m$, $C = D = E = 0$, and m and ω are the (time independent) particle's mass and frequency, respectively. Another example is a charged particle that couples to an external time-dependent electromagnetic field $A^\mu(t,\vec{x}) = (\phi_C, \vec{A})$. In this case the interaction is time dependent (and Lorentz invariant),

$$L_{\rm int}(t) = -\frac{e}{c}\left[\gamma\vec{v}\cdot\vec{A} - \phi_C\gamma\right]\,, \tag{20}$$

where $\vec{v} = d\vec{x}/dt$ is particle's velocity, $\gamma = (1 - \vec{v}^2/c^2)^{-1/2}$, c is the speed of light and e the electric charge. In the non-relativistic limit and in one dimension the coupling (20) reduces to a bi-linear coupling,

$$\hat{H}_{\rm int}(t) \to \frac{e}{c}\Big(\frac{p}{m}A - \phi_C\Big)\,,$$

such that in this case $E(t) \approx eA(t)/(mc)$, and $e\phi_C/c$ constitutes a trivial contribution.

In this subsection we discuss what kind of evolution is generated by the Hamiltonian (19). In particular we shall see how an initial Gaussian state changes due to the evolution induced by (19). The meaning of a general Gaussian state is made more precise below. The Hamiltonian (19) implies the following Heisenberg equations (in Heisenberg picture),

$$\begin{aligned}\frac{d\hat{Q}}{dt} &= C\hat{Q} + B\hat{P} + E \\ \frac{d\hat{P}}{dt} &= -A\hat{Q} - C\hat{P} - D\,,\end{aligned} \tag{21}$$

Let us now shift $\hat{Q}$ and $\hat{P}$ as follows,

$$\hat{Q} = \hat{q} + \bar{Q}\,, \qquad \hat{P} = \hat{p} + \bar{P} \tag{22}$$

where

$$\bar{Q}(t) = \langle\hat{Q}\rangle \equiv {\rm Tr}[\hat{\rho}_H\hat{Q}]\,, \qquad \bar{P}(t) = \langle\hat{P}\rangle \equiv {\rm Tr}[\hat{\rho}_H\hat{P}] \tag{23}$$

are the expectation values of $\hat{Q}$ and $\hat{P}$, which obey the classical equations of motion (because the Hamiltonian is quadratic) and $\hat{\rho}_H$ is the density operator in the Heisenberg picture. When $\hat{\rho}_H$ is pure, the following simplified equations, Eqs. (23) reduce to the usual formulae, *e.g.* ${\rm Tr}[\hat{\rho}_H\hat{Q}] \to \langle\psi, t_0|\hat{Q}(t)|\psi, t_0\rangle$. Inserting (22) into (21) yields

$$\frac{d\hat{q}}{dt} = C\hat{q} + B\hat{p} \tag{24}$$

$$\frac{d\hat{p}}{dt} = -A\hat{q} - C\hat{p}\,, \tag{25}$$

where we made use of the fact that the expectation values $\{\bar{Q}, \bar{P}\}$ obey the classical equations of motion,

$$\begin{aligned}\frac{d\bar{Q}}{dt} &= C\bar{Q} + B\bar{P} + E \\ \frac{d\bar{P}}{dt} &= -A\bar{Q} - C\bar{P} - D\,.\end{aligned} \tag{26}$$

We have thus shown that the operators for fluctuations around the mean, $\hat{q}$ and $\hat{p}$, obey simplified equations (25), in which $D(t)$ and $E(t)$ are set to zero. Conversely, the role of the terms $D(t)\hat{Q} +$

$E(t)\hat{P}(t)$ in the Hamiltonian (19) is to shift the expectation value of $\hat{Q}$ and $\hat{P}$. When $\hat{Q}$ and $\hat{P}$ are operators whose fluctuations are Gaussian (the meaning of which is made more precise below), then the linear terms in the Hamiltonian generate a displacement (shift) in $\hat{Q}$ and $\hat{P}$.

The quantum Gaussian states that describe motion of a quantum particle with non-vanishing $\bar{Q}$ and $\bar{P}$ are known as *coherent states*, and were originally introduced by Glauber [1, 2] to study the effects of a charged distribution on the photon (vacuum) state, and their properties are summarized in Appendix A. Essentially, a coherent state is defined as the eigenstate of the annihilation operator $\hat{a}$,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle\,, \tag{27}$$

where, for a simple harmonic oscillator (SHO),

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left[\hat{Q} + \imath\frac{\hat{P}}{m\omega}\right], \quad \alpha = \sqrt{\frac{m\omega}{2\hbar}}\left[\bar{Q} + \imath\frac{\bar{P}}{m\omega}\right].$$

Glauber observed that the canonical coupling of a photon field $\hat{A}^\mu(t,\vec{x}) = \int d^3x \mathrm{e}^{\imath\vec{k}\cdot\vec{x}/\hbar}\,\hat{A}^\mu(t,\vec{k}\,)$ to a (classical) distribution of moving charges described by a 4-current, $j^\mu(t,\vec{x}) = (c\rho,\vec{j})(t,\vec{x}\,) = \int d^3x \mathrm{e}^{\imath\vec{k}\cdot\vec{x}/\hbar}\,j^\mu(t,\vec{k}\,)$, induces an interaction of the form,

$$\int dt\hat{H}_{\rm int}(t) = -\frac{1}{c^2}\int d^4x j^\mu(x)\hat{A}_\mu(x) = -\frac{1}{c^2}\int \frac{d^3k}{(2\pi\hbar)^3}\Big[-\hat{A}^0(t,\vec{k}\,)\rho(t,\vec{k}\,) + \hat{A}^i(t,\vec{k}\,)j^i(t,\vec{k}\,)\Big]\,, \tag{28}$$

where $\hat{A}^\mu(t,\vec{k}\,) = \sum_{p=\pm}\epsilon^\mu_p(\vec{k})\hat{A}^p(t,\vec{k})$, and $\epsilon^\mu_p(\vec{k})$ $(p=\pm)$ are the two transverse polarization vectors of the dynamical photon. Since for each individual $\{\vec{k},p\}$ the interaction (28) is linear in the field operator $\hat{A}^p(t,\vec{k})$, a time dependent current $j^i(t,\vec{k}\,)$ will generate light in a coherent state, making thus coherent states a very natural choice for the description of photons in laser beams. The coherent state formalism is not only of fundamental importance for quantum optics; it has also many other applications such as optical lattices in quantum condensed matter. Since electromagnetism is a quantum field theory, one can represent a coherent photon state as a product state over the momenta of coherent states $|\alpha_{\vec{k},p}(t)\rangle$, each of which describes a photon of definite polarization p and momentum $\vec{k}$. Of course, as light propagates through medium (such as air, water or various types of transparent materials such as glasses and crystals), the photons in the beam will scatter off particles in the medium, thus decohering the light beam. These type of effects cannot be described within the formalism of coherent states, but they *can* be described by the more general Gaussian states which also contain information about the phase decoherence between individual photons in the beam. The Gaussian density operator introduced below is suitable for the description of photons in partially decohered laser beams.

An interesting question is what kind of physical effects can be produced by the remaining terms (A, B, C) in Eqs. (24–25). These time dependent terms can in general produce the effect of *squeezing*, and for that the theory of squeezed states has been developed (for a review see Refs. [3, 4] and for an interesting application to cosmology consult [5]). In general one can say that squeezed states are those which exhibit correlations between $\hat{Q}$ and $\hat{P}$, *i.e.* those for which the correlator $\langle\{\hat{q}, \hat{q}\}\rangle$ does not vanish. For example, for a simple harmonic oscillator, that implies that the fluctuations in $\hat{Q}$ and $\hat{P}/m\omega$ are not equal. From Eqs. (29–31) below it follows that a squeezing is dynamically produced if $A(t)\langle\hat{q}^2\rangle \neq B(t)\langle\hat{p}^2\rangle$, which will be quite generally the case when either A, B or C are time dependent. Some basic properties of *squeezed states* are presented in Appendix B, and their applications range from quantum optics (when laser beams propagate through transparent, anisotropic media) to early cosmology (where they are used to describe (scalar and tensor) cosmological perturbations, see *e.g.* [5]). Squeezed states are characterized by the squeeze factor $r(t)$ (roughly speaking $\mathrm{e}^{r(t)}$ measures the size of quantum fluctuations in $\hat{q}$ and in $\hat{p}$) and the squeeze phase $\phi(t)$ (which is the time dependent phase along which the squeezing occurs). Squeezed states are pure states, which means that (at each moment in time) there exists an axis which defines $\hat{q}'$ and $\hat{p}'$ and along which $\Delta q' \Delta p' = \hbar/2$. Squeezed coherent states represent the most general Gaussian pure states. In order to characterize the most general Gaussian state we need another parameter which characterizes state impurity, and that we discuss next. As an example where general Gaussian states can be relevant is the squeezed light obtained by passing a laser light through medium, which necessarily induces some amount of phase decoherence, thus making the light impure. Squeezed laser light has a wide range of applications since it allows for a more precise determination of the photon phase. For that reason gravitational wave observatories – such as the advanced LIGO – are planning to use squeezed light in order to improve the measurement precision of the gravitational wave amplitude.

By inspecting the Heisenberg equations (24–25) for $\hat{q}$ and $\hat{p}$, one may observe that they can be equivalently recast as a set of *three* Heisenberg equations for the (composite, equal time) operators $\hat{q}^2$, $\hat{p}^2$ and $\{\hat{q}, \hat{p}\}$ (of course, the commutator $[\hat{q}, \hat{p}] = \imath\hbar$ is trivial and needs no special consideration). These equations are,

$$\frac{d\hat{q}^2}{dt} = 2C\hat{q}^2 + B\{\hat{q}, \hat{p}\} \tag{29}$$

$$\frac{d\hat{p}^2}{dt} = -A\{\hat{q}, \hat{p}\} - 2C\hat{p}^2 \tag{30}$$

$$\frac{d\{\hat{q}, \hat{p}\}}{dt} = -2A\hat{q}^2 + 2B\hat{p}^2\,. \tag{31}$$

The first equation is obtained by multiplying (24) from the left by $\hat{q}$ and from the right by $\hat{p}$ and by adding the two; analogous procedures give the other two equations. Eqs. (29–31) are the Heisenberg picture operator equations, so they are identical to those written for the corresponding expectation values (which are obtained by multiplying with a time independent $\hat{\rho}_H(t) = \hat{\rho}(t_0)$ and taking a trace. At first it seems puzzling that we have started with *two* Heisenberg equations for the linear canonical operators and got an equivalent system of *three (sic!)* equations for the quadratic operators. So, one equation must be redundant. This is indeed so, and an inspection of Eqs. (29–31) shows that they contain a *conserved quantity*, namely, [3]

$$\frac{d\Delta^2(t)}{dt} = 0\,, \qquad \left(\frac{\hbar\Delta(t)}{2}\right)^2 = \langle(\hat{q})^2\rangle\langle(\hat{p})^2\rangle - \left[\left\langle\frac{1}{2}\{\hat{q},\hat{p}\}\right\rangle\right]^2. \tag{32}$$

$\Delta(t)$ is known as the Gaussian invariant of a (Gaussian) state. An interesting question is, of course, what is the physical significance of Δ. To answer that question let us introduce a new concept of the (Gaussian) entropy for quantum systems.

But before we do that, first observe that the form of $\Delta(t)$ in Eq. (32) suggests that it has something to do with the amount of fluctuations in a quantum system (with one degree of freedom). Indeed, one can show that for a general Gaussian state, $\hbar\Delta/2$ is limited from below by $\hbar/2$. More precisely, one can show that for pure states $\Delta = 1$ and for mixed states $\Delta > 1$. Thus we can formulate the following *generalized uncertainty relation*,

$$\begin{aligned} \texttt{pure state}: \ \langle\hat{q}^2\rangle\langle\hat{p}^2\rangle - \left[\left\langle\tfrac{1}{2}\{\hat{q},\hat{p}\}\right\rangle\right]^2 &= \frac{\hbar^2}{4} \\ \texttt{mixed state}: \ \langle\hat{q}^2\rangle\langle\hat{p}^2\rangle - \left[\left\langle\tfrac{1}{2}\{\hat{q},\hat{p}\}\right\rangle\right]^2 &> \frac{\hbar^2}{4}\,. \end{aligned} \tag{33}$$

Below we show that these relations are indeed correct. But before we do that, we need to introduce the concept of entropy for quantum systems.

D. Entropy of Quantum Systems

Eq. (33) is helpful, in that it tells us that Δ can be used to parametrize state purity and since entropy is a measure for (im-)purity in a state, it is reasonable to demand that the entropy for quantum states should satisfy the following properties:

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[3] The conserved composite operator exists and it is equal to, $\frac{1}{2}\{\hat{q}^2,\hat{p}^2\} - \frac{1}{4}\{\hat{q},\hat{p}\}^2$. For our purpose, Δ defined in (32) is a more useful quantity.

- the entropy S should be a function of Δ, $S = S(\Delta)$;
- the entropy of a pure state ($\Delta = 1$) should vanish;
- the entropy of a mixed state ($\Delta > 1$) should be strictly positive;
- the entropy should be a monotonically increasing function of Δ.

In fact, such an entropy exists. Namely, inspired by the Boltzmann's definition of entropy, $S = -f \ln(f)$ in terms of the (classical) phase space distribution function $f(t; q, p)$, von Neumann introduced the following definition of the entropy suitable for quantum systems, [4]

$$S_{\rm vN} = -\langle \ln(\hat{\rho}) \rangle = -{\rm Tr}\left[\hat{\rho}\ln(\hat{\rho})\right] . \tag{34}$$

We shall see that the von Neumann entropy of a Gaussian state satisfies all of the requirements mentioned above. While the definition (34) seems appealing, is is not as such very useful, since it is conserved, *i.e.* it carries no dynamical information about the system. This can be shown by making use of Eq. (11) and performing some simple operations which are legitimate under a trace,

$$\frac{d}{dt}S_{\rm vN} = -{\rm Tr}\left[\frac{d\hat{\rho}(t)}{dt}\ln(\hat{\rho}) + \frac{d\hat{\rho}(t)}{dt}\right] = \frac{\imath}{\hbar}{\rm Tr}\left\{[\hat{H}(t)\hat{\rho}(t) - \hat{\rho}(t)\hat{H}(t)](\ln(\hat{\rho}) + 1)\right\} = 0\,, \tag{35}$$

such that $S_{\rm vN}$ =const. Since this holds for general systems, it must also hold for Gaussian systems of interest to us. The von Neumann entropy has some of the right properties. Namely, when viewed in a diagonal basis $\{|i\rangle\}$, Eq. (34) implies, $S_{\rm vN} = \sum_i \rho_i \ln(1/\rho_i)$. Since all ρ_i are positive and less than unity, $0 \leq \rho_i \leq 1$, $S_{\rm vN} \geq 0$, and moreover $S_{\rm vN} = 0$ for a pure state (for which $\rho_i = \delta_{ji}$ for some j).

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[4] More generally, one can define the quantum Rényi entropy as,

$$S_{\rm Renyi} = -\frac{1}{\varepsilon}\langle \rho^{\varepsilon} \rangle\,, \quad (0 < \varepsilon < 1)\,.$$

The von Neumann entropy (34) can be then viewed as the special case of the Rényi entropy. Indeed, up to an irrelevant constant, the von Neumann entropy can be thought as the $\varepsilon \to 0$ limit of the Rényi entropy (show that!). (In fact, $S_{\rm Renyi} = -\frac{1}{\varepsilon}\langle \rho^{\varepsilon} - 1 \rangle$ reduces to the von Neumann entropy when $\varepsilon \to 0$.) One can show that the Rényi entropy satisfies all of the requirements (spelled out at the top of the page) that one imposes on a well defined entropy. The Rényi entropy is used because it is often easier to evaluate than the von Neumann entropy.

E. General Gaussian States

The density operator of a *general Gaussian state* is of the form,

$$\hat{\rho}(t) = \frac{1}{Z}\exp\left\{-\frac{1}{2}\left[\alpha(t)\hat{Q}^2 + \beta(t)\{\hat{Q},\hat{P}\} + \gamma(t)\hat{P}^2 + \delta(t)\hat{Q} + \eta(t)\hat{P}\right]\right\}, \tag{36}$$

where here $\hat{Q}$ and $\hat{P}$ are the Schrödinger picture operators and $\alpha(t), \beta(t), \gamma(t), \delta(t)$ and $\eta(t)$ are real functions of time (the reality follows immediately from $\hat{\rho}^\dagger = \hat{\rho}$). The (inverse) norm Z (also known as the partition function) is chosen such that $\mathrm{Tr}[\hat{\rho}] = 1$. Just as in the case of the quadratic Hamiltonian, $\delta(t)$ and $\eta(t)$ can be removed by a suitable shift of the operators, $\hat{Q} = \hat{q} + \bar{Q}$ and $\hat{P} = \hat{p} + \bar{P}$, where $\bar{Q} = \langle\hat{Q}\rangle$, $\bar{P} = \langle\hat{P}\rangle$ (show that!), resulting in a simpler density operator (see problem 3),

$$\hat{\rho}(t) = \frac{1}{Z}\exp\left\{-\frac{1}{2}\left[\alpha(t)\hat{q}^2 + \beta(t)\{\hat{q},\hat{p}\} + \gamma(t)\hat{p}^2\right]\right\}. \tag{37}$$

Next, the following (time dependent) transformation diagonalizes $\hat{\rho}(t)$,

$$\hat{b}(t) = \sqrt{\frac{\sigma}{2\alpha\hbar}}\left[\left(1 + \frac{\imath\beta}{\sigma}\right)\hat{q} + \imath\frac{\alpha}{\sigma}\hat{p}\right], \qquad \hat{b}^\dagger(t) = \sqrt{\frac{\sigma}{2\alpha\hbar}}\left[\left(1 - \frac{\imath\beta}{\sigma}\right)\hat{q} - \imath\frac{\alpha}{\sigma}\hat{p}\right], \tag{38}$$

where $\sigma = \sqrt{\alpha\gamma - \beta^2}$. Note that $\hat{b}(t)$ and $\hat{b}^\dagger(t)$ are Schrödinger picture operators that are in general time dependent and are not the creation and annihilation operators that diagonalize the Hamiltonian. The density operator then becomes simply,

$$\hat{\rho}(t) = \frac{1}{Z'}\exp\left(-\hbar\sigma(t)\hat{N}(t)\right), \qquad \hat{N}(t) = \hat{b}^\dagger(t)\hat{b}(t), \qquad Z' = \mathrm{e}^{\hbar\sigma/2}Z, \tag{39}$$

the operators $\hat{b}^\dagger(t)$ and $\hat{b}(t)$ define a Fock basis $\{|n\rangle\}$ in the usual way, $\hat{N}(t)|n\rangle = n|n\rangle$, $\hat{b}^\dagger(t)|n\rangle = \sqrt{n+1}|n+1\rangle$, $\hat{b}|n\rangle = \sqrt{n}|n-1\rangle$ $(n > 0)$ and $\hat{b}|0\rangle = 0$. $\hat{\rho}$ is diagonal in this basis,

$$\hat{\rho}(t) = \frac{1}{Z'}\sum_{n=0}^{\infty}\mathrm{e}^{-\hbar\sigma(t)n}|n\rangle\langle n|. \tag{40}$$

Requiring $\mathrm{Tr}[\hat{\rho}] = 1$ implies immediately,

$$Z' = 1 + \bar{n}(\sigma), \quad \bar{n} = \frac{1}{\mathrm{e}^{\hbar\sigma} - 1} \implies Z = \frac{1}{2\sinh[\hbar\sigma/2]}, \tag{41}$$

where $\bar{n}$ is the (average) *statistical particle number* defined by,

$$\bar{n}(t) = \langle\hat{N}(t)\rangle = \mathrm{Tr}[\hat{\rho}\hat{N}(t)]. \tag{42}$$

From this we see that, in thermal equilibrium $\hbar\sigma$ reduces to $\hbar\omega/(k_B T)$, but in general σ has nothing to do with temperature. Indeed, $\sigma(t)$ and $\bar{n}$ are well defined for time dependent, far from equilibrium systems with a non-vanishing squeezing and expectation values $(\bar{Q}, \bar{P})$. while temperature is well defined only for thermal systems which are invariant under time translations. More generally, thermal equilibrium can be meaningfully defined also for systems in which temperature varies slowly (adiabatically) in time, such as in the early universe setting. The statistical particle number $\bar{n}$ should not be confused with the usual particle number n that corresponds to the particle number operator associated with the creation and annihilation operators that diagonalize the Hamiltonian (these operators are usually defined for a SHO, but can also be generalized). In the Hamiltonian case, the number of particles associated with these operators tell us how much energy is pumped into the system (in units of the fundamental excitation energy $\hbar\omega$, ω being the angular frequency at which the state rotates in phase space). Indeed, the operators $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ in (38) do not necessarily diagonalize the Hamiltonian (19). An important example of such a situation is a squeezing Hamiltonian with a $C(t) \neq 0$, and an initial density operator of a thermal state (zero squeezing, $\beta = 0$). It is obviously not possible to simultaneously diagonalize both the Hamiltonian and the density operator neither at the initial time nor at any later time.

One can show that the statistical particle number of a Gaussian state whose evolution is generated by the general quadratic Hamiltonian (19) is conserved. In fact, it is closely related to the Gaussian invariant $\Delta(t)$ discussed above (32),

$$\bar{n}(t) = \frac{\Delta(t) - 1}{2}\,. \tag{43}$$

This can be easily shown from the identities,

$$\langle \hat{q}^2 \rangle = -2\hbar\partial_\alpha \ln[Z] = \hbar\left(\bar{n} + \frac{1}{2}\right)\frac{\gamma}{\sigma}\,,$$

etc., see problem 3. In order to get a better understanding of the physical significance of $\bar{n}$, let us calculate the von Neuman entropy (34) of the Gaussian state (36). The calculation is easily perfomed in the Fock basis $\{|n\rangle\}$ in which $\hat{\rho}$ is diagonal (40). The result is,

$$S_{\rm vN} = (\bar{n}+1)\ln(\bar{n}+1) - \bar{n}\ln(\bar{n})\,. \tag{44}$$

(As an exercise, calculate the Rényi entropy defined at the end of section II D.) This is the well known result from statistical physics for the entropy of $\bar{n}$ identical, noninteracting (Bose) particles. This begs the following interpretation of $\bar{n}$:

The statistical particle number $\bar{n}$ signifies the number of uncorrelated (decohered) regions in a Gaussian state.

Therefore, the knowledge of $\bar{n}$ is of crucial importance for determining whether a state is pure or mixed. Indeed, Eqs. (43) and (33) imply the following *generalized uncertainly relation*,

$$\langle\hat{q}^2\rangle\langle\hat{p}^2\rangle - \left[\left\langle \tfrac{1}{2}\{\hat{q},\hat{p}\}\right\rangle\right]^2 = \left(\bar{n}+\frac{1}{2}\right)^2\hbar^2\,, \tag{45}$$

such that a state is pure (mixed) if $\bar{n}=0, \sigma\to\infty$ ($\bar{n}>0, 0<\sigma<\infty$). A simple calculation gives, $\mathrm{Tr}[\hat{\rho}^2] = 1/(1+2\bar{n})$, from which we conclude that $\hat{\rho}$ is pure when $\bar{n}=0$, and it is mixed when $\bar{n}>0$, in agreement with the conclusions reached above. The functional form $\bar{n} = 1/[\exp(\hbar\sigma)-1]$ with $\sigma\geq 0$ tells us that the generalized uncertainty relation is saturated when $\bar{n}=0$ (pure state), which also proves that for Gaussian states $\Delta\geq 1$, which is what we have promised above to prove.

By replacing $\hbar^2$ by $|[\hat{q},\hat{p}]|^2$ on the right hand side of (45) one gets the following inequality,

$$\bar{n}\gg 1 \implies \langle\hat{q}^2\rangle\langle\hat{p}^2\rangle - \left[\left\langle\frac{1}{2}\{\hat{q},\hat{p}\}\right\rangle\right]^2 \gg \left|\left[\hat{q},\hat{p}\right]\right|^2, \tag{46}$$

which can be understood as the criterion for *classicality* of a quantum state. Eq. (46) tells us that, if a state contains a large amount of mixing, operator ordering is not important, such that in dynamical equations for n-point functions one can (a) ignore operator ordering ambiguities and (b) neglect and contributions that contain commutators of fields. When this is the case one speaks of classical (stochastic) limit of quantum theory. The resulting classical theory is stochastic because it still contains (large) fluctuations which can be described by weighing classical evolution with suitable probability distributions.

This completes our description of Gaussian states. To summarize, a general Gaussian state of a particle is characterized by five functions (which are in the density operator (36) represented by real functions of time α, β, γ, δ and η), whose physical meaning is as follows:

1. $\bar{Q}(t)$ and $\bar{P}(t)$ characterizing the classical (expectation) values of $\hat{Q}$ and $\hat{P}$ and which define the coherent state parameter $\alpha(t)$ through $\alpha(t) = \sqrt{m\omega/(2\hbar)}\,[\bar{Q}+\imath\bar{P}/(m\omega)]$;

2. the squeeze factor $r(t)$ and the squeeze phase $\phi(t)$;

3. the statistical particle number $\bar{n}(t) = (\mathrm{e}^{\hbar\sigma}-1)^{-1}$, $\sigma = \sqrt{\alpha\gamma-\beta^2}$.

More details about Gaussian states can be found in *e.g.* Refs. [6, 7], where one can also find how to study simple quantum Gaussian fermionic systems.

The main conclusions of this subsection are: the Gaussian nature of the general Gaussian state is preserved by the evolution generated by the general quadratic Hamiltonian, and moreover there are quantities – the statistical particle number and the von Neumann entropy – that are conserved under the evolution. The question that deserves a clarification is: What is, if any, usefulness of the concept of the von Neumann entropy, if it is conserved? In order to answer this question, it is useful to introduce the concept of open systems. Namely, realistic physical systems are so complex that it is not conceivable that one would be able to measure all independent observables. For example, in presence of (cubic, quadratic, *etc.*) interactions, the knowledge of an infinite set of expectation values of Hermitean operators, $\langle \hat{Q}^n \rangle$ ($n \in \mathbb{N}$) (plus all other correlators that involve powers of $\hat{P}$) is generally required to fully specify the system at a given moment in time. It is not conceivable that an observer would be able to construct an apparatus that is able to measure all of these correlators. Hence, not all information about such a system is accessible to the observer. In this case it is natural to split the system into two parts: the part of the system accessible to the observer (which defines the *system*) and the part that is inaccessible (which defines the *environment*). This split the system+environment defines an open system, see figure 1, in the sense that not all information about the system is accessible to an observer. What the observer's apparatus can measure then defines the split. Now, there will be non-vanishing correlations between the system and the environment (which are often represented by the entanglement), which will not be accessible to the observer. The inability to measure these correlators induces a non-vanishing, time dependent, von Neumann entropy of this reduced system, which provides a useful information about the evolution of the system, bringing thus the concept of entropy of quantum systems back to the realm of useful physical quantities. A simple example is the following interaction Hamiltonian, $\hat{H}_{\rm int} = \sum_i^N \lambda_i \hat{x}_i \hat{q}$, where $\hat{x}_i$ is a position of an environmental oscillator i, and λ_i is a coupling strength. In this case one can show that Δ is not in general conserved. Indeed, $(d/dt)\Delta^2 \propto \lambda_i \langle \hat{x}_i \hat{q} \rangle, \lambda_i \langle \hat{x}_i \hat{p} \rangle$, and these correlators do not in general vanish. While this interaction is illustrative, it is bi-linear, and by using conventional methods of linear algebra, one can diagonalize the problem, and find a set of variable in which the problem is diagonal, such that the total entropy (which is the sum of the entropies of the individual degrees of freedom in the diagonal basis) will be conserved. However, when a genuine cubic or other higher order interaction is present, conventional diagonalisation methods do not work. Then the inaccessible higher order (non-Gaussian) correlators (that generally do not vanish) define in a natural way the $S - E$ split. Such higher order interactions are ubiquitous in quantum field theories (QFTs) such as the standard model of elementary particles and the BCS Hamiltonian

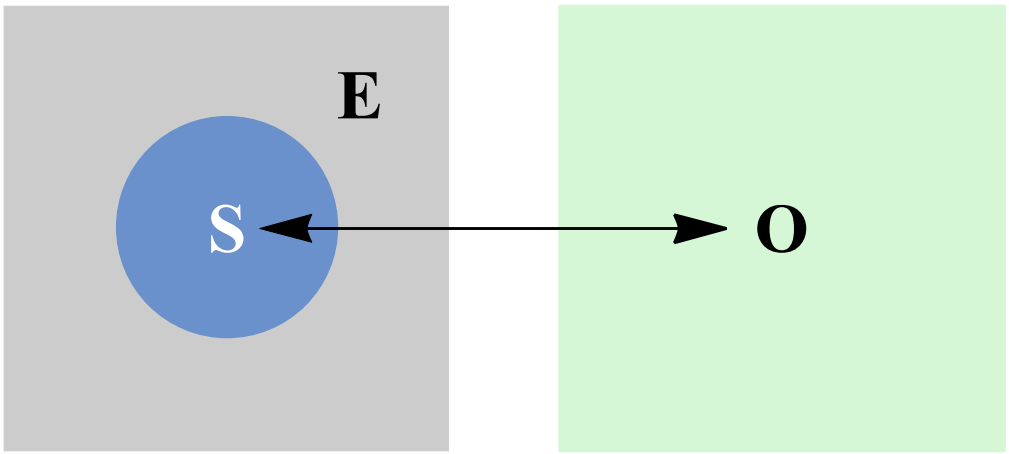

FIG. 1: An open system (S) couples to an environment E. An observer O couples to S, but it does not couple (directly) to E, and thus has no access to the information stored in the S-E correlators (which are at the level of the density operator stored in the S-E entanglement).

describing super-conductors.

III. PROPAGATORS AND TWO-POINT FUNCTIONS

In this section we discuss various two point functions which are useful for addressing time dependent problems both in quantum mechanics and in quantum field theory (QFT). We begin with the Feynman propagator for simple harmonic oscillator (SHO), and then we discuss the (positive and negative frequency) Wightman functions, the causal (Pauli-Jordan) and statistical (Hadamard) two point function and, last but not least, the retarded and advanced propagators. The two point functions are of essential importance for perturbative calculations, as they constitute essential part of the Feynman rules for non-equilibrium quantum (field) theories.

A. The Feynman Propagator and Wightman Functions in the Operator Formalism

Let us consider the following operator (in the Heisenberg picture),

$$\hat{\mathcal{O}}_F(t) = T[\hat{q}(t)\hat{q}(t')] = \Theta(t-t')\hat{q}(t)\hat{q}(t') + \Theta(t'-t)\hat{q}(t')\hat{q}(t)\,. \tag{47}$$

which satisfies the Heisenberg equation (14) and it is symmetric under the exchange of times, $t \leftrightarrow t'$. The *Feynman propagator* is defined as the expectation value of this operator,

$$\imath\Delta_F(t;t') \equiv \langle T[\hat{q}(t)\hat{q}(t')]\rangle = \mathrm{Tr}\Big[\hat{\rho}_H T[\hat{q}(t)\hat{q}(t')]\Big]\,. \tag{48}$$

It can be solved analytically for Gaussian states and quadratic Hamiltonians and for some integrable models, but in general it can be determined only within some approximation scheme, such as perturbation theory. Assume, for simplicity, that the evolution is generated by the Hamiltonian of an SHO,

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{q}^2\,. \tag{49}$$

Using the canonical quantization we have,

$$[\hat{q},\hat{p}] = \imath\hbar\,, \tag{50}$$

resulting in the Heisenberg equations of motion (14),

$$\frac{d\hat{q}}{dt} = \frac{\hat{p}}{m}\,, \qquad \frac{d\hat{p}}{dt} = -m\omega^2\hat{q} \;\Rightarrow\; \Big(-\frac{d^2}{dt^2} - \omega^2\Big)\hat{q} = 0\,, \tag{51}$$

which are solved by,

$$\begin{aligned}\hat{q}(t) &= \hat{q}_0\cos(\omega t) + \frac{\hat{p}_0}{m\omega}\sin(\omega t)\,,\\ \hat{p}(t) &= \hat{p}_0\cos(\omega t) - m\omega\hat{q}_0\sin(\omega t)\,,\end{aligned} \tag{52}$$

where $\hat{q}_0 = \hat{q}(t_0 = 0)$, $\hat{p}_0 = \hat{p}(t_0 = 0)$ and $[\hat{q}_0, \hat{p}_0] = \imath\hbar$. In order to find out what is the equation of motion for the operator (47) (and equivalently for the Feynman propagator (48)), let us act with the time derivative on $\mathcal{O}_F$,

$$\frac{d}{dt}\mathcal{O}_F(t) = \delta(t-t')[\hat{q}(t), \hat{q}(t')]_{t'=t} + \Theta(t-t')\frac{d\hat{q}(t)}{dt}\hat{q}(t') + \Theta(t'-t)\hat{q}(t')\frac{d\hat{q}(t)}{dt}\,. \tag{53}$$

The first term obviously vanishes, and we can take a second time derivative to obtain,

$$\frac{d^2}{dt^2}\mathcal{O}_F(t) = \delta(t-t')\Big[\frac{d\hat{q}(t)}{dt}, \hat{q}(t')\Big]_{t'=t} + \Theta(t-t')\frac{d^2\hat{q}(t)}{dt^2}\hat{q}(t') + \Theta(t'-t)\hat{q}(t')\frac{d^2\hat{q}(t)}{dt^2}\,. \tag{54}$$

Now, the first term can be evaluated from the canonical quantization (50), such that, upon making use of the Heisenberg equation of motion (51), we get

$$\Big(-\frac{d^2}{dt^2} - \omega^2\Big)T[\hat{q}(t)\hat{q}(t')] = \frac{\imath\hbar}{m}\delta(t-t')\,, \tag{55}$$

which is a C-number result. Because $\hat{\rho}_H$ in (48) is time independent, the Feynman propagator (48) obeys an identical equation of motion. Conversely, we can take the following as the defining equations for motion for the Feynman propagator,

$$\Big(-\frac{d^2}{dt^2} - \omega^2\Big)\imath\Delta_F(t;t') = \frac{\imath\hbar}{m}\delta(t-t')\,, \qquad \imath\Delta_F(t;t') = \imath\Delta_F(t';t) \tag{56}$$

with suitable boundary conditions, which are to be discussed below. Notice that the symmetry requirement in (56) can be equivalently replaced by a second differential equation,

$$\Big(-\frac{d^2}{dt'^2} - \omega^2\Big)\imath\Delta_F(t;t') = \frac{\imath\hbar}{m}\delta(t-t')\,. \tag{57}$$

The Feynman propagator depends on the choice of the (initial) state, such that the general solution to (56) is far from unique.

We shall illustrate this non-uniqueness of the Feynman propagator by looking in more detail at the solution of the problem (56). For clarity, we shall first seek the solution in the operator formalism, and then by transforming to frequency space. For that, it is convenient to reexpress the solutions (52) in terms of the standard annihilation and creation operators,

$$\hat{q}_0 = \sqrt{\frac{\hbar}{2m\omega}}[\hat{a} + \hat{a}^\dagger] \;\to\; \hat{q}(t) = \sqrt{\frac{\hbar}{2m\omega}}\Big[\hat{a}\mathrm{e}^{-\imath\omega t} + \hat{a}^\dagger\mathrm{e}^{\imath\omega t}\Big]\,. \tag{58}$$

$\hat{a}$ and $\hat{a}^\dagger$ define a Fock basis $\{|n\rangle\}$ in the usual way ($\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$, $\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$) that diagonalizes the Hamiltonian (49), $\hat{H} \to \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)$. Inserting this into Eq. (48) one gets,

$$\begin{aligned}\imath\Delta_F(t;t') \;=\; & \Theta(t-t')\frac{\hbar}{2m\omega}\Big[\langle\hat{a}\hat{a}^\dagger\rangle\mathrm{e}^{-\imath\omega(t-t')} + \langle\hat{a}^\dagger\hat{a}\rangle\mathrm{e}^{\imath\omega(t-t')} + \langle\hat{a}^2\rangle\mathrm{e}^{-\imath\omega(t+t')} + \langle(\hat{a}^\dagger)^2\rangle\mathrm{e}^{\imath\omega(t+t')}\Big] \\ & + \;\Theta(t'-t)\frac{\hbar}{2m\omega}\Big[\langle\hat{a}\hat{a}^\dagger\rangle\mathrm{e}^{\imath\omega(t-t')} + \langle\hat{a}^\dagger\hat{a}\rangle\mathrm{e}^{-\imath\omega(t-t')} + \langle\hat{a}^2\rangle\mathrm{e}^{-\imath\omega(t+t')} + \langle(\hat{a}^\dagger)^2\rangle\mathrm{e}^{\imath\omega(t+t')}\Big]\,. \end{aligned} \tag{59}$$

The terms multiplying the second Heaviside function $\Theta(t'-t)$ are the complex conjugate of those multiplying $\Theta(t-t')$, and hence contain no additional information. Therefore, the general structure of the Feynman propagator can be written as,

$$\imath\Delta_F(t;t') = \Theta(t-t')\imath\Delta^+(t;t') + \Theta(t'-t)\imath\Delta^-(t;t')\,, \tag{60}$$

where (see Eq. (47))

$$\imath\Delta^+(t;t') = \langle\hat{q}(t)\hat{q}(t')\rangle\,, \qquad \imath\Delta^-(t;t') = \langle\hat{q}(t')\hat{q}(t)\rangle \tag{61}$$

denote the positive and negative frequency *Wightman functions*, respectively, and in general we have,

$$\imath\Delta^-(t;t') = [\imath\Delta^+(t;t')]^*\,. \tag{62}$$

Unlike the Feynman propagator (56), the Wightman functions satisfy a homogeneous equation of motion and they are not symmetric in $(t;t')$, instead they transform to each other under $t \leftrightarrow t'$ (see Eq. (61)),

$$\Big(-\frac{\partial^2}{\partial t^2} - \omega^2\Big)\imath\Delta^\pm(t;t') = 0\,, \qquad \imath\Delta^\pm(t;t') = \imath\Delta^\mp(t';t)\,. \tag{63}$$

Another useful propagator (that occurs in the evolution operator $\hat{U}^\dagger(t;t_0)$ of the anti-time ordered Dyson series) is the Dyson (anti-Feynman, anti-time ordered) propagator, which can as well be expressed in terms of the Wightman functions as,

$$\imath\Delta_{\bar{F}}(t;t') \equiv \imath\Delta_D(t;t') = \Theta(t-t')\imath\Delta^-(t;t') + \Theta(t'-t)\imath\Delta^+(t;t')\,. \tag{64}$$

The Dyson propagator obeys the equation,

$$m(-\partial_t^2 - \omega^2)\imath\Delta_{\bar{F}}(t;t') = -\imath\hbar\delta(t-t')\,, \tag{65}$$

$(\partial_t^2 \equiv \partial^2/\partial t^2)$ which is symmetric under the exchange of times,

$$\imath\Delta_{\bar{F}}(t;t') = \imath\Delta_{\bar{F}}(t';t) \tag{66}$$

and in the operator formalism it can be defined as the anti-time ordered product of two position operators,

$$\imath\Delta_{\bar{F}}(t;t') = \langle\bar{T}[\hat{q}(t)\hat{q}(t')]\rangle\,. \tag{67}$$

Let us now try to understand the origin of the name 'positive' and 'negative' frequency in the Wightman functions. From (59) and (60) we see that in the vacuum $|0\rangle$, in which,

$$\hat{a}|0\rangle = 0\,, \qquad \langle 0|\hat{a}^\dagger = 0\,, \tag{68}$$

and in which the only non-vanishing contribution comes from, $\langle \hat{a}\hat{a}^\dagger \rangle = 1$, the positive and negative frequency Wightman functions are,

$$[\imath\Delta^+(t;t')]_{\rm vac} = \frac{\hbar}{2m\omega} {\rm e}^{-\imath\omega(t-t')}\,, \qquad [\imath\Delta^-(t;t')]_{\rm vac} = \frac{\hbar}{2m\omega} {\rm e}^{\imath\omega(t-t')}\,. \tag{69}$$

From this result we see that the positive (negative) frequency refers to the form of the time evolution, $\propto {\rm e}^{-\imath\omega(t-t')}$ ($\propto {\rm e}^{+\imath\omega(t-t')}$), which corresponds to the evolution generated at the positive (negative) frequency pole of the Wightman function (more precise statements are made below when we discuss the Wightman functions in frequency space). While in quantum mechanics the Wightman functions and propagators of the vacuum are time-translation invariant, the corresponding symmetry in QFT is the much larger Poincaré group.

In a more general (Gaussian) state (when the system is not in the vacuum state $|0\rangle$) there will be more contributions in (59). For example, if the state is (initially) mixed, but not squeezed, then

$$\langle \hat{a}\hat{a}^\dagger \rangle = 1 + \bar{n}_0\,, \qquad \langle \hat{a}^\dagger\hat{a} \rangle = \bar{n}_0\,, \tag{70}$$

where

$$\bar{n}_0 = \langle \hat{N}_0 \rangle = {\rm Tr}[\hat{\rho}_H \hat{N}_0] \tag{71}$$

denotes the average particle number $n_0 = \bar{n}(t_0 = 0)$ in the initial state $\hat{\rho}_H$ associated with the number operator $\hat{N}_0 = \hat{a}^\dagger\hat{a}$. In this case the Wightman functions (69) generalize to,

$$\imath\Delta^+(t;t') = \frac{\hbar}{2m\omega}\Big[(1+\bar{n}_0){\rm e}^{-\imath\omega(t-t')} + \bar{n}_0 {\rm e}^{\imath\omega(t-t')}\Big]\,, \qquad \imath\Delta^-(t;t') = [\imath\Delta^+(t;t')]^*\,. \tag{72}$$

These Wightman fuctions are time translation invariant, *i.e.* they are invariant under the translations $\hat{T}(c) = {\rm e}^{c\partial_{\bar{t}}}$ generated by the average time derivative operator, $\partial_{\bar{t}}$, where $\bar{t} = (t+t')/2$ and c is a real constant (representing the time shift of $\bar{t}$). In simple words, $\imath\Delta^\pm(t;t')$ are functions of the relative time $\Delta t = t - t'$ only. This property is satisfied, for example, by thermal states. And in fact, the Wightman functions (72) (and the corresponding Feynman propagator) are an example of a (non-interacting) thermal state, whereby one can introduce a temperature T_0 by writing,

$$\bar{n}_0 = \frac{1}{{\rm e}^{\beta_0\hbar\omega} - 1}\,, \qquad \beta_0 = \frac{1}{k_B T_0}\,, \tag{73}$$

where k_B is the Boltzmann constant. Finally, there are two more contribution in (59), those that are proportional to $\langle \hat{a}^2 \rangle$ and those $\propto \langle (\hat{a}^\dagger)^2 \rangle$. These contributions do not vanish when a state is squeezed (since the fluctuations in $\hat{q}$ and $\hat{p}$ (appropriately rescaled) are not equal), and violate time translation invariance of the two point functions (they do not depend on the relative time

but instead on the average time $\bar{t} = (t+t')/2$). This was to be expected, since squeezed states do not look symmetric in phase space, and as they rotate they look like a two-winged windmill in phase space. Because of the fundamental relation between the creation and annihilation operators of squeezed and un-squeeded states (see Appendix B),

$$\hat{a} \to \hat{a}(r,\phi) = \hat{a}\cosh(r) + \hat{a}^\dagger \mathrm{e}^{2\imath\phi}\sinh(r)\,, \tag{74}$$

we have,

$$\langle \hat{a}(r_0,\phi_0)^2\rangle = \cosh(r_0)\sinh(r_0)\,\mathrm{e}^{2\imath\phi_0}\,, \qquad \langle [\hat{a}^\dagger(r_0,\phi_0)]^2\rangle = \cosh(r_0)\sinh(r_0)\,\mathrm{e}^{-2\imath\phi_0}\,, \tag{75}$$

and hence the Wightman functions for the general initial Gaussian state are,

$$\begin{aligned} \imath\Delta^+(t;t') =&\ \frac{\hbar}{2m\omega}\Big[(1+\bar{n}_0)\,\mathrm{e}^{-\imath\omega(t-t')} + \bar{n}_0\,\mathrm{e}^{\imath\omega(t-t')} + \sinh(2r_0)\cos[\omega(t+t') - 2\phi_0]\Big] \\ \imath\Delta^-(t;t') =&\ [\imath\Delta^+(t;t')]^*\,. \end{aligned} \tag{76}$$

The corresponding Feynman propagator follows immediately from (60). From (76) we learn that, not surprisingly, the SHO Hamiltonian generates a rather trivial evolution on a Gaussian state: the associated particle number $\bar{n}_0$ and the squeeze factor r_0 do not change in time; the squeeze phase evolves in a rather trivial way, $\phi_0 \to \phi(\bar{t}\,) = \phi_0 - \omega\bar{t}$, just like it is the case with coherent states. In Eq. (76) we have not included the possibility that the initial state is coherent, *i.e.* that the initial expectation values $\bar{q}_0$ and $\bar{p}_0$ are non-vanishing. Of course, these can be easily included into the Wightman functions (76), and we leave it as an exercise to the reader.

Up to now we have constructed the Wightman functions and the corresponding Feynman and Dyson propagators for general Gaussian states generated by the SHO Hamiltonian (49). Of course, the general quadratic Hamiltonian (19) generates a more complicated evolution and in general $i\Delta^\pm(t;t')$ can be found only numerically. Nevertheless, Wightman functions are (approximately) known in many dynamical situations of interest and these can then be used as the basic building blocks for non-equilibrium perturbation theory.

B. The Feynman Propagator and Wightman Functions in Frequency Space

Here we show how to make the connection between the usual frequency space representation of the Feynman propagator and that in the operator formalism. In particular, we discuss how to fix the homogeneous part of the propagator, and how to relate it to the Wightman functions.

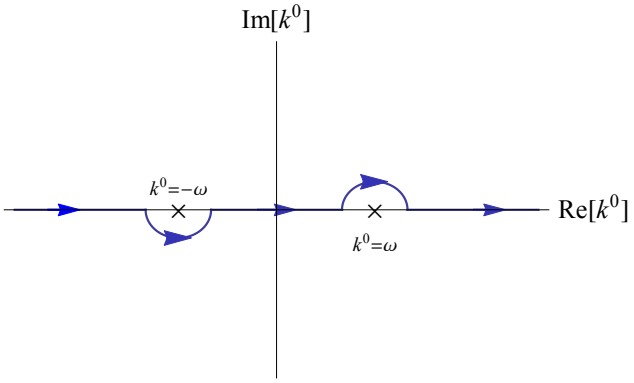

FIG. 2: The Feynman propagator integration contour in the complex k^0-plane.

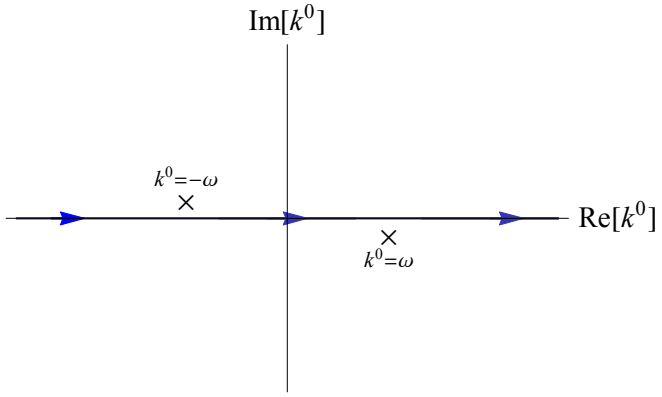

FIG. 3: The Feynman propagator integration contour, for which the negative and positive frequency poles are shifted by $\pm\imath\epsilon$, respectively, where $\epsilon > 0$ is infinitesimal.

The propagator equation (56) can be transformed into the frequency space by a Wigner transform, under which the propagator transforms as,

$$\imath\Delta_F(t;t') = \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \mathrm{e}^{-\imath k^0(t-t')} \imath\tilde{\Delta}_F(\bar{t};k^0) \tag{77}$$

(the ground state energy E of an oscillator is $E = \hbar\omega$, and the corresponding units are $[E] = \mathrm{J}$ and $[\omega] = [k^0] = \mathrm{s}^{-1}$) such that the equation of motion (56) becomes,

$$\left(k^{0\,2} - \omega^2\right)\imath\tilde{\Delta}_F(\bar{t};k^0) = \frac{\imath\hbar}{m}\,, \qquad \imath\tilde{\Delta}_F(\bar{t};-k^0) = \imath\tilde{\Delta}_F(\bar{t};k^0)\,, \qquad \bar{t} = \frac{1}{2}(t+t')\,. \tag{78}$$

The general solution to this equation is a distribution of the form,

$$\imath\tilde{\Delta}_F(\bar{t};k^0) = \mathrm{P}\frac{\imath\hbar/m}{k^{0\,2} - \omega^2} + \tilde{g}_F(\bar{t};k^0)\delta(k^{0\,2} - \omega^2)\,, \tag{79}$$

where P denotes a principal value, [5] and $g(\bar{t};k^0)$ is a general function that represents a homogeneous solution. As can be found in most of quantum field theory textbooks, the vacuum part of the Feynman propagator is determined by adding to the principal part a homogeneous contribution from 1/2 of the pole contributions, as indicated in figure 2. The easiest way to evaluate the

[5] The principal value defines how a distribution acts on a test function. For example, the principal part in the Feynman propagator (79) means that it acts on a (complex) test function $f(k^0)$ as

$$\mathrm{P}\int_{-\infty}^{\infty} dk^0 \frac{\imath\hbar/m}{k^{0\,2} - \omega^2} f(k^0) = \lim_{\epsilon\to 0}\left[\int_{-\infty}^{-\omega-\epsilon} dk^0 \frac{\imath\hbar/m}{k^{0\,2} - \omega^2} f(k^0) + \int_{-\omega+\epsilon}^{\omega-\epsilon} dk^0 \frac{\imath\hbar/m}{k^{0\,2} - \omega^2} f(k^0) + \int_{\omega+\epsilon}^{\infty} dk^0 \frac{\imath\hbar/m}{k^{0\,2} - \omega^2} f(k^0)\right].$$

integral is to make use the residue theorem in complex calculus, which tells us how to evaluate an integral along a closed, complex contour of a mero-morphic function. A mero-morphic function is a function on the complex plane that is analytic (C^∞) everywhere except possibly at a finite set of isolated points, where it has poles of finite order. Our function has two simple (of order one) poles at $k^0 = \pm\omega$, and therefore the residue theorem applies. According to the residue theorem, one can evaluate the integral in figure 2 by closing the contour with a semicircle of a large radius R below the real axis when $t > t'$ and above the real axis when $t < t'$ (the contours are chosen such that the contribution from the semicircles vanish in the limit when $R \to \infty$). Since the integral of a mero-morphic function along a closed contour depends on the pole contributions only, one can shift the contour at will (as long as it does not cross a pole) without affecting the result. Therefore the contour shown in figure 3 (see also figure 4), in which the negative frequency pole is shifted to $k^0 = -\omega + \imath\epsilon$ and the positive frequency pole is shifted to $k^0 = \omega - \imath\epsilon$, where $\epsilon > 0$ is an infinitezimal positive constant, is completely equivalent to the contour in figure 2. Hence, the vacuum part of the Feynman propagator can be obtained schematically from the contour integrals in figure 4,

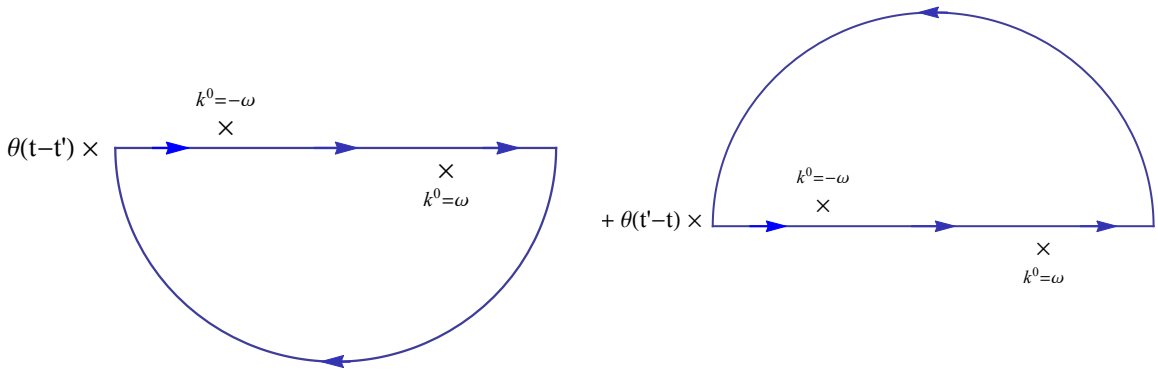

FIG. 4: The positive and negative pole contours that contribute to the vacuum Feynman propagator.

Of course, both of these contributions contribute to the Feynman propagator and thus we have,

$$\begin{aligned}
[\imath\Delta_F(t;t')]_{\text{vac}} &= \Theta(t-t')(-2\pi\imath)\text{Res}\left[\frac{\imath\hbar \mathrm{e}^{-\imath k^0(t-t')}}{2\pi m(k^{0^2}-\omega^2)}, k^0=\omega\right] \\
&+ \Theta(t-t')(2\pi\imath)\text{Res}\left[\frac{\imath\hbar \mathrm{e}^{-\imath k^0(t-t')}}{2\pi m(k^{0^2}-\omega^2)}, k^0=-\omega\right] \\
&= \Theta(t-t')\frac{\hbar}{2m\omega}\mathrm{e}^{-\imath\omega(t-t')} + \Theta(t'-t)\frac{\hbar}{2m\omega}\mathrm{e}^{\imath\omega(t-t')}\,, \qquad (80)
\end{aligned}$$

which agrees withe the result (69), (60) obtained by making use of the operator formalism.

From this result we have learned the following two facts: the boundary prescription for the vacuum part of the Feynman propagator (in frequency space) can be mnemonically written as,

$$[\imath\tilde{\Delta}_F(k^0)]_{\rm vac} = \frac{\imath\hbar/m}{k^{0\,2} - (\omega - \imath\epsilon)^2} = \frac{\imath\hbar/m}{k^{0\,2} - \omega^2 + \imath\tilde{\epsilon}}\,, \qquad (\tilde{\epsilon} = 2\omega\epsilon)\,. \tag{81}$$

where we dropped the ϵ^2 contribution. The $\imath\epsilon-$prescription is used in order to make it easier to memorize what is the homogeneous contribution to the vacuum Feynman propagator. Indeed, by making use of the Plemelj-Sokhotski (or Chochotski) identity,

$$\frac{1}{x + \imath\epsilon} = {\rm P}\frac{1}{x} - \imath\pi\delta(x)\,, \tag{82}$$

the Feynman propagator becomes,

$$[\imath\tilde{\Delta}_F(k^0)]_{\rm vac} = {\rm P}\frac{\imath\hbar/m}{k^{0\,2} - \omega^2} + \frac{\pi\hbar}{m}\delta\Big(k^{0\,2} - \omega^2\Big)\,, \tag{83}$$

from which one can simply read off the homogeneous part of the vacuum Feynman propagator in Eq. (79),

$$[\tilde{g}_F(t; k^0)]_{\rm vac} = \frac{\pi\hbar}{m}\,. \tag{84}$$

This homogeneous part corresponds to a Feynman propagator where the initial state is chosen to be the pure Gaussian vacuum state. For a more general initial Gaussian state, the Feynman propagator has a more complicated homogeneous part, which can be determined by performing a Wigner transform on the solution (76) and (60), and we leave this as an exercise to the reader.

Secondly, Eq. (60) tells us that the vacuum part of the positive and negative frequency Wightman functions can be obtained from the contours shown in figure 4. These two contours can be continuously deformed to the contours around the positive and negative frequency poles shown in figure 5. Therefore, the vacuum part of the positive (negative) frequency Wightman function is obtained from the residue of the positive (negative) frequency pole, explaining the origin of the names. We also know that the Wightman functions come from the homogeneous solution of the equation of motion, which in frequency space reads,

$$\Big(k^{0\,2} - \omega^2\Big)\imath\tilde{\Delta}^{\pm}(\bar{t}; k^0) = 0 \tag{85}$$

By rewriting the homogeneous part of (83) as,

$$\frac{\pi\hbar}{2m\omega}[\delta\big(k^0 - \omega\big) + \delta\big(k^0 + \omega\big)]\,, \tag{86}$$

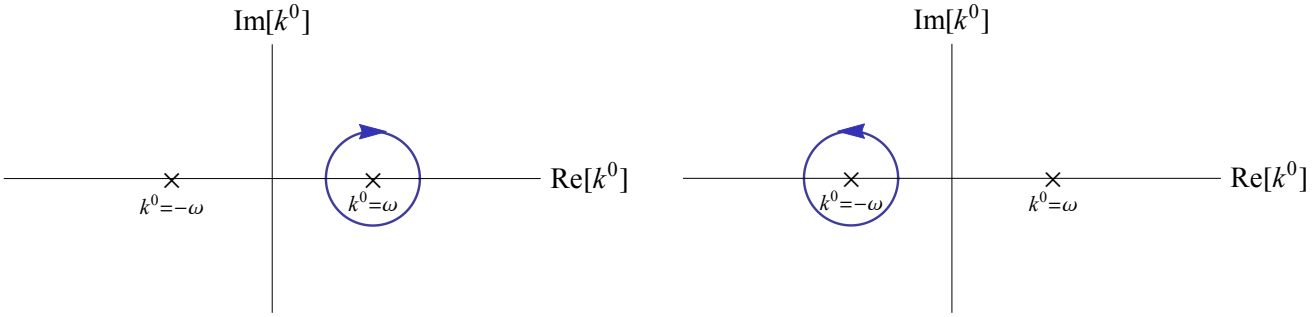

FIG. 5: The integration contours for the vacuum parts of the positive frequency (left panel) and negative frequency (right panel) Wightman functions.

which equals the half of the sum of the two Wightman functions (recall the integration contour in figure 2), we can read off the vacuum part of the positive and negative frequency Wightman functions,

$$\begin{aligned}\left[\imath\tilde{\Delta}^{+}(k^0)\right]_{\rm vac} &= \frac{\pi\hbar}{m\omega}\delta\big(k^0-\omega\big) = \Theta(k^0)\frac{2\pi\hbar}{m}\delta\Big(k^{0\,2}-\omega^2\Big)\\ \left[\imath\tilde{\Delta}^{-}(k^0)\right]_{\rm vac} &= \frac{\pi\hbar}{m\omega}\delta\big(k^0+\omega\big) = \Theta(-k^0)\frac{2\pi\hbar}{m}\delta\Big(k^{0\,2}-\omega^2\Big)\,.\end{aligned} \tag{87}$$

Of course, the same result is obtained by performing a Wigner transform of Eq. (69) (check that!).

For the general Gaussian initial state the Wightman functions are given in Eq. (76), from which it is clear that the Wightman functions (and consequently also the Feynman propagator) are sensitive to the choice of the initial state. Eq. (76) can be trasformed to frequency space by performing an inverse Wigner transform (*cf.* Eq. (77)),

$$\imath\tilde{\Delta}^{\pm}(\bar{t};k^0) = \int_{-\infty}^{\infty} d\Delta t \mathrm{e}^{\imath k^0\Delta t}\imath\Delta^{\pm}(t;t')\,, \tag{88}$$

from which we get,

$$\begin{aligned}\imath\tilde{\Delta}^{+}(\bar{t};k^0) &= \frac{\pi\hbar}{m\omega}\Big[(1+\bar{n}_0)\delta(k^0-\omega)+\bar{n}_0\delta(k^0+\omega)+\sinh(2r_0)\cos(2\phi(\bar{t}\,))\delta(k^0)\Big]\\ \imath\tilde{\Delta}^{-}(\bar{t};k^0) &= \frac{\pi\hbar}{m\omega}\Big[\bar{n}_0\delta(k^0-\omega)+(1+\bar{n}_0)\delta(k^0+\omega)+\sinh(2r_0)\cos(2\phi(\bar{t}\,))\delta(k^0)\Big]\,,\end{aligned} \tag{89}$$

where $\phi(\bar{t}\,) = \phi_0-\omega\bar{t}$ is the time dependent squeeze phase. From this result we learn that: if the initial state is mixed (which is characterized by a non-vanishing initial particle number $\bar{n}_0$, where here $\bar{n}_0$ is the particle number associated with the creation and annihilation operators that diagonalize the Hamiltonian), then $\bar{n}_0$ contributes equally to both positive and negative frequency poles. If the initial state is squeezed, then it breaks time translation invariance, and

gives an off-shell contribution to the Wightman functions that is $\propto \delta(k^0)$, and whose strength is given by the squeeze factor ($\sinh(2r_0)$), modulated harmonically by the (time dependent) squeeze angle $\phi(\bar{t}\,)$ as $\sin(\phi(\bar{t}\,))$.

An analogous calculation for the Feynman propagator (60) gives (show that!),

$$\imath\tilde{\Delta}_F(\bar{t};k^0) = \frac{\hbar}{m}\left[\frac{\imath}{k^{0\,2}-(\omega-\imath\epsilon)^2} + 2\pi\bar{n}_0\delta\Big(k^{0\,2}-\omega^2\Big) + \frac{\pi}{\omega}\sinh(2r_0)\cos(2\phi(\bar{t}\,))\delta(k^0)\right]. \qquad (90)$$

The first contribution is, of course, the vacuum contribution, and the latter two come from the initial state correction: one due to the mixed and one due to the squeezed nature of the initial state. Now, comparing (90) with (89) we see that the homogeneous part of the Feynman propagator in frequency space, $[\imath\tilde{\Delta}_F]_{\rm hom}$, satisfies the same equality as in the pure vacuum state, namely,

$$[\imath\tilde{\Delta}_F]_{\rm hom} = \frac{1}{2}[\imath\tilde{\Delta}^+ + \imath\tilde{\Delta}^-]\,, \qquad (91)$$

which is in agreement with the contours in figure 5. The above considerations will be helpful when considering the general boundary conditions that are needed to fully define the Feynman propagator. Note that the exact frequency space relation between the Feynman propagator and Wightman functions can be obtained by taking the inverse Wigner transform of (60). The general result can be written as a convolution and requires performing an integral over the frequencies (derive that relation!).

Note that the Wightman functions (89) as well as the corresponding Feynman (90) and Dyson propagator of general Gaussian states cannot be represented by the *quasiparticle approximation*, according to which knowledge of the occupation numbers on the positive ($k^0=\omega$) and negative ($k^0=-\omega$) frequency poles suffices. Indeed, from (89) and (90) we see that, for states that are not invariant under time translations and/or for states with a non-trivial squeezing (for which $\langle\hat{a}^2\rangle \neq 0$), the third, $k^0=0$, shell also gets populated. This shell lies in between the positive and negative frequency shells, and its population (when viewed from the quantum field theoretic perspective) signifies particle pair creation. An important example of such a process is the creation of particle pairs (of opposite spatial momenta) out of the vacuum of the inflaton quanta during cosmic inflation. The physical origin of the effect is in the coupling of the inflaton field (which is the scalar field that drives inflation) to gravity generated by the Universe's rapid expansion. Cosmologists believe that we can see that process imprinted as the temperature fluctuations in the relic cosmic microwave background radiation (CMB).

C. The Kubo-Martin-Schwinger Condition

So far we have considered the properties of general Gaussian states. An important special case are thermal states, whose density operator is given in Eq. (18). The corresponding positive frequency Wightman functions is then,

$$\begin{aligned} \imath\Delta^{+}_{\rm th}(t;t') &= \frac{1}{Z}{\rm Tr}\left[{\rm e}^{-\beta\hat{H}}\hat{q}(t)\hat{q}(t')\right] \\ &= \frac{1}{Z}{\rm Tr}\left[{\rm e}^{(\imath/\hbar)\hat{H}(t+\imath\hbar\beta)}\hat{q}(0){\rm e}^{-(\imath/\hbar)\hat{H}(t+\imath\hbar\beta)}{\rm e}^{-\beta\hat{H}}\hat{q}(t')\right] \\ &= \frac{1}{Z}{\rm Tr}\left[{\rm e}^{-\beta\hat{H}}\hat{q}(t')\hat{q}(t+\imath\beta)\right] , \end{aligned} \tag{92}$$

where in the last step we used the cyclic property under the trace. From this one easily obtains the following Kubo-Martin-Schwinger (KMS) condition,

$$\imath\Delta^{+}_{\rm th}(t;t') = \imath\Delta^{-}_{\rm th}(t;t'-\imath\beta) = \imath\Delta^{-}_{\rm th}(t+\imath\beta;t') , \tag{93}$$

for thermal Wightman functions. The last condition in (93) is obtained by making use of the time translation symmetry of thermal states. Since our proof applies for a general Hamiltonian, the KMS condition is true for general (Gaussian and non-Gaussian) states/Hamiltonians. Unlike in the case of non-equilibrium states, the Hamiltonian and the thermal density operator are simultaneously diagonalized. It is convenient to write the KMS in frequency space. A simple integration gives,

$$\imath\tilde{\Delta}^{+}_{\rm th}(k^0) = {\rm e}^{\beta\hbar k^0}\imath\tilde{\Delta}^{-}_{\rm th}(k^0) . \tag{94}$$

When applied to the Wightman functions in Eq. (89) (with $r_0 = 0$ as required by the time translation invariance of thermal states) the KMS condition implies,

$$\bar{n}_0 = \frac{1}{{\rm e}^{\beta\hbar\omega}-1} , \tag{95}$$

which is the famous Bose-Einstein formula for the thermal occupancy of bosons, with $\beta = 1/(k_B T)$. In fact, the KMS condition is so powerful, that one can take it as the defining formula for thermal equilibrium. [6]

[6] Alternatively, one can obtain the KMS condition (94) by considering the quantum evolution equations for the Wightman functions (also known as the Kadanoff-Baym equations) (which generalize the classical kinetic Boltzmann equation). Demanding that interactions do not induce a change in time of the Wightman functions yields the KMS condition (94). This means that the KMS condition can be also derived from quantum dynamics.

D. The Statistical and Causal Two-Point Functions

The statistical (Hadamard) (F) and causal (Pauli-Jordan, spectral) ($\imath\Delta^c$) two-point functions are defined (in terms of the real and imaginary parts of the Wightman functions) as follows,

$$F(t;t') = \frac{1}{2}\left(\imath\Delta^+(t;t') + \imath\Delta^-(t;t')\right) , \qquad \imath\Delta^c(t;t') = \imath\Delta^-(t;t') - \imath\Delta^+(t;t') , \tag{96}$$

such that

$$\imath\Delta^\pm = F \mp \frac{1}{2}\imath\Delta^c . \tag{97}$$

For the Gaussian state (76) we have in real space (time domain),

$$\begin{aligned} F(t;t') &= \frac{\hbar}{2m\omega}\Big[(1+2\bar{n}_0)\cos[\omega(t-t')] + \sinh(2r_0)\cos(2\phi(\bar{t}))\Big] \\ \imath\Delta^c(t;t') &= \frac{\imath\hbar}{m\omega}\sin[\omega(t-t')] \end{aligned} \tag{98}$$

while in frequency space (see (89)),

$$\begin{aligned} \tilde{F}(\bar{t};k^0) &= \frac{\pi\hbar}{m}\Big[(1+2\bar{n}_0)\delta\Big(k^{0\,2}-\omega^2\Big) + \frac{1}{2\omega}\sinh(2r_0)\cos(2\phi(\bar{t}))\delta(k^0)\Big] \\ \imath\tilde{\Delta}^c(\bar{t};k^0) &= -\frac{2\pi\hbar}{m}\mathrm{sign}(k^0)\,\delta\Big(k^{0\,2}-\omega^2\Big) , \end{aligned} \tag{99}$$

where $\mathrm{sign}(k^0) = \Theta(k^0) - \Theta(-k^0)$. The relevant integration contours are shown in figure 6. While

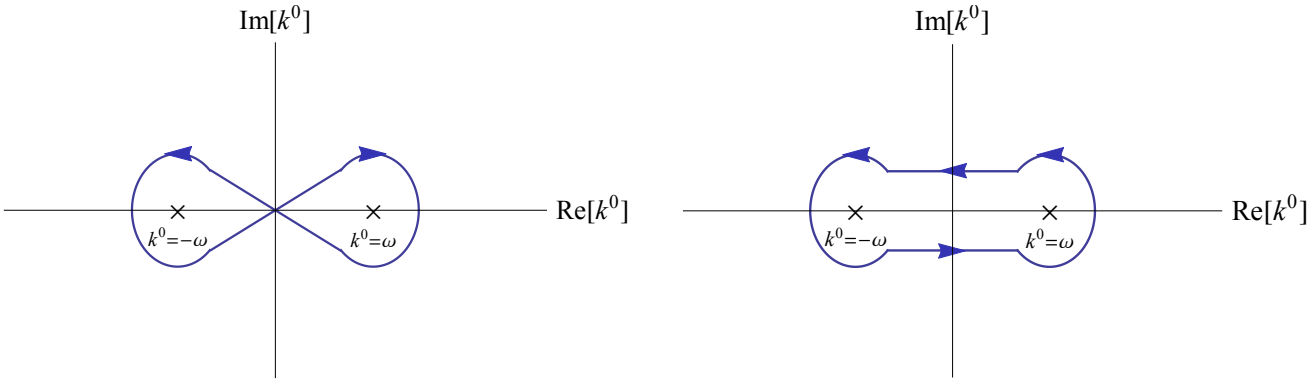

FIG. 6: The integration contours for the vacuum part of the Hadamard (statistical) two-point function (left panel) and of the spectral (Pauli-Jordan) two-point function (right panel).

the statistical two-point function contains a complete information about the (initial) state (it is, namely, sensitive to both $\bar{n}_0$ and r_0) and interactions (medium), the spectral function does not. Indeed, it only records what states are available in the system, but it contains no information

on how are these states populated. [7] On the other hand, the Hadamard function F contains the information on how the states are populated both due to the way the initial state is prepared as well as due to interactions. Indeed, from the frequency space form of F we clearly see that one can easily distinguish the squeezing contribution from the particle number contribution, as they contribute to different poles. For those reasons, it is the Hadamard function and not the spectral function which figures in the formula for the (von Neumann) entropy (44). [8]

The origin of the name spectral function can be traced to the fact that it obeys a *spectral sum rule,* which can be derived as follows. Upon acting with a time derivative on $\imath\Delta^c$ and setting $t' \to t$, one gets,

$$[\partial_t \imath\Delta^c(t;t')]_{t'\to t} = \frac{\imath\hbar}{m} = \langle[\hat{q}(t), \dot{\hat{q}}(t)]\rangle \,. \tag{100}$$

Because this result follows from the commutator, it is true not just for Gaussian states, but also more generally for the spectral function for arbitrary states and for arbitrary Hamiltonians containing interactions. One often quotes the spectral sum rule (100) in frequency space as,

$$\int \frac{dk^0}{2\pi} k^0 [-\imath\tilde{\Delta}^c(\bar{t}; k^0)] = \frac{\hbar}{m} \,. \tag{101}$$

The spectral sum rule is useful for the normalisation of the spectral function, which then in turn can be useful for determining the retarded and advanced propagators, which we discuss next.

E. The Retarded and Advanced Propagators

The retarded and advanced propagators are useful for studying the motion of particles and evolution of quantum fields (such as under the influence of external sources). The retarded and advanced propagators can be obtained from the Feynman and Dyson propagators and Wightman functions as follows,

$$\begin{aligned} \imath\Delta^r(t;t') &= \imath\Delta_F(t;t') - \imath\Delta^-(t;t') = -\imath\Delta_{\bar{F}}(t;t') + \imath\Delta^+(t;t') \\ \imath\Delta^a(t;t') &= \imath\Delta_F(t;t') - \imath\Delta^+(t;t') = -\imath\Delta_{\bar{F}}(t;t') + \imath\Delta^-(t;t') \,. \end{aligned} \tag{102}$$

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[7] Of course, in presence of interactions this is only approximately true and the spectral function gets modified by medium (interaction) effects. Arguably, the simplest such effect is a local (singular) contribution to the self-energy, which can change the mass of a particle (or, equivalently, the frequency of the oscillator).

[8] One can show [6, 7] that the statistical particle number $\bar{n}$ in (44) can be expressed in terms of F as,

$$\left(\bar{n}(t) + \frac{1}{2}\right)^2 = \frac{m^2}{\hbar^2}\Big[F(t;t)[\partial_t\partial_{t'}F(t;t')]_{t'=t} - \Big([\partial_t F(t;t')]_{t'=t}\Big)^2\Big] \,.$$

The later equalities are a consequence of the following relation (show that!),

$$\imath\Delta_F(t;t') + \imath\Delta_{\bar{F}}(t;t') = \imath\Delta^+(t;t') + \imath\Delta^-(t;t')\,. \tag{103}$$

Upon inserting Eq. (60) (or (64)) into (102) one finds that the retarded and advanced propagators are naturally expressed in terms of the spectral function as,

$$\begin{aligned}
\imath\Delta^r(t;t') &= \Theta(t-t')[\imath\Delta^+(t;t') - \imath\Delta^-(t;t')] = -\Theta(t-t')\imath\Delta^c(t;t') \\
\imath\Delta^a(t;t') &= \Theta(t'-t)[\imath\Delta^-(t;t') - \imath\Delta^+(t;t')] = \Theta(t'-t)\imath\Delta^c(t;t')\,.
\end{aligned} \tag{104}$$

Below in III F, where we discuss quantum field theory of a real scalar field, we shall see that this representation of the retarded and advanced propagator is manifestly causal (due to the fact that the spectral or causal function vanishes outside the light-cones). The remnant of causality in quantum mechanics are the Θ-functions in (104), which tell us that the retarded (advanced) propagator vanishes in the future (past) of t. For example, one can use these propagators to write down general solutions of the initial value problem where a particle couples to an external current,

$$m\Big(-\frac{d^2}{dt^2} - \omega^2\Big)\hat{q}(t) = \hat{j}(t)\,. \tag{105}$$

The general solution can be written in two ways,

$$\begin{aligned}
\hat{q}(t) &= \hat{q}_{h,r}(t) + \frac{1}{\hbar}\int_{t_0}^{\infty} dt'\Delta^r(t;t')\hat{j}(t') \\
\hat{q}(t) &= \hat{q}_{h,a}(t) + \frac{1}{\hbar}\int_{t_0}^{-\infty} dt'\Delta^a(t;t')\hat{j}(t')\,,
\end{aligned} \tag{106}$$

where $\hat{q}_{h,r}(t)$ and $\hat{q}_{h,a}(t)$ are the appropriate homogeneous solutions for the retarded and advanced boundary conditions. Namely, $\hat{q}_{h,r}(t)$ defines the solution at a past boundary $t_0 < t$, while $\hat{q}_{h,a}(t)$ defines the solution at a future boundary $t_0 > t$.

Because the retarded and advanced propagators are expressed in terms of the spectral function, they are to a large extent insensitive to the nature of the initial state and to interactions with an external medium. This can be also seen from the frequency space form of these propagators, which for the general Gaussian state are,

$$\imath\tilde{\Delta}^r(k^0) = \frac{\imath\hbar/m}{(k^0+\imath\epsilon)^2 - \omega^2}\,, \qquad \imath\tilde{\Delta}^a(k^0) = \frac{\imath\hbar/m}{(k^0-\imath\epsilon)^2 - \omega^2} = [\imath\tilde{\Delta}^r(k^0)]^*\,. \tag{107}$$

The corresponding integration contours are shown in figure 7. Eqs. (107) can be also used

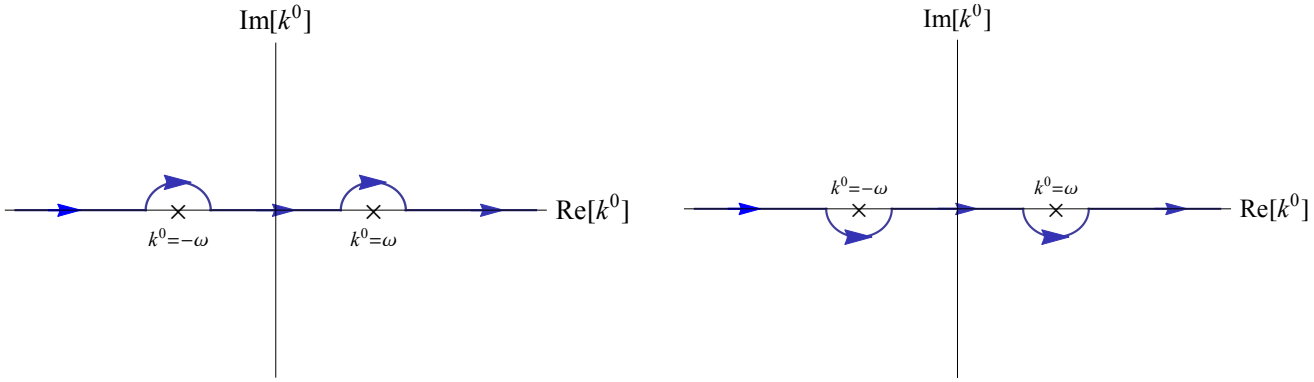

FIG. 7: The integration contours for the vacuum part of the retarded (left panel) and advanced (right panel) propagator.

to express the retarded and advanced propagator in frequency space in terms of the spectral function,

$$\imath\tilde{\Delta}^r(k^0) = -\int_{-\infty}^{\infty} \frac{dk'^0\,\tilde{\Delta}^c(k'^0)}{k'^0 - k^0 - \imath\epsilon}\,, \qquad \imath\tilde{\Delta}^a(k^0) = -\int_{-\infty}^{\infty} \frac{dk'^0\,\tilde{\Delta}^c(k'^0)}{k'^0 - k^0 + \imath\epsilon}\,. \tag{108}$$

Because Eqs. (107) are generally true, so are these relations. We leave it to the reader to derive these relations. As a hint, note that the Heaviside functions $\Theta(\Delta t) = \int_{-\infty}^{\Delta t} dy\delta(y)$ and $\Theta(-\Delta t)$ can be written as,

$$\Theta(\Delta t) = \int_{-\infty}^{\infty} \frac{d\kappa^0}{2\pi} \frac{\imath}{\kappa^0 + \imath\epsilon} \mathrm{e}^{-\imath\kappa^0\Delta t}\,, \qquad \Theta(-\Delta t) = \int_{-\infty}^{\infty} \frac{d\kappa^0}{2\pi} \frac{-\imath}{\kappa^0 - \imath\epsilon} \mathrm{e}^{-\imath\kappa^0\Delta t}\,, \tag{109}$$

where the $\imath\epsilon-$prescription is introduced to define the contribution from the pole at $\kappa^0 = 0$.

F. Two-Point Functions in Quantum Field Theory and Causality

Here we briefly outline how to generalise our results for the two-point functions in quantum mechanics to quantum field theories. As an example, we discuss a real scalar field, but leave fermions as an exercise to the reader. For simplicity, here we set $\hbar = 1 = c$. We begin by quoting the action for a real scalar field $\phi(x)$ in curved space-times, which in the case of a canonical kinetic term is,

$$S[\phi] = \int d^4\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]\,, \tag{110}$$

where $g^{\mu\nu}$ is the inverse of the metric tensor $g_{\mu\nu}$, *i.e.* $g^{\mu\rho}g_{\rho\nu} = \delta^{\mu}_{\nu}$, $g = \det[g_{\mu\nu}]$ and $V = V(\phi)$ is a potential, which we shall assume to have the following simple form, [9]

$$V(\phi) = \frac{m^2}{2}\phi^2 + V_{\rm int}\,, \qquad V_{\rm int}(\phi) = \frac{\lambda}{4!}\phi^4\,, \tag{111}$$

where m denotes the field's mass and λ is the strength of the quartic self-coupling. Here we take for the metric signature to be $(-,+,+,+)$, which is commonly used in the general relativity community; the other common and completely equivalent signature, that is used mostly by field theorists, is $(+,-,-,-)$. The action (110) is covariant, in the sense that it transforms as a scalar under general coordinate transformations. The corresponding canonical momentum is obtained by varying the action (110) with respect to $\dot{\phi} = \partial_0\phi(x)$,

$$\pi(x) \equiv \frac{\delta S}{\delta\dot{\phi}(x)} = -\sqrt{-g}g^{0\mu}\partial_\mu\phi\,, \tag{112}$$

and $\phi(x)$ and $\pi(x)$ satisfy canonical Poisson brackets, $\{\phi(t,\vec{x}), \pi(t,\vec{x}\,')\} = \delta^3(\vec{x}-\vec{x}\,')$, where we use a short-hand notation $\phi(x) = \phi(t,\vec{x})$. The canonical quantisation then dictates that the corresponding operators $\hat{\phi}(t,\vec{x})$ and $\hat{\pi}(t,\vec{x}\,')$ obey the canonical commutation relations,

$$[\hat{\phi}(t,\vec{x}), \hat{\pi}(t,\vec{x}\,')\,] = \imath\delta^3(\vec{x}-\vec{x}\,')\,, \qquad [\hat{\phi}(t,\vec{x}), \hat{\phi}(t,\vec{x}\,')\,] = 0 = [\hat{\pi}(t,\vec{x}), \hat{\pi}(t,\vec{x}\,')\,]\,. \tag{113}$$

On the other hand, varying the action (110) with respect to the field ϕ, and promoting ϕ to an operator $\hat{\phi}$ (such that it satisfies (113)) results in the following equation of motion (see also problem 6),

$$\delta_\phi S = 0 \implies \Box\hat{\phi} - \frac{dV}{d\phi}(\hat{\phi}) = 0\,, \qquad \Box = g^{\mu\nu}\nabla_\mu\nabla_\nu = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\,, \tag{114}$$

where $\Box$ denotes the d'Alembertian operator as it acts on a scalar field. Since in general $g_{\mu\nu} = g_{\mu\nu}(x)$ is space-time dependent, ϕ couples through the d'Alembertian to a space-time dependent background resulting in a non-trivial dependence on space and time of the quantum field and the corresponding quantum state, which is the subject of non-equilibrium quantum field theory.

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[9] The Higgs sector of standard model is endowed with such a potential. More precisely, $V(H) = \mu^2 HH^\dagger + \lambda(HH^\dagger)^2$, where $\mu^2 < 0$ is a mass parameter and H is the physical Higgs field which consists of two complex scalar fields. The Higgs field can be represented in terms of four real fields as, $H^T = (1/\sqrt{2})(\phi_1 + i\phi_2, \phi_3 + i\phi_4)$, in terms of which $V(\phi_a) = (\mu^2/2)\sum_{a=1}^4 \phi_a^2 + (\lambda/4)\big[\sum_{a=1}^4 \phi_a^2\big]^2$. Note that the potential $V(\phi_a)$ is $O(4)$ symmetric. Moreover, one can show that the form of the potential is maintained when quantum corrections are included (because standard model is a renormalisable theory).

In this subsection, for simplicity we consider the scalar field propagators and two-point functions in a trivial (space-time independent) Minkowski background; for a discussion of a more general (cosmological de Sitter space) background, we refer to problem 6.

One can formally rewrite Eq. (114) as an integral Yang-Feldman equation,

$$\hat{\phi}(x) = \hat{\phi}_h(x) + \int d^4x' G^r(x;x') \frac{dV_{\rm int}}{d\phi}(\hat{\phi}(x')) \,, \tag{115}$$

where $G^r(x;x')$ is the retarded Green function of the free problem,

$$\sqrt{-g}(\Box - m^2) G^r(x;x') = \delta^4(x-x') \,. \tag{116}$$

The Yang-Feldman equation (115) is a convenient starting form for building perturbative expansions for solving for $\hat{\phi}(x)$.

Let us now for simplicity assume that $\lambda \to 0$ and that $g_{\mu\nu} \to \eta_{\mu\nu}$, where $\eta_{\mu\nu} = {\rm diag}(-1,1,1,1)$ is the Minkowski (flat space) metric. In this case the equation of motion (114) simplifies to,

$$(\partial^2 - m^2)\hat{\phi} = 0 \,, \qquad \partial^2 = \eta^{\mu\nu}\partial_\mu\partial_\nu = -\partial_t^2 + \nabla^2 \,, \qquad \nabla^2 = \partial_i\partial_i \,, \tag{117}$$

where $i,j,.. = 1,2,3$ denote spatial indices, and $\mu,\nu,.. = 0,1,2,3$ denote space-time indices (as usually, repeated indices imply summation).

The Feynman propagator for the scalar field

$$\imath\Delta_F(x;x') = \langle T[\hat{\phi}(x)\hat{\phi}(x')]\rangle \tag{118}$$

satisfies the expected equation,

$$(\partial^2 - m^2)\imath\Delta_F(x;x') = \imath\delta^4(x-x') \tag{119}$$

while the Wightman functions

$$\imath\Delta^+(x;x') = \langle\hat{\phi}(x)\hat{\phi}(x')\rangle \,, \qquad \imath\Delta^-(x;x') = \langle\hat{\phi}(x')\hat{\phi}(x)\rangle \,, \tag{120}$$

satisfy the homogeneous equation,

$$(\partial^2 - m^2)\imath\Delta^\pm(x;x') = 0 \,. \tag{121}$$

The boundary conditions for the Feynman propagator dictate the same form in terms of the Wightman functions as in quantum mechanics (show that(!), see Eq. (60)),

$$\imath\Delta_F(x;x') = \Theta(t-t')\imath\Delta^+(x;x') + \Theta(t'-t)\imath\Delta^-(x;x') \,, \tag{122}$$

with the difference that now the Feynman propagator (and also the Wightman and other two-point functions) are functions of two space-time points x^μ and x'^μ, in total of eight coordinates. Because of this finding the general solution for the Feynman propagator is (very) hard. Nevertheless, there are many situations of physical interest in which one can find explicit analytical results for the Feynman propagator (and other two-point functions). The simplest example is the propagator in the free (non-interacting) Minkowski vacuum, in which case the solution must be Poincaré invariant. The (observed) translational invariance of the vacuum state dictates that $\imath\Delta_F$ is a function of $x^\mu - x'^\mu$ only, while Lorentz invariance (the invariance under boosts and spatial rotations) dictates that it must be a function of the invariant distance only, $(x-x')^2 \equiv \eta_{\mu\nu}(x^\mu - x'^\mu)(x^\nu - x'^\nu)$. If the propagator is invariant under space-time translations, it pays off to transform into the Wigner space,

$$\imath\Delta_F(x;x') = \int \frac{d^4k}{(2\pi)^4} \mathrm{e}^{\imath k\cdot(x-x')} \imath\tilde{\Delta}_F(k^\mu)\,. \tag{123}$$

The equation of motion (119) then becomes, [10]

$$(-k_\mu k^\mu - m^2)\imath\tilde{\Delta}_F(k^\mu) = \imath\,, \qquad k_\mu k^\mu = \eta_{\mu\nu}k^\mu k^\nu = -k^{0\,2} + \vec{k}^{\,2} \tag{125}$$

and it is solved by

$$\imath\tilde{\Delta}_F(k^\mu) = \mathrm{P}\frac{\imath}{-k_\mu k^\mu - m^2} + \tilde{g}(k^\mu)\delta(k_\mu k^\mu + m^2)\,, \tag{126}$$

where $-k_\mu k^\mu - m^2 = k^{0\,2} - \omega^2(k)$, $\omega^2(k) = \|\vec{k}\,\|^2 + m^2$ and here $k = \|\vec{k}\,\|$. As usually, the homogeneous part of the Feynman propagator is fixed by the vacuum boundary conditions which, just as in the case of the quantum mechanical oscillator, can be mnemonically represented in terms of the integration contour in the complex k^0-plane shown in figures 2 and 3. Therefore, a

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[10] In the case when the propagator is not invariant under space-time translations, it may still be useful to perform the Wigner transform if the dependence of the average coordinate $\bar{x} = (x + x')/2$ is weak. Namely, in this case the propagator equation (119) becomes more complicated,

$$\Big(-k_\mu k^\mu + \imath k^\mu \frac{\partial}{\partial \bar{x}^\mu} + \frac{1}{4}\frac{\partial}{\partial \bar{x}^\mu}\frac{\partial}{\partial \bar{x}_\mu} - m^2\Big)\imath\tilde{\Delta}_F(k^\mu;\bar{x}^\nu) = \imath\,, \tag{124}$$

where this result is obtained by noting that ∂_μ transforms into Wigner space as, $\partial/\partial x^\mu \to \imath k_\mu + (1/2)\partial/\partial\bar{x}^\mu$. When the dependence of $\imath\tilde{\Delta}_F$ on $\bar{x}^\nu$ is weak, one can use equation (124) as a starting point for a gradient expansion (expansion in derivatives of $\partial/\partial\bar{x}^\mu$). When applied to the statistical two-point function $\tilde{F} = \tilde{F}(k^\mu;\bar{x}^\nu)$, and when interactions are also taken into account, this procedure leads to a generalized quantum kinetic equation. The usual Boltzmann kinetic equation is then obtained by projecting on-shell, which is tantamount to integrating the statistical 2-point function $\tilde{F}(k^\mu;\bar{x}^\nu)$ over k^0.

convenient way of writing (126) is (see Eqs. (83–84)),

$$\imath\tilde{\Delta}_F(k^\mu) = \frac{\imath}{k^{0\,2} - \omega^2(k) + \imath\epsilon} + 2\pi n(k^\mu)\delta\Big(k^{0\,2} - \omega^2\Big)\,, \tag{127}$$

where the first part corresponds to the vacuum (ground state) contribution, while the second part containing $n(k^\mu)$ represents a medium contribution (coming from a mixed state or interactions) that equally contributes to both the positive and negative frequency poles.

Here we are interested in the Feynman propagator in direct (physical) space, thus we need to perform the Wigner transform on the solution (127). We have already performed the k^0-integral; the result is given in (76) and (60). What remains to be done is to integrate over the spatial momenta,

$$\begin{aligned}
\imath\Delta_F(x;x') &= \Theta(t-t')\imath\Delta^+(x;x') + \Theta(t'-t)\imath\Delta^-(x;x') \\
\imath\Delta^\pm(x;x') &= \int \frac{d^3k}{(2\pi)^3}\mathrm{e}^{\imath\vec{k}\cdot(\vec{x}-\vec{x}')}\frac{1}{2\omega}\Big[\mathrm{e}^{\mp\imath\omega(t-t')} + 2n(\vec{k})\cos[\omega(t-t')]\Big]\,,
\end{aligned} \tag{128}$$

where here $n(\vec{k})$ is the pole contribution from the mixed state (for rotationally invariant states, $n(\vec{k}) \to n(\|\vec{k}\|) \equiv n(k)$, which can be equivalently represented as $n(\omega)$. (In the special case of thermal equilibrium, $n(\omega) \to 1/[\exp(\beta\omega) - 1]$). What we see from (128) is that a free scalar field theory can be thought of as a bunch (of mutually non-interacting) harmonic oscillators with a momentum dependent frequency, $\omega(k) = \sqrt{k^2 + m^2}$, such that the Feynman propagator (and other two-point functions) can be obtained by summing over all allowed momenta (which in the case of a field theory in an infinite volume reduces to an integral over $\vec{k}$). The corresponding state can be written as a direct product over the momenta of states of single oscillators (in order to avoid mathematical problems with defining products over continuous label $\vec{k}$, it is advisable to perform a finite volume quantization, in which case the momenta become discrete).

The angular integral in (128) can be easiest performed by choosing $\vec{x} - \vec{x}'$ to be in the $z-$direction, *i.e.* $\vec{x} - \vec{x}' \equiv r(0,0,1)$, where $r = \|\vec{x} - \vec{x}'\|$. In this case, $\mathrm{e}^{\imath\vec{k}\cdot(\vec{x}-\vec{x}')} = \mathrm{e}^{\imath kr\cos(\theta)}$, where θ is the spherical angle of $\vec{k} = k\big(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)\big)$, such that,

$$\int \frac{d^3k}{(2\pi)^3}\mathrm{e}^{\imath\vec{k}\cdot(\vec{x}-\vec{x}')} \to \frac{1}{2\pi^2}\int_0^\infty dk k^2 \frac{\sin(kr)}{kr}\,,$$

and Eq. (128) simplifies to,

$$\imath\Delta^\pm(x;x') = \frac{1}{4\pi^2 r}\int_0^\infty \frac{dkk}{\omega}\sin(kr)\Big[\mathrm{e}^{\mp\imath\omega(t-t')} + 2n(\omega)\cos[\omega(t-t')]\Big]\,. \tag{129}$$

This integral is ultraviolet (UV) divergent, *i.e.* it diverges when $k \to \infty$, and – as it is written – it is not simple to evaluate. So, we shall further simplify it by assuming $n(\omega) = 0$ (pure vacuum

state) and by taking $m \to 0$ [11] A simple way to regulate the integral in (129) is to complexify time, by adding it a small imaginary part according to,

$$t-t' \;\to\; t-t' \mp \imath\epsilon\,. \tag{130}$$

With this, the $k-$integral in (129) can be evaluated to yield,

$$\imath\Delta^{\pm}(x;x') \;=\; \frac{1}{8\pi^2 r}\left[\frac{1}{r \mp \Delta t + \imath\epsilon} + \frac{1}{r \pm \Delta t - \imath\epsilon}\right] = \frac{1}{4\pi^2}\frac{1}{\Delta x^2_{\pm}(x;x')}\,, \tag{131}$$

where $\Delta t = t-t'$, and

$$\Delta x^2_{\pm}(x;x') = -(t-t' \mp \imath\epsilon)^2 + \|\vec{x}-\vec{x}'\|^2\,, \tag{132}$$

such that the Wightman functions of the scalar field vacuum appear to violate (boost) Lorentz invariance. Indeed, $\Delta x^2_{\pm} = \Delta x^2 + \epsilon^2 \pm \tilde{\epsilon}\text{sign}[\Delta t]$, $(\tilde{\epsilon} = 2|\Delta t|\epsilon)$ where

$$\Delta x^2 = -(t-t')^2 + \|\vec{x}-\vec{x}'\|^2$$

is Lorentz invariant, but sign$[\Delta t]$ is not. Recall, that it was necessary to complexify time in order to make the $k-$ integral convergent. The resolution must be that the part that violates Lorentz invariance in $\imath\Delta^{\pm}(x;x')$ is not directly observable. By making use of the identities between the two-point functions discussed above, one can easily construct other two-point functions from the Wightman functions (132). For example, the Feynman and Dyson propagators are given by,

$$\begin{aligned}
\imath\Delta_F(x;x') \;&=\; \Theta(\Delta t)\imath\Delta^{+}(x;x') + \Theta(-\Delta t)\imath\Delta^{-}(x;x') = \frac{1}{4\pi^2}\frac{1}{\Delta x^2_F(x;x')} \\
\imath\Delta_D(x;x') \;&\equiv\; \imath\Delta_{\bar{F}}(x;x') = \frac{1}{4\pi^2}\frac{1}{\Delta x^2_{\bar{F}}(x;x')} = \left[\imath\Delta_F(x;x')\right]^*
\end{aligned} \tag{133}$$

where

$$\begin{aligned}
\Delta x^2_F(x;x') \;&=\; -(|t-t'| - \imath\epsilon)^2 + \|\vec{x}-\vec{x}'\|^2 \\
\Delta x^2_{\bar{F}}(x;x') \;&=\; -(|t-t'| + \imath\epsilon)^2 + \|\vec{x}-\vec{x}'\|^2 = \left[\Delta x^2_F(x;x')\right]^*,
\end{aligned} \tag{134}$$

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[11] Show that the non-zero mass case can be evaluated and that the resulting Feynman propagator can be written in terms of the modified Bessel function of the second kind $K_\nu(z)$ as,

$$i\Delta_F(x;x') = \frac{m^2}{4\pi^2}\frac{K_1\big(\sqrt{m^2\Delta x^2}\big)}{\sqrt{m^2\Delta x^2}}\,, \qquad \Delta x^2_F = -(|t-t'| - i\epsilon)^2 + \|\vec{x}-\vec{x}'\|^2\,.$$

such that both of these propagators are Lorentz invariant. Based on these results, one can easily check that the sum of the Feynman and Dyson propagator equals the sum of the Wightman functions.

By making use of the Plemelj-Sokhotski identity (82), one can decompose the Wightman functions (140) into the principal and homogeneous parts as,

$$\imath\Delta^{\pm}(x;x') = P\frac{1}{4\pi^2}\frac{1}{\Delta x^2(x;x')} \pm \frac{\imath}{4\pi}\mathrm{sign}(\Delta t)\delta(\Delta x^2)\,, \tag{135}$$

such that the spectral (causal) two-point function

$$\imath\Delta^{c}(x;x') = \imath\Delta^{-}(x;x') - \imath\Delta^{+}(x;x') = -\frac{\imath}{2\pi}\mathrm{sign}(\Delta t)\delta(\Delta x^2) \tag{136}$$

and the retarded and advanced propagators are,

$$\begin{aligned} \imath\Delta^{r}(x;x') &= -\Theta(\Delta t)\imath\Delta^{c}(x;x') = \frac{\imath}{2\pi}\Theta(\Delta t)\delta(\Delta x^2) \\ \imath\Delta^{a}(x;x') = \Theta(-\Delta t)\imath\Delta^{c}(x;x') &= \frac{\imath}{2\pi}\Theta(-\Delta t)\delta(\Delta x^2)\,. \end{aligned} \tag{137}$$

We are now ready to discuss the causal properties embedded into various two-point functions. Recall that the retarded propagator is used to solve an inhomogeneous equation of the type (see also the Yang-Feldman equation (115)),

$$\partial^2\hat{\phi}(x) = \hat{j}_{\phi}(x)\,, \tag{138}$$

whose general solution can be written as,

$$\hat{\phi}(x) = \hat{\phi}_h(x) + \int d^4x'\Delta^{r}(x;x')\hat{j}_{\phi}(x') = \hat{\phi}_h(x) + \frac{1}{4\pi}\int d^3x'\frac{\hat{j}_{\phi}(t_r,\vec{x}')}{\|\vec{x}-\vec{x}'\|}\,, \tag{139}$$

where

$$t_r = t - \|\vec{x}-\vec{x}'\|$$

is the retarded time and $\partial^2\hat{\phi}_h(x) = 0$, $\hat{\phi}_h(x)$ specifies the initial data, *i.e.* the field operator at some initial hyper-surface defined by $t = t_0$. From the last expression in (139) we see that the field $\hat{\phi}(t;\vec{x}\,)$ at a time t is influenced by the current $\hat{j}_{\phi}(t',\vec{x}\,')$ only if t' is in the past of t, and if $(t',\vec{x}\,')$ lies on the past light-cone of $(t,\vec{x})$, *i.e.* if $t' = t_r = t - \|\vec{x}-\vec{x}'\|$, see figure 8. Figure 8 illustrates the future and past light-cones, where for simplicity we represent the two dimensional sphere S^2 covered by angular coordinates θ and ϕ by a point. From this discussion we see that causality is an essential feature in quantum field theory, and that building covariant actions has

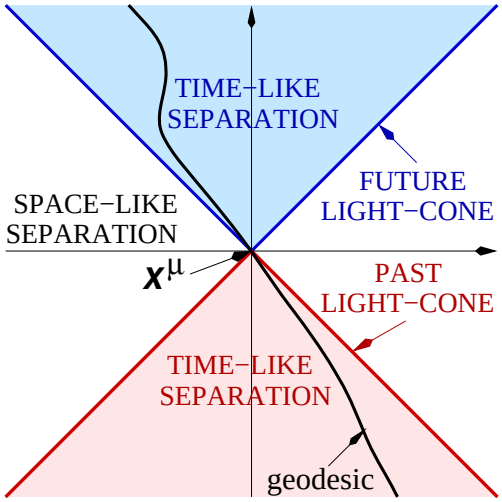

FIG. 8: The past and future light-cones in Minkowski space-time separate time-like from space-like distances. Time is on the vertical axis, and space (radial distance) on the horizontal axis. Each point on the diagram corresponds to a two-dimensional sphere S^2 of the spatial section of space-time. The diagram also shows a typical time-like geodesic on which (massive) particles can move, and which lies within the past and future light-cones.

as a consequence causality at the level of solutions. The causal properties of other two-point functions can be summarized as follows. The Feynman and Dyson propagators do not respect causality, in the sense that, when a test function is integrated against the Feynman or Dyson propagator, one gets influences from outside the past or future light-cones, *i.e.* from points that lie on some space-like curve in figure 8 can also contribute to the integral. Similar properties hold for the Wightman functions, as well as for the statistical (Hadamard) functions. The Wightman functions and the statistical function F pick up medium effects outside the light-cone, but these effects do not indicate physical correlations: they have the property of a random noise, and hence cannot be used to transfer any information faster than light. To see this, let us act with (140) on a test function $f(x)$,

$$\int d^4x \imath\Delta^{\pm}(x;x')f(x') = \frac{1}{4\pi^2}P\int d^4x' \frac{f(x')}{\Delta x^2(x;x')} \pm \frac{\imath}{4\pi}\int d^3x' \Big[f(t_r,\vec{x}\,') - f(t_a,\vec{x}\,')\Big]\,, \quad (140)$$

where $t_{r,a} = t \mp \|\vec{x}-\vec{x}'\|$ denote the retarded and advanced time that lie on the past and future light-cone, respectively. This means that the first integral (representing a real contribution) picks in general contributions from within as well as from outside the light-cones, while the latter contributions (representing imaginary contributions) come either from within the past or

from within the future light-cone.

The proper causal functions are: the spectral function, and the retarded and advanced propagators, and their influence comes from within the light-cones only. The story of causal two-point functions and propagators is carried over to general properties of evolution of observables in quantum field theories. Namely, evolution of physical quantities such as (transverse components) of the vector potential, a (mean) Higgs field and the corresponding two-point functions (that carry essential information about the fluctuations about the mean value) obey in quantum field theory equations with a well defined causal structure that is similar to that shown in (115) and in which the influence is limited to lie within the past light-cone. This can be rigorously shown by making use of the Schwinger-Keldysh formalism, which is the subject of section IV.

IV. THE SCHWINGER-KELDYSH FORMALISM

The Schwinger-Keldysh, *in-in* or closed-time-path (CTP), [12] formalism is constructed to facilitate studying evolution of observables (Hermitean operators) in quantum systems (usually in the Heisenberg or interaction picture). From section II we already know how to represent evolution of a Hermitean operator $\hat{O}(t)$ (see Eq. (14)); the Schwinger-Keldysh formalism is designed to represent evolution such that it can be neatly organized into a perturbation theory in some suitably chosen small parameter. On top of it, there are resummation techniques that have been developed and that are particularly suitable for non-equilibrium problems that go under the name of two-particle irreducible (2PI) methods, in which the basic object is a 2PI effective action, and the corresponding equation of motion. The 2PI effective action is a double Legendre transform of the classical (tree level) action (one with respect to the field and one with respect to the suitable two-point functions). Numerical experiments show that the resulting equations of motion (when solved within an appropriate perturbative scheme) give good late time behaviour in many non-equilibrium systems of physical interest. Namely, the hope (some aspects of which need to be verified) is that these equations accurately describe the dynamics of (many relevant) quantum systems. A good, but more advanced, review of the in-in formalism, together with the 2PI and more generally nPI resummation techniques, can be found in [8].

Examples where the in-in formalism can be (and has been) applied abound. In particular, it has been applied in cosmology, condensed matter problems, and in studying heavy ion collisions. In cosmology, it has been used for studying quantum effects during cosmological inflation, for preheating (heating of the Universe after inflation), for studying the dynamics of phase transitions in the early universe (an important example being the electroweak transition); for baryogenesis (dynamical matter-antimatter creation); for the problem of dark matter creation and evolution in the early Universe setting, *etc.* In condensed matter for studying the dynamics of phase transi-

[12] The name *in-in* derives from the fact that one follows evolution of an operator from some initial state at $t = t_0$ to some final state at $t > t_0$ without knowing what the suitable late time states are. Evolving an operator from t_0 to t requires a time-ordered evolution from t_0 to t and the corresponding anti-time-ordered evolution, which can be thought of as evolution from t back to t_0, hence the in-in formalism. When these two integration paths are joined together, one gets the closed-time-path (CTP), which explains the origin of the third name. This is in contrast to the more common *in-out* formalism, where one studies a transition amplitude between some well defined initial (or in) state and some well defined final (or out) state. The in-in formalism is needed for many non-equilibrium problems because we do not know *a priori* what are the suitable out states to use (hence one uses instead a complete set of out-states).

tions, quantum transport (in *e.g.* spintronics and systems with randomly distributed impurities). In heavy ion collisions various approximations have been attempted that range from (classical) hydrodynamics to the full quantum 2PI evolution truncated in some perturbative scheme.

This section is organized as follows. In the next subsection we give a brief introduction to the in-out formalism in which the transition amplitude between some initial and some final state (generally in presence of interactions) is the object of fundamental interest. Many of these techniques are useful and carry over to the in-in formalism, which is what we discuss thereafter.

A. The in-out Formalism

The basic object of interest in the in-out formalism is the *transition amplitude*, which can be formulated in terms of the time dependent position eigenstates

$$|q_0, t_0\rangle = \hat{U}^\dagger(t_0, 0)|q_0\rangle = \bar{T} \exp\left(\frac{\imath}{\hbar}\int_0^{t_0} d\tilde{t}\hat{H}(\tilde{t})\right)|q_0\rangle\,, \tag{141}$$

where $|q_0\rangle \equiv |q_0, 0\rangle$ is the eigenstate of $\hat{q}(0)$ and $|q_0, t_0\rangle$ is the eigenstate of

$$\hat{q}(t_0) = \hat{U}^\dagger(t_0, 0)\hat{q}(0)\hat{U}(t_0, 0)\,. \tag{142}$$

It seems as if $|q_0, t_0\rangle$ in (141) evolves in the opposite sense. In fact this has nothing to do with evolution of the state $|q_0, t_0\rangle$. Instead it is derived from evolution of the position operator in the Heisenberg picture (142). The transition amplitude $F(q_*, t_*|q_0, t_0)$ is then defined as the probability amplitude that a particle, which is at an initial time t_0 at a position q_0, will be at a time t_* at a position q_*,

$$F(q_*, t_*|q_0, t_0) = \langle q_*, t_*|q_0, t_0\rangle = \langle q_*|\hat{U}(t_*, 0)\hat{U}^\dagger(t_0, 0)|q_0\rangle = \langle q_*|\hat{U}(t_*, t_0)|q_0\rangle\,, \tag{143}$$

where in the last step we made use of the composition property of the evolution operator, $\hat{U}(t_*, t_0) = \hat{U}(t_*, 0)\hat{U}(0, t_0) = \hat{U}(t_*, 0)\hat{U}^\dagger(t_0, 0)$. With this Eq. (143) simplifies to,

$$F(q_*, t_*|q_0, t_0) = \langle q_*|T \exp\Big(-\frac{\imath}{\hbar}\int_{t_0}^{t_*} dt\hat{H}(t)\Big)|q_0\rangle\,. \tag{144}$$

From these considerations it then follows that one can also think of the in-out transition amplitude as the position space representation of the evolution operator.

Let us for definiteness consider the simple quantum mechanical case where the Hamiltonian is given by,

$$\hat{H} = H(\hat{q}, \hat{p}) = \frac{\hat{p}^2}{2m} + V(\hat{q})\,, \tag{145}$$

where $V(\hat{q}) = (m\omega^2/2)\hat{q}^2 + V_{\rm int}(\hat{q})$. It is convenient to represent $F(q_*, t_*|q_0, t_0)$ in the path integral formalism, in which one divides the time interval $t_* - t_0 > 0$ into N infinitesimal steps δt, according to $t_* - t_0 = N\delta t$ (one subsequently takes the limit $N \to \infty$) and then one inserts $(N-1)$ times a complete set of states,

$$\mathbb{I} = \int dq_\ell |q_\ell\rangle\langle q_\ell| \tag{146}$$

to get,

$$F(q_*, t_*|q_0, t_0) = \langle q_*, t_*| \left(\prod_{\ell=1}^{N-1} \int dq_\ell |q_\ell, t_\ell\rangle\langle q_\ell, t_\ell| \right) |q_0, t_0\rangle \,, \tag{147}$$

where

$$\begin{aligned} \langle q_{\ell+1}, t_{\ell+1}|q_\ell, t_\ell\rangle &= \langle q_{\ell+1}|\hat{U}(t_{\ell+1}, t_\ell)|q_\ell\rangle \\ &= \langle q_{\ell+1}| \exp\Big(-\frac{\imath}{\hbar}\delta t\Big[\frac{\hat{p}^2}{2m} + V(\hat{q})\Big] + \mathcal{O}(\delta t^2)\Big)|q_\ell\rangle \end{aligned} \tag{148}$$

and $q_N = q_*, t_N = t_*$. Using the Zassenhuis formula (which is easily derived from the Baker-Campbell-Hausdorff formula),

$$\mathrm{e}^{t(X+Y)} = \mathrm{e}^{tX}\mathrm{e}^{tY}\mathrm{e}^{\frac{t^2}{2}[X,Y]}\mathrm{e}^{\mathcal{O}(t^3)} \tag{149}$$

we find,

$$\exp\left(-\frac{\imath}{\hbar}\delta t\Big[\frac{\hat{p}^2}{2m} + V(\hat{q})\Big]\right) = \exp\left(-\frac{\imath}{\hbar}\delta t\frac{\hat{p}^2}{2m}\right)\exp\left(-\frac{\imath}{\hbar}\delta t V(\hat{q})\right)\exp\Big(\mathcal{O}(\delta t^2)\Big) \tag{150}$$

and hence

$$\langle q_{\ell+1}, t_{\ell+1}|q_\ell, t_\ell\rangle = \int \frac{dp_\ell}{2\pi\hbar}\langle q_{\ell+1}| \exp\left(-\frac{\imath}{\hbar}\delta t\frac{p_\ell^2}{2m}\right)|p_\ell\rangle\langle p_\ell| \exp\left(-\frac{\imath}{\hbar}\delta t V(q_\ell)\right)\Big)|q_\ell\rangle + \mathcal{O}(\delta t^2) \tag{151}$$

where we inserted a complete set of momentum states.

$$\mathbb{I} = \int \frac{dp_\ell}{2\pi\hbar}|p_\ell\rangle\langle p_\ell| \tag{152}$$

Now, upon using $\langle q_{\ell+1}|p_\ell\rangle = \exp\big(\imath p_\ell q_{\ell+1}/\hbar\big)$ and inserting (151) into (147) we obtain,

$$F(q_*, t_*|q_0, t_0) = \left(\prod_{\ell=1}^{N-1}\int dq_\ell\right)\left(\prod_{\ell=0}^{N-1}\int\frac{dp_\ell}{2\pi\hbar}\right)\exp\left(\frac{\imath\delta t}{\hbar}\sum_{\ell=0}^{N-1}\Big[p_\ell\frac{q_{\ell+1}-q_\ell}{\delta t}\Big] - H(p_\ell, q_\ell)\right). \tag{153}$$

Taking the limit $N \to \infty$, this can be symbolically written as,

$$F(q_*, t_*|q_0, t_0) = \int_{q(t_0)=q_0}^{q(t_*)=q_*} \mathcal{D}q\mathcal{D}p\, \exp\left(\frac{\imath}{\hbar}\int_{t_0}^{t_*} dt\,\big[p\,\dot{q} - H(p, q)\big]\right). \tag{154}$$

The exponent is just the action in disguise (where the Lagrangian is written as a Legendre transform of the Hamiltonian). Indeed, upon performing the momentum integration in (154) (this can be done quite easily by starting with the discretized form (153)), Eq. (154) gives the more familiar form for the transition amplitude,

$$\begin{aligned} F(q_*, t_*|q_0, t_0) &= \int_{q(t_0)=q_0}^{q(t_*)=q_*} \mathcal{D}q \exp\left(\frac{\imath}{\hbar} S[q]\right) \\ S[q] &\equiv \int_{t_0}^{t_*} dt L(q, \dot{q}) = \int_{t_0}^{t_*} dt \Big[\frac{m\dot{q}^2}{2} - V(q)\Big] , \end{aligned} \tag{155}$$

where $S[q]$ denotes the tree level action, which in general contains interactions and the discrete version of the path integral in (155) is now,

$$\int_{q(t_0)=q_0}^{q(t_*)=q_*} \mathcal{D}q = \sqrt{\frac{m}{2\pi\imath\hbar\delta t}} \prod_{\ell=1}^{N-1} \left(\int dq_\ell \sqrt{\frac{m}{2\pi\imath\hbar\delta t}} \right) . \tag{156}$$

We have thus learned that, when written in the Feynman path integral formalism, the evolution operator of the operator formalism gets replaced by the complex 'weight factor' $\exp\left(\frac{\imath}{\hbar} S[q, \dot{q}]\right)$. We emphasize that Eqs. (153–154) is more fundamental and that, if there is any doubt, one should start with the transition amplitude as written in Eqs. (153–154).

Eqs. (154–155) represent our starting point for further analysis, where we discuss the in-out partition function. One is often interested in an in-out transition amplitude between more general in and out states, so the above analysis requires a generalization to arbitrary states. A further generalization that is very useful is to couple some operator (whose expectation value we are interested in) to an external current, and calculate the transition amplitude in presence of a source current. This can be considered as a mathematical trick, but it also can correspond to a physical situation. Indeed, electromagnetic vector potential $A_\mu(x)$ couples linearly to a charge current density $j^\mu(x)$ *via* the following interaction action (*cf.* Eq. (28)),

$$S_{\rm int} = -\frac{1}{c^2} \int d^4x j^\mu A_\mu \, . \tag{157}$$

The in-out partition function (also known as the generating functional) between an *in* state ψ_0 at t_0 and an *out* state ψ at t_* is defined as,

$$Z[J] = \langle \psi, t_*|\psi_0, t_0\rangle_J = \langle \psi| T \exp\left(-\frac{\imath}{\hbar} \int_{t_0}^{t_*} dt \left[\hat{H}(t) - J(t)\hat{q}(t)\right] \right) |\psi_0\rangle \, . \tag{158}$$

When reformulated in the path integral formalism, analogous steps as above result in,

$$Z[J] = \int dq_* \int dq_0 \, \psi^*(q_*)\psi_0(q_0) F_J(q_*, t_*|q_0, t_0) \, , \tag{159}$$

where $\psi_0(q) = \langle q|\psi, t_0\rangle = \langle q, t_0|\psi, 0\rangle$ ($\psi(q_*) = \langle q_*|\psi, t_*\rangle = \langle q_*, t_*|\psi, 0\rangle$) is the initial (final) state onto which we project and $F_J(q_*, t_*|q_0, t_0)$ is the transition amplitude in presence of a current J (see Eqs. (154) and (158)),

$$F_J(q_*, t_*|q_0, t_0) = \int_{q(t_0)=q_0}^{q(t_*)=q_*} \mathcal{D}q\mathcal{D}p \exp\left(\frac{\imath}{\hbar}\int_{t_0}^{t_*} dt\Big[p\dot{q} - H(q,p) + Jq\Big]\right) . \tag{160}$$

For notational simplicity we shall often use the simpler (albeit less fundamental) form for the transition amplitude given in Eq. (155), such that Eq. (158) can be written more compactly as,

$$Z[J] \equiv \mathrm{e}^{\frac{i}{\hbar}W[J]} = \int dq_* \int dq_0\, \psi^*(q_*)\psi_0(q_0) \int_{q(t_0)=q_0}^{q(t_*)=q_*} \mathcal{D}q \exp\left(\frac{\imath}{\hbar}\int_{t_0}^{t_*} dt\Big[L(q,\dot{q}) + Jq\Big]\right) , \tag{161}$$

where $\int_{t_0}^{t_*} dt L(q,\dot{q}) = S[q,\dot{q}]$ is the (classical) action, which contains free (quadratic) parts and interaction (cubic, quartic, *etc*) parts,

$$S[q,\dot{q}] = S_0[q,\dot{q}] + S_{\rm int}[q] . \tag{162}$$

Even though we started with a time-independent Hamiltonian, the result (161) is applicable to general Lagrangians which are explicitly time dependent. Furthermore, the form (161) is correct for Hamiltonians with at most quadratic dependence on the canonical momentum. When the Hamiltonian contains higher powers of the momentum one should use the more general phase-space formulation given in (154).

The in-out partition function can be used, for example, to calculate the expectation value of (time ordered) $n-$point functions. For example, the $1-$point function $\bar{q}_J(t) = \langle\hat{q}(t)\rangle_J$ is obtained by a single variation of $Z[J]$ with respect to $J(t)$,

$$\begin{aligned}\langle\hat{q}(t)\rangle_J &= \frac{1}{Z[J]}\frac{\hbar\delta Z[J]}{\imath\delta J(t)} \\ &= \frac{1}{Z[J]}\int dq_* \int dq_0\, \psi^*(q_*)\psi_0(q_0) \int_{q(t_0)=q_0}^{q(t_*)=q_*} \mathcal{D}q q(t) \exp\left(\frac{\imath}{\hbar}\int_{t_0}^{t_*} dt\Big[L(q,\dot{q}) + Jq\Big]\right) ,\end{aligned} \tag{163}$$

while a time-ordered two-point function is obtained by an additional variation of $Z[J]$ with respect to $J(t)$,

$$\begin{aligned}\langle T[\hat{q}(t)\hat{q}(t')]\rangle_J &= \frac{\hbar^2}{Z[J]}\frac{\delta^2 Z[J]}{\imath\delta J(t')\imath\delta J(t)} \\ &= \frac{1}{Z[J]}\int dq_* \int dq_0 \psi^*(q_*)\psi_0(q_0) \int_{q(t_0)=q_0}^{q(t_*)=q_*} \mathcal{D}q q(t)q(t') \exp\left(\frac{\imath}{\hbar}\int_{t_0}^{t_*} dt\Big[L(q,\dot{q}) + Jq\Big]\right) ,\end{aligned} \tag{164}$$

and an analogous formula can be written for a time ordered $n-$point function. An important application of the partition function,

$$Z[J] = \mathrm{e}^{\imath W[J]/\hbar}$$

is in that one can use it as a starting point for construction of the effective action, which is defined as a Legendre transform of $W[J]$,

$$\Gamma[\bar{q}] = W[J] - \int_{t_0}^{t_*} dt J\bar{q}[J]\,, \tag{165}$$

where $J[\bar{q}]$ is defined as the inverse of,

$$\frac{\delta W[J]}{\delta J} = \bar{q}[J]\,. \tag{166}$$

The effective action contains the information about the dynamics of the *mean position* $\bar{q}$ [13] which is determined by the variation principle,

$$\frac{\delta\Gamma[\bar{q}]}{\delta\bar{q}} = 0\,. \tag{167}$$

The principal difficulty in calculating the effective action is in finding the inverse of the functional differential equation (166), and in general it is possible to find it only within some approximation scheme.

One problematic feature of the in-out formalism is that $Z[J]$ generates $n-$point functions [14] only if the initial and final states are the eigenstates of the Hamiltonian. To see that, let us consider the case when the in- and out-states are a super-position of the ground state $|0\rangle$ and the first excited state $|1\rangle$ (of an SHO or some other system Hamiltonian) or of any two eigenstates of the Hamiltonian,

$$|\psi_0\rangle = \alpha|0\rangle + \beta|1\rangle\,, \quad (|\alpha|^2 + |\beta|^2 = 1)\,, \tag{168}$$

defined at t_0. The expectation value of $\hat{O}_H(t)$ is

$$\langle\psi_0|\hat{O}_H(t)|\psi_0\rangle = |\alpha|^2\langle 0|\hat{O}_H(t)|0\rangle + |\beta|^2\langle 1|\hat{O}_H(t)|1\rangle + \alpha\beta^*\langle 1|\hat{O}_H(t)|0\rangle + \alpha^*\beta\langle 0|\hat{O}_H(t)|1\rangle\ . \tag{169}$$

[13] In literature $\bar{q} \equiv \langle\hat{q}\rangle$ is referred to as classical or mean field. One should keep in mind however, that the dynamics of $\bar{q}(t)$ is equal to the dynamics of $\langle\hat{q}\rangle(t)$ and therefore it captures *all* quantum effects in the theory. This is also true in QFT: a variation of the effective action generates the dynamics of expectation value of quantum field operators, and thus it incorporates all quantum effects of the theory.

[14] A second, commonly used, name for $Z[J]$ is the generating functional, but here we opt for the partition function, because generating functional is a generic name that refers to the generating functional of any mathematical problem.

The in-out generating functional when the out state is $|\psi\rangle$ as well is

$$Z[J] = \langle\psi|\hat{U}_J(t_*, t_0)|\psi_0\rangle \ . \tag{170}$$

Assuming as usually that $\hat{O}$ couples to an external current $J(t)$ linearly, $S_{\rm int} = \int_{t_0}^{t_*} dt J(t)\mathcal{O}_H(t)$, the first functional derivative of (170) yields,

$$\begin{aligned} \frac{\hbar}{i}\frac{\delta Z}{\delta J(t)} &= \langle\psi|\hat{U}_J(t_*, t)\hat{O}_S(t)\hat{U}_J(t, t_0)|\psi_0\rangle \\ &= \langle\psi|\hat{U}_J(t_*, t)\hat{U}(t, t_0)\hat{O}_H(t)|\psi_0\rangle = \langle\psi|\hat{U}_J(t_*, t_0)\hat{O}_H(t)|\psi_0\rangle \ . \end{aligned} \tag{171}$$

Now set the current to zero and take the in and out states to be identical, $\langle\psi| \to \langle\psi_0|$, one arrives at,

$$\frac{\hbar}{i}\frac{\delta Z}{\delta J(t)}\bigg|_{J=0,\psi=\psi_0} = \langle\psi_0|\hat{U}(t_*, t_0)\hat{O}_H(t)|\psi_0\rangle \ . \tag{172}$$

Obviously this is not the expectation value we are looking for because of the extra evolution factor $\hat{U}(t_*, t_0)$. Let us see what it does to the state,

$$\langle\psi_0|\hat{U}(t_*, t_0) = \alpha^* e^{-\frac{i}{\hbar}E_0(t_*-t_0)}\langle 0| + \beta^* e^{-\frac{i}{\hbar}E_1(t_*-t_0)}\langle 1| \, . \tag{173}$$

Then (172) becomes,

$$\begin{aligned} \frac{\hbar}{i}\frac{\delta Z}{\delta J(t)}\bigg|_{J=0,\psi=\psi_0} &= |\alpha|^2 e^{-\frac{i}{\hbar}E_0(t_*-t_0)}\langle 0|\hat{O}_H(t)|0\rangle + |\beta|^2 e^{-\frac{i}{\hbar}E_1(t_*-t_0)}\langle 1|\hat{O}_H(t)|1\rangle \\ &\quad + \alpha^*\beta e^{-\frac{i}{\hbar}E_0(t_*-t_0)}\langle 0|\hat{O}_H(t)|1\rangle + \alpha\beta^* e^{-\frac{i}{\hbar}E_1(t_*-t_0)}\langle 1|\hat{O}_H(t)|0\rangle \end{aligned} \tag{174}$$

Now, the claim is that by dividing this by $Z[0]$,

$$Z[0] = \langle\psi|U(t_*, t_0)|\psi\rangle = |\alpha|^2 e^{-\frac{i}{\hbar}E_0(t_*-t_0)} + |\beta|^2 e^{-\frac{i}{\hbar}E_1(t_*-t_0)} \ , \tag{175}$$

it becomes a genuine quantum mechanical expectation value,

$$\begin{aligned} \frac{1}{Z[J]}\frac{\hbar}{i}\frac{\delta Z}{\delta J}\bigg|_{J=0,\psi=\psi_0} &= \frac{1}{|\alpha|^2 e^{-\frac{i}{\hbar}E_0(t_*-t_0)} + |\beta|^2 e^{-\frac{i}{\hbar}E_1(t_*-t_0)}}\Big(|\alpha|^2 e^{-\frac{i}{\hbar}E_0(t_*-t_0)}\langle 0|\hat{O}_H(t)|0\rangle \\ &\quad + |\beta|^2 e^{-\frac{i}{\hbar}E_1(t_*-t_0)}\langle 1|\hat{O}_H(t)|1\rangle + \alpha^*\beta e^{-\frac{i}{\hbar}E_0(t_*-t_0)}\langle 0|\hat{O}_H(t)|1\rangle + \alpha\beta^* e^{-\frac{i}{\hbar}E_1(t_*-t_0)}\langle 1|\hat{O}_H(t)|0\rangle\Big) \ . \end{aligned} \tag{176}$$

But comparing this quantity with (169), which is the genuine expectation value, shows that they clearly do not match. The case when they do match is when the state in question reduces to the eigenstate of the Hamiltonian, for example when $\alpha \to 0$ (then $\beta \to 1$). In that case both expressions reduce to $\langle 1|\hat{O}_H(t)|1\rangle$. Needless to say, when the out state differs from the in state,

then the discrepancy becomes even more obvious. Hence the in-out partition function generates correctly $n-$point functions (and more generally expectation values of Hermitian operators) only when both the in- and out-states are eigenstates of the Hamiltonian. Albeit this is quite a stringent requirement, it is to high accuracy satisfied by scattering experiments in particle physics. Indeed, in these experiments one can approximately set, $t_0 \to -\infty$ and $t_* \to +\infty$. In both of these limits the Hamiltonian reduces to a free (time-independent) Hamiltonian for which particle number is well defined and conserved. In other words, any initial state with a definite particle number is an eigenstate of the (asymptotic) Hamiltonian, and such states are most often taken to be the in- and out- states in scattering experiments. On the other hand, in typical non-equilibrium situations these requirements are most of the time not met, and therefore we need a more general framework.

Next, upon an inspection of Eq. (163) we see that it is possible to rewrite the partition function $Z[J]$ in the form in which the interaction terms $S_{\rm int}$ (see Eq. (162)) appear as the suitable functional derivative acting on the partition function of the free (non-interacting) theory (162) as follows,

$$Z[J] = \exp\left(\frac{\imath}{\hbar} S_{\rm int}\Big[\frac{\hbar}{\imath}\frac{\delta}{\delta J}\Big]\right) Z_0[J] \tag{177}$$

where

$$Z_0[J] = \int dq_* \int dq_0\, \psi^*(q_*)\psi_0(q) \int_{q(t_0)=q_0}^{q(t_*)=q_*} \mathcal{D}q \exp\left(\frac{\imath}{\hbar}\int_{t_0}^{t_*} dt\Big[L_0(q,\dot{q}) + Jq\Big]\right) , \tag{178}$$

where L_0 is a quadratic (free) Lagrangian, for which Z_0 can be explicitly evaluated. When a small parameter can be identified in $S_{\rm int}$, it is useful to expand the exponential term in (177) as,

$$\exp\left(\frac{\imath}{\hbar} S_{\rm int}\Big[\frac{\hbar}{\imath}\frac{\delta}{\delta J}\Big]\right) = 1 + \frac{\imath}{\hbar} S_{\rm int}\Big[\frac{\hbar}{\imath}\frac{\delta}{\delta J}\Big] + \cdots . \tag{179}$$

This form is suitable as a starting point for a perturbative expansion.

One can use Eq. (164) to define the Feynman propagator by the limiting procedure,

$$\imath\Delta_F(t;t') \equiv \lim_{J\to 0}\langle T[\hat{q}(t)\hat{q}(t')]\rangle_J = \langle T[\hat{q}(t)\hat{q}(t')]\rangle\,. \tag{180}$$

Since the order of taking functional derivatives in (164) is irrelevant, it is clear that the two-point function (164) is symmetric under the exchange of times, $t \leftrightarrow t'$, which is precisely what is required by the Feynman propagator. At first sight it is not however clear how this is realised in detail and therefore we shall now show that the symmetry indeed holds for the two-point

function (164) or, equivalently, for (180). For simplicity, let us assume that the free action S_0 is that of the simple harmonic oscillator (SHO),

$$S_0 = \int_{t_0}^{t_*} dt \left(\frac{m\dot{q}^2}{2} - \frac{m\omega^2}{2} q^2 \right) \tag{181}$$

and that the initial and final states are the vacuum states of SHO,

$$\psi_0(q_0) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega q_0^2}{2\hbar}\right), \qquad \psi(q_*) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega q_*^2}{2\hbar}\right) \tag{182}$$

It is convenient to partially integrate the action (181) to get,

$$S_0 = \frac{m}{2}\left(q_* \frac{dq}{dt}(t_*) - q_0 \frac{dq}{dt}(t_0) \right) + \int_{t_0}^{t_*} dt\, q(t) \frac{m}{2} \left(-\partial_t^2 - \omega^2\right) q(t)\,, \tag{183}$$

or equivalently,

$$S_0 = \int_{t_0}^{t_*} dt \int_{t_0}^{t_*} dt' q(t) \frac{m}{2} \left[-\partial_t^2 - \omega^2 + \delta(t-t_*)\partial_t - \delta(t-t_0)\partial_t\right] \delta(t-t') q(t')\,. \tag{184}$$

When the contributions from the initial states (182) are included, the free part of the partition function can be compactly written as,

$$Z_0[J] = \int \mathcal{D}q \exp\left(\frac{\imath}{\hbar} \int_{t_0}^{t_*} dt \int_{t_0}^{t_*} dt' \left[\frac{1}{2} q(t) D_0(t;t') q(t') + J(t)\delta(t-t') q(t') \right] \right), \tag{185}$$

where

$$D_0(t;t') = m\left[-\partial_t^2 - \omega^2 + \delta(t-t_*)(\partial_t + \imath\omega) + \delta(t-t_0)(-\partial_t + \imath\omega)\right] \delta(t-t') \tag{186}$$

and (from (156) and (182))

$$\int \mathcal{D}q = \sqrt{2\imath\omega\delta t} \prod_{\ell=0}^{N} \left(\int dq_\ell \frac{m}{2\pi\imath\hbar\delta t} \right), \tag{187}$$

whereby the integrations over the initial and final states have been absorbed in the definition of the path integral. It is now easy to integrate (185) by completing the square, which is done by shifting $q(t)$ as,

$$q(t) \to \tilde{q}(t) = q(t) + \frac{1}{\hbar} \int d\tilde{t}\, \Delta_F(t;\tilde{t}) J(\tilde{t}) \tag{188}$$

where $\Delta_F/\hbar$ is the operator inverse of D_0 in (186),

$$\int d\tilde{t}\, D_0(t;\tilde{t}) \imath \Delta_F(\tilde{t};t') = \imath\hbar\delta(t-t')\,. \tag{189}$$

With this, Eq. (185) can be reduced to

$$Z_0[J] \;=\; \int \mathcal{D}\tilde{q}\exp\left(\frac{\imath}{\hbar}\int_{t_0}^{t_*}dt\int_{t_0}^{t_*}dt'\frac{1}{2}\tilde{q}(t)D_0(t;t')\tilde{q}(t')\right) \\ \times\exp\left(-\frac{\imath}{2\hbar^2}\int_{t_0}^{t_*}dt\int_{t_0}^{t_*}dt'J(t)\Delta_F(t;t')J(t')\right)\,. \tag{190}$$

This can be integrated to yield (up to a $J-$independent factor $\tilde{Z}_0$),

$$Z_0[J] = \tilde{Z}_0(t_0;t_*)\exp\left(-\frac{1}{2\hbar^2}\int_{t_0}^{t_*}dt\int_{t_0}^{t_*}dt'J(t)\imath\Delta_F(t;t')J(t')\right)\,, \tag{191}$$

From the form of Eq. (191) we see immediately that the (free) Feynman propagator follows from Eqs. (164) and (180). Eq. (189) can be also considered as the definition of Feynman propagator, which can be also written in a more transparent way as,

$$\begin{aligned} m(-\partial_t^2-\omega^2)\imath\Delta_F(t;t') \;&=\; \imath\hbar\delta(t-t') \\ \delta(t-t_0)(-\partial_t+\imath\omega)\imath\Delta_F(t;t') \;&=\; 0 \\ \delta(t-t_*)(\partial_t+\imath\omega)\imath\Delta_F(t;t') \;&=\; 0\,. \end{aligned} \tag{192}$$

In fact, these equations provide a rigorous mathematical definition of the Feynman propagator. While the first equation is the dynamical (bulk) equation, the latter two are the equations that determine the Feynman propagator at the early ($t=t_0$) and late time ($t=t_*$) boundaries. We shall now show that equations (192) uniquely determine the Feynman propagator, and that their solution agrees with the one obtained by the operator formalism.

The first equation in (192) tells us that the solution is of the form,

$$\imath\Delta_F(t;t') = \Theta(\Delta t)\imath\Delta^+(t;t') + \Theta(-\Delta t)\imath\Delta^-(t;t') \tag{193}$$

where $\Delta t = t-t'$ and $\imath\Delta^\pm(t;t')$ are arbitrary functions of t and t' (not to be identified with the positive and negative frequency Wightman functions, even though the name is suggestive) that satisfy the homogeneous equations,

$$m(-\partial_t^2-\omega^2)\imath\Delta^\pm(t;t') = 0\,, \tag{194}$$

whose general solution is of the form,

$$\imath\Delta^\pm(t;t') = A^\pm(t')\mathrm{e}^{-\imath\omega t} + B^\pm(t')\mathrm{e}^{\imath\omega t}\,, \tag{195}$$

where $A^{\pm}(t')$ and $B^{\pm}(t')$ are arbitrary functions of time t' (which plays the role of a parameter in this solution). Since $\Theta(\Delta t) = 0$ at $t = t_0$ and $\Theta(-\Delta t) = 0$ at $t = t_*$, the second and the third equations (for boundary conditions) in (192) can be recast as,

$$\delta(t-t_0)(-\partial_t + \imath\omega)\imath\Delta^-(t;t') = 0\,, \qquad \delta(t-t_*)(\partial_t + \imath\omega)\imath\Delta^+(t;t') = 0\,. \tag{196}$$

from which it follows that $A^-(t') = 0$ and $B^+(t') = 0$, and hence,

$$\imath\Delta^+(t;t') = A^+(t')\mathrm{e}^{-\imath\omega t}\,, \qquad \imath\Delta^-(t;t') = B^-(t')\mathrm{e}^{\imath\omega t}\,. \tag{197}$$

Next, upon acting with the first time derivative on (193) one gets,

$$\partial_t\imath\Delta_F(t;t') = \delta(\Delta t)[\imath\Delta^+(t;t') - \imath\Delta^-(t;t')] + \Theta(\Delta t)\partial_t\imath\Delta^+(t;t') + \Theta(-\Delta t)\partial_t\imath\Delta^-(t;t')\,. \tag{198}$$

Since the second time derivative must not produce a $\partial_t\delta$ function, Eq. (198) implies that $\Delta^{\pm}$ must be equal at coincidence,

$$\imath\Delta^+(t';t') - \imath\Delta^-(t';t') = 0\,. \tag{199}$$

Acting the second time derivative on (198) and taking account that both Δ^+ and Δ^- obey homogeneous equations results in,

$$\delta(t-t')\Big[\partial_t\imath\Delta^+(t;t') - \partial_t\imath\Delta^-(t;t') + \frac{\imath\hbar}{m}\Big] = 0\,. \tag{200}$$

Inserting (197) into (199–200) one finally gets,

$$\imath\Delta^+(t;t') = \frac{\hbar}{2m\omega}\mathrm{e}^{-\imath\omega(t-t')}\,, \qquad \imath\Delta^-(t;t') = \frac{\hbar}{2m\omega}\mathrm{e}^{\imath\omega(t-t')}\,, \tag{201}$$

which are precisely the positive and negative frequency Wightman functions (69) obtained in the operator formalism.

To summarize, we have shown that the Feynman propagator is uniquely determined from the path integral formalism by Eqs. (164) and the result is (193), where $\imath\Delta^{\pm}(t;t')$ in (201) represent the positive and negative frequency Wightman functions of the SHO vacuum, respectively. Even though we have shown this for the case of simple initial and final states (given by the ground states of the simple harmonic oscillator), Eq. (193) holds true for general initial and final states, with $\imath\Delta^{\pm}$ being the Wightman functions that correspond to those states.

B. The in-in formalism

The *Schwinger-Keldysh*, or *in-in*, formalism is suitable for calculation of expectation value of (Heisenberg picture) operators,

$$\mathcal{O}(t) = \langle \hat{\mathcal{O}}_H(t) \rangle = \mathrm{Tr}[\hat{\rho}_H(t)\hat{\mathcal{O}}_H(t)] \,, \tag{202}$$

where $\hat{\mathcal{O}}_H(t)$ obeys the Heisenberg equation (14) whose formal solution (15) can be written in terms of the evolution operator $\hat{U}(t,t_0)$ in Eq. (6). In most situations of physical interest it is not possible to solve exactly for the evolution operator. However it is possible to find a perturbative solution. For that, one ought to split the Hamiltonian into the part which generates the evolution we know how to solve exactly (usually the quadratic part of the Hamiltonian) $\hat{H}_0(t)$ and the rest, in which we hope to identify a small parameter that controls a perturbative expansion and to which we refer to as the interaction Hamiltonian, $\hat{H}_{\rm int}(t)$, such that in the Schrödinger picture we have,

$$\hat{H}(t) = \hat{H}_0(t) + \hat{H}_{\rm int}(t) \,. \tag{203}$$

We can then define an interaction picture operator and the corresponding Hamiltonian as,

$$\hat{\mathcal{O}}_I(t) = \hat{U}_0^\dagger(t;t_0)\hat{\mathcal{O}}_S(t)\hat{U}_0(t;t_0) \,, \qquad \hat{H}_I(t) = \hat{U}_0^\dagger(t;t_0)\hat{H}_{\rm int}(t)\hat{U}_0(t;t_0) \,, \tag{204}$$

where $\hat{H}_{\rm int}(t)$ ($\hat{H}_I(t)$) is the interaction Hamiltonian in the Schrödinger (interaction) picture and the (free) evolution operator,

$$\hat{U}_0(t,t_0) = T\exp\left(-\frac{\imath}{\hbar}\int_{t_0}^{t} dt' \hat{H}_0(t')\right) \tag{205}$$

is assumed to be known. The Heisenberg equation (14) for $\hat{\mathcal{O}}_{\rm I}(t)$ then becomes,

$$\frac{d}{dt}\hat{\mathcal{O}}_I(t) = \frac{\imath}{\hbar}[\hat{H}_I(t), \hat{\mathcal{O}}_I(t)] + \left[\frac{\partial}{\partial t}\hat{\mathcal{O}}_S(t)\right]_I \,. \tag{206}$$

and it is formally solved by,

$$\hat{\mathcal{O}}_H(t) = \hat{U}_I^\dagger(t,t_0)\hat{\mathcal{O}}_I(t)\hat{U}_I(t,t_0) \,, \tag{207}$$

where

$$\hat{U}_I(t,t_0) = T\exp\left(-\frac{\imath}{\hbar}\int_{t_0}^{t} dt' \hat{H}_I(t')\right) \,. \tag{208}$$

Upon expanding the evolution operator in powers of $\hat{H}_{\rm int}$ and reorganising the terms, one arrives at the following 'perturbative' expression for $\mathcal{O}(t)$,

$$\mathcal{O}(t) \equiv \langle \hat{\mathcal{O}}_H(t) \rangle \;=\; \langle \hat{\mathcal{O}}_I(t) \rangle + \frac{\imath}{\hbar} \int_{t_0}^{t} dt' \langle [\hat{H}_I(t'), \hat{\mathcal{O}}_I(t)] \rangle \\ + \; \left(\frac{\imath}{\hbar}\right)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \langle [\hat{H}_I(t'), [\hat{H}_I(t''), \hat{\mathcal{O}}_I(t)]] \rangle + \cdots \tag{209}$$

where the remaining terms follow the pattern of a time-ordered product of nested commutators of the interaction picture Hamiltonians with the (interaction picture) operator $\hat{\mathcal{O}}_I(t)$. If $\hat{H}_I$ is linear in some small, dimensionless (e.g. coupling) parameter λ, then we can write $\hat{H}_I \to \lambda \hat{H}_I$, and we see that the series (209) generates a perturbative expansion in powers of λ. The convergence properties of this series are not guaranteed even if $\lambda \ll 1$ because a time dependent perturbative expansion generates other dimensionless parameters. For example, in the case of a SHO, ωt is a dimensionless parameter, and when $\omega t \gg 1$, it can compensate for the smallness of λ (such that eventually $\lambda \omega t \gg 1$, indicating a possible breakdown of the perturbative expansion (209). The reasons for breakdown of a perturbative expansion in a time dependent setting can be multifold. These reasons can range from collective effects that environmental degrees of freedom can have on the characteristic frequency of the system, these degrees of freedom can induce dissipative effects (and induce damping of oscillations), and they source viscosity, fluctuations, *etc.* There is no ready made rule how to prevent a perturbative expansion from eventually breaking down, but currently the method of choice seems to be the equations of motion based on two-particle irreducible (2PI) effective action, especially when considered in the regime of a large number of fields (also known as the large N expansion). When treated in this way, one obtains thermalization towards the correct thermal equilibrium state (characterized by the Bose-Einstein or Fermi-Dirac distribution) even starting from far-from-equilibrium initial conditions, and there is numerical evidence that thermalization occurs at the right time scale. Furthermore, Eq. (209) has the right structure to generate causal evolution. Indeed, the nested commutators are made up of causal (Pauli-Jordan) correlators and we have shown in section III F (see Eq. (136)) that the simplest Pauli-Jordan two-point function vanishes outside the (past and future) light-cones. This property is generally true for any correlator appearing in the evolution equation for operators (such as (209)), illuminating the sense in which quantum physical systems are causal. For any candidate theory for non-equilibrium quantum processes, causality is a powerful test of its correctness. As we shall see below, the in-in formalism passes this test with flying colors.

1. The in-in generating functional

The fundamental object in the in-in formalism is the partition function (also known as the generating functional),

$$Z[J_+, J_-] = \mathrm{e}^{\imath W[J_+,J_-]/\hbar} = \sum_\alpha \mathrm{Tr}\left[\hat{\rho}(t_0)|\alpha, t_*\rangle_{J_-} \otimes \langle \alpha, t_*|_{J_+}\right] \tag{210}$$

where (in absence of currents $J_\pm$) the basis $\{|\alpha\rangle\}$ forms a complete set of states,

$$\mathbb{I} = \sum_\alpha |\alpha, t_*\rangle\langle\alpha, t_*| \,. \tag{211}$$

In order to get a better grip on the meaning of Eq. (210), let us assume that the density operator is pure, and that $\{|\alpha\rangle\}$ are position eigenstates, $\{|q_*\rangle\}$. In this case Eq. (210) simplifies to,

$$Z_{\mathrm{pure}}[J_+, J_-] = \int dq_* \langle \psi, t_0|q_*, t_*\rangle_{J_-} \langle q_*, t_*|\psi, t_0\rangle_{J_+} \,. \tag{212}$$

Upon inserting two more times a complete set of position states, this can be rewritten in terms of the transition amplitudes as follows,

$$Z_{\mathrm{pure}}[J_+, J_-] = \int dq_* \int dq_0^+ \int dq_0^- \psi_0^*(q_0^-)\psi_0(q_0^+) F^*(q_*, t_*|q_0^-, t_0|J_-) F(q_*, t_*|q_0^+, t_0|J_+) \,, \tag{213}$$

where

$$F(q_*, t_*|q_0^\pm, t_0|J_\pm) = \langle q_*|T \exp\left(-\frac{\imath}{\hbar}\int_{t_0}^{t_*} dt\big[\hat{H}(t) - J_\pm \hat{q}\big]\right)|q_0^\pm\rangle \tag{214}$$

define the transition amplitudes in (213). By borrowing from our discussion of the in-out transition amplitude (160), we can now easily go over to the path integral formulation of the in-in generating functional (213), which for general (mixed) states equals,

$$\begin{aligned} Z[J_+, J_-] \;=\; & \int dq_* \int dq_0^+ \int dq_0^- \rho_0(q_0^+, q_0^-) \int_{q_\pm(t_0)=q_0^\pm}^{q_\pm(t_*)=q_*} \mathcal{D}q_+ \mathcal{D}q_- \\ & \times \int \mathcal{D}p_+ \mathcal{D}p_- \exp\left(\frac{\imath}{\hbar}\int_{t_0}^{t_*} dt \sum_\pm \pm\Big[p_\pm \dot{q}_\pm - H(q_\pm, p_\pm) + J_\pm q_\pm\Big]\right) \,, \end{aligned} \tag{215}$$

where we returned to the general case with a mixed state density operator $\hat{\rho}_0 = \hat{\rho}(t_0)$ and $\rho_0(q_0^+, q_0^-) = \langle q_0^-|\hat{\rho}(t_0)|q_0^+\rangle$ denotes a matrix element of $\hat{\rho}(t_0)$ in the position basis; for pure states, $\rho_0(q_0^+, q_0^-) \to \psi_0^*(q_0^-)\psi_0(q_0^+)$ as in (213). If the Hamiltonian has a simple (quadratic) dependence on the momentum, the momentum integrals in (215) can be performed, resulting in a simpler in-in partition function,

$$Z[J_+, J_-] = \int dq_* \int dq_0^+ dq_0^- \rho_0(q_0^+, q_0^-) \int_{q_\pm(t_0)=q_0^\pm}^{q_\pm(t_*)=q_*} \mathcal{D}q_+ \mathcal{D}q_- \exp\left(\frac{\imath}{\hbar}\int_{t_0}^{t_*} dt \sum_\pm \pm\Big[L(q_\pm, \dot{q}_\pm) + J_\pm q_\pm\Big]\right) \,, \tag{216}$$

where $L(q_\pm, \dot{q}_\pm)$ is the Lagrangian. A notable difference between the in-in partition function and the in-out one is in that the in-in partition function does not make any reference to the *out* state. This is particularly important when one has no idea what good out states are. In fact, the in-in formalism is tailored to study how hermitean operators, or equivalently states, evolve as time goes on, and thus an important part of solving the problem will consist of finding a suitable late time state to expand in. This is of a particular importance for problems involving non-equilibrium quantum field theory, where a failure to construct proper *out* states may lead to un-physical divergences, which sometimes occur when the in-out formalism is attempted in situations where we have no good understanding of what suitable late time states are, *e.g.* some correlators on (exponentially expanding) de Sitter space have that property.

Just as in the in-out formalism (see Eqs. (164)), the in-in generating functional can be used for construction of expectation values of an $n-$point function. While in the in-out formalism one automatically generates time-ordered products, in the in-in formalism the situation is more subtle. Namely, varying with respect to J_+ (J_-) generates time-ordered (anti-time ordered) products, while varying with respect to a mixture of J_+ and J_- currents generates a sequence of an anti-time ordered product and a time-ordered product (the time ordered product being placed at the right). [15] For example, varying with respect to $J_\pm$ once generates a one-point function,

$$\begin{aligned}\hbar\left(\frac{\pm\delta Z}{\delta\imath J_\pm(t)}\right)_{J_\pm=0} &= \int dq_* \int dq_0^+ dq_0^- \rho_0(q_0^+, q_0^-) \int_{q_\pm(t_0)=q_0^\pm}^{q_\pm(t_*)=q_*} \mathcal{D}q_+ \mathcal{D}q_- \mathrm{e}^{(\imath/\hbar)\int_{t_0}^{t_*} dt \sum_\pm \pm(L(q_\pm,\dot{q}_\pm)+J_\pm q_\pm)} q_\pm(t) \\ &= \langle\hat{q}_\pm(t)\rangle = \langle\hat{q}(t)\rangle \equiv \bar{q}(t)\,, \end{aligned} \tag{217}$$

which is independent of the polarity $\pm$, since $\hat{q}$ is a Hermitian operator (namely, taking $\mathrm{e}^{\imath S/\hbar}$ or $\mathrm{e}^{-\imath S/\hbar}$ as a weight factor does not affect the result). In (217) we took account of the fact that $Z[J,J] = 1$, which can be easily seen from the definition of Z (210). This is to be contrasted with the in-out partition function (161), which is not equal to one in absence of currents. On the other hand, taking two variations with respect to J_+ and J_- give answers that depend on the polarities. Thus we claim that,

$$\begin{aligned}\left(\frac{\hbar^2\delta^2 Z[J_+,J_-]}{\delta\imath J_+(t)\delta\imath J_+(t')}\right)_{J_\pm=0} &= \langle T[\hat{q}_+(t)\hat{q}_+(t')]\rangle = \langle T[\hat{q}(t)\hat{q}(t')]\rangle = \imath\Delta_F(t;t') \\ \left(\frac{\hbar^2\delta^2 Z[J_+,J_-]}{\delta[-\imath J_-(t)]\delta[-\imath J_-(t')]}\right)_{J_\pm=0} &= \langle \bar{T}[\hat{q}_-(t)\hat{q}_-(t')]\rangle = \langle \bar{T}[\hat{q}(t)\hat{q}(t')]\rangle = \imath\Delta_D(t;t') \end{aligned} \tag{218}$$

[15] In the unlikely situation when someone is interested in a more baroque time-ordered product, one would have to invent a generating functional that contains more currents and possibly introduce a more baroque integration contour.

and

$$\left(\frac{\hbar^2\delta^2 Z[J_+,J_-]}{\delta\imath J_+(t)\delta[-\imath J_-(t')]}\right)_{J_\pm=0} = \langle\hat{q}_-(t')\hat{q}_+(t)\rangle = \langle\hat{q}(t')\hat{q}(t)\rangle = \imath\Delta^-(t;t')$$
$$\left(\frac{\hbar^2\delta^2 Z[J_+,J_-]}{\delta[-\imath J_-(t)]\delta\imath J_+(t')}\right)_{J_\pm=0} = \langle\hat{q}_-(t)\hat{q}_+(t')\rangle = \langle\hat{q}(t)\hat{q}(t')\rangle = \imath\Delta^+(t;t')\,. \tag{219}$$

That the above variations indeed produce the propagators (218) and the two point functions (219) need to be shown, which is what we argue in what follows.

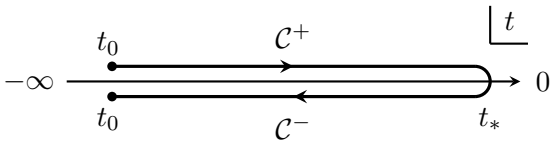

FIG. 9: The closed time path (CTC) integration contour, $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-$. The upper branch $\mathcal{C}^+$ goes forward from the initial time ($t = t_0$) to the final time ($t = t_*$) and the lower branch $\mathcal{C}^-$ goes backward from the final time to the initial time.

Notice first that the operator ordering in the above expressions is such that the operator $\hat{q}_-$ always comes to the left from the operator $\hat{q}_+$, suggesting that the time of $\hat{q}_-$ is always 'later' than the time of $\hat{q}_+$. This was used by Schwinger to motivate the introduction of a complex integration contour shown in figure 9, where the integrals in $S[q_+] - S[q_-]$ are replaced by one contour integral on the complex $t-$plane (the first integral is shifted a little bit up, while the second is shifted a little bit down). Thus we have for the complex contour integration,

$$S_{\mathcal{C}} = \int_{\mathcal{C}} dtL(q,\dot{q}) = \int_{\mathcal{C}_+} dtL(q,\dot{q}) + \int_{\mathcal{C}_-} dtL(q,\dot{q})\,, \qquad \mathcal{C} = \mathcal{C}_+ \cup \mathcal{C}_-\,, \tag{220}$$

where the two contours $\mathcal{C}_+$ and $\mathcal{C}_-$ are continuously joined at $t = t_*$. This is important since we want to make sure that $q_+(t_*) = q_-(t_*)$, as it is imposed by the definition of the in-in partition function. We have dropped the now redundant $(\pm)-$indices from the integrands in (220), as the integration contours specify uniquely the variable q. Notice that events on the $\mathcal{C}_-$ contour are always chronologically later than those on $\mathcal{C}_+$; also operators on $\mathcal{C}_+$ are time-ordered, while those on $\mathcal{C}_-$ are anti-time ordered. Thus one can say: there is time ordering along the complex contour $\mathcal{C}$.

We now return to the more explicit $(\pm)-$notation. It is often convenient to split the action S into the free (quadratic) action S_0 (for which we know the evolution operator) and the interaction part $S_{\rm int}$ (which contains cubic and other higher order vertices) as in (162). The partition function

can then be written as (see Eq. (177)),

$$Z[J_+,J_-] = \exp\left(\sum_\pm \pm\frac{\imath}{\hbar}S_{\rm int}\left[\hbar\frac{\delta}{\pm\imath\delta J_\pm}\right]\right) Z_0[J_+,J_-] \tag{221}$$

where

$$\begin{aligned} Z_0[J_+,J_-] \;&=\; \int dq_* \int dq_0^+ \int dq_0^- \rho_0(q_0^-,q_0^+) \\ &\times \int_{q_\pm(t_0)=q_0^\pm}^{q_\pm(t_*)=q_*} \mathcal{D}q_+\mathcal{D}q_- \exp\left(\sum_\pm \frac{\pm\imath}{\hbar}\int_{t_0}^{t_*} dt\left[L_0(q_\pm,\dot q_\pm)+J_\pm q_\pm\right]\right). \end{aligned} \tag{222}$$

As in the in-our formalism, this form can be a good starting point for an in-in perturbation theory (if one can identify a small parameter in $S_{\rm int}$). When writing Eqs. (221–222) we have assumed that $S_{\rm int}$ is independent on $\dot q$. When the interactions depend also on $\dot q$, which is the case *e.g.* in gravity and in systems with frictive interactions, $S_{\rm int}=S_{\rm int}[q,\dot q]$, one ought to start with the phase space formulation of the in-in partition function, in which both q and p acquire their source currents. We encourage the reader to write the phase-space path-integral expression for the partition function.

2. The Keldysh propagator

We shall now show that (for a simple initial state) $Z_0[J_+,J_-]$ in (222) defines uniquely the (free) Keldysh propagator, which is the basic building block for the Feynman rules of the in-in formalism. The steps are analogous to those in the in-out formalism, so we shall be a bit less explicit in the derivation. Let us for simplicity assume that S_0 is the action of an SHO. This action can be partially integrated to get,

$$\begin{aligned} \pm S_0[q_\pm] \;&=\; \pm\int_{t_0}^{t_*} dt\left(\frac{m\dot q_\pm^2}{2}-\frac{m\omega^2 q_\pm^2}{2}\right) \\ &=\; \pm\frac{m}{2}\left(q_*\dot q_\pm(t_*)-q_\pm(t_0)\dot q_\pm(t_0)\right) \pm\int_{t_0}^{t_*} dtdt' q_\pm(t)\frac{m}{2}\left(-\partial_t^2-\omega^2\right)\delta(t-t')q_\pm(t'). \end{aligned} \tag{223}$$

There is an ambiguity in (223), and a careful investigation of the discretised version of the path integral (see Appendix D) shows that the q_* in the second line should be interpreted as $q_*\to[q_+(t_*)+q_-(t_*)]/2$. Next, let us for simplicity assume that the initial states are the ground state of the SHO,

$$\rho_0(q_0^+,q_0^-)\to\psi_0^*(q_0^-)\psi_0(q_0^+)=\sqrt{\frac{m\omega}{2\hbar}}\exp\left(-\frac{m\omega}{2\hbar}\left[(q_0^+)^2+(q_0^-)^2\right]\right). \tag{224}$$

With these remarks we can write Z_0 in (222) as

$$
\begin{aligned}
Z_0[J_+,J_-] \;&=\; \int dq_* \int dq_0^+ \int dq_0^- \qquad (225)\\
&\times \int_{q_\pm(t_0)=q_0^\pm}^{q_\pm(t_*)=q_*} \mathcal{D}q_+\mathcal{D}q_- \exp\left(\sum_{a,b=\pm}\frac{\imath}{\hbar}\int_{t_0}^{t_*} dtdt'\frac{1}{2}q_a(t)D_0^{ab}(t;t')q_b(t') + \sum_{a=\pm}\int_{t_0}^{t_*} dtaJ_aq_a\right),
\end{aligned}
$$

where

$$
\begin{aligned}
D_0^{++}(t;t') \;&=\; m\left[-\partial_t^2-\omega^2+\delta(t-t_0)(-\partial_t+\imath\omega)+\frac{1}{2}\delta(t-t_*)\partial_t\right]\delta(t-t')\\
D_0^{-+}(t;t') \;&=\; \frac{m}{2}\delta(t-t_*)\partial_t\delta(t-t')\\
D_0^{--}(t;t') \;&=\; -m\left[-\partial_t^2-\omega^2+\delta(t-t_0)(-\partial_t-\imath\omega)+\frac{1}{2}\delta(t-t_*)\partial_t\right]\delta(t-t') = -[D_0^{++}(t;t')]^*\\
D_0^{+-}(t;t') \;&=\; -\frac{m}{2}\delta(t-t_*)\partial_t\delta(t-t') = -D_0^{-+}(t;t')\,, \qquad (226)
\end{aligned}
$$

Note the (anti-hermiticity) relationships between the upper two and the lower two derivative operators in (226), which are used below to simplify the analysis. In fact, there is another piece contributing to the derivative operator that comes from the condition $q_+(t_*) = q_-(t_*)$ in Eq. (215). Indeed, we can write the integral over q_* in (225) as,

$$
\int dq_* = \int dq_+^* \int dq_-^*\,\delta\Big(q_+(t_*)-q_-(t_*)\Big)
$$

and the delta function can be represented as, [16]

$$
\delta\Big(q_+(t_*)-q_-(t_*)\Big) = \lim_{\epsilon\to 0}\sqrt{\frac{\imath}{\pi\hbar\epsilon}}\exp\left(-\frac{\imath}{\hbar\epsilon}(q_+(t_*)-q_-(t_*))^2\right)\,, \qquad (227)
$$

from which we get,

$$
\Delta D(t;t') = \begin{pmatrix} \Delta D^{++} & \Delta D^{+-}\\ \Delta D^{-+} & \Delta D^{--}\end{pmatrix} = \lim_{\epsilon\to 0}\begin{pmatrix} -\frac{2}{\epsilon} & \frac{2}{\epsilon}\\ \frac{2}{\epsilon} & -\frac{2}{\epsilon}\end{pmatrix}\delta(t-t_*)\delta(t'-t_*)\,, \qquad (228)
$$

such that we have

$$
D(t;t') = D_0(t;t') + \Delta D(t;t')\,; \qquad D_0(t;t') = \begin{pmatrix} D_0^{++} & D_0^{+-}\\ D_0^{-+} & D_0^{--}\end{pmatrix}(t;t')\,. \qquad (229)
$$

[16] There is ambiguity in the representation of the $\delta-$function in (227) which has no consequence on the final result. The one we use in Eq. (227) is chosen such to naturally contribute to the derivative operator D in (226) and it respects the hermiticity property of the operator $D\sigma^3$ ($\sigma^3 = \mathrm{diag}(1,-1)$).

We are now ready to calculate Z_0 in Eq. (225). As in the in-out formalism, it is useful to shift $\vec{q} = (q_+, q_-)^T$ as,

$$\vec{q} \to \vec{\tilde{q}} = \vec{q} + \frac{1}{\hbar} \int d\tilde{t} \Delta_K(t;\tilde{t}) \cdot \vec{J}(\tilde{t}) \,, \qquad \vec{J}(t) = \begin{pmatrix} J_+(t) \\ -J_-(t) \end{pmatrix} \,, \tag{230}$$

where

$$i\Delta_K(t;t') = \begin{pmatrix} i\Delta^{++} & i\Delta^{+-} \\ i\Delta^{-+} & i\Delta^{--} \end{pmatrix} (t;t') \tag{231}$$

is the Keldysh propagator, and $\Delta_K(t;t')/\hbar$ is defined as the operator inverse of $D(t;t')$,

$$\int_{t_0}^{+\infty} d\tilde{t} D(t;\tilde{t}) i\Delta_K(\tilde{t};t') = i\hbar\delta(t-t') \,. \tag{232}$$

This defines the free Keldysh propagator, *i.e.* it excludes interactions. One can generalize the definition (232) to include interactions, in which case one can get a definition that includes loop corrections truncated at some loop order. The relevant equation that the full (dressed) Keldysh propagator obeys can be obtained by varying the 2PI effective action and it corresponds to the Schwinger-Dyson equation suitably adapted to the in-in formalism.

When the shift (230) is inserted into (225) and the integrals over $q_\pm$ are performed, one obtains

$$Z_0[J_+, J_-] = \exp\left(-\frac{1}{2\hbar^2} \int_{t_0}^{t_*} dt dt' \vec{J}^T(t) \imath\Delta_K(t;t') \vec{J}(t') \right) \tag{233}$$

Now, when Eqs. (231) and (229) are inserted into (232) one obtains the following set of equations that fully define the Keldysh propagator,

$$\begin{aligned} D^{++}(t)\imath\Delta^{++}(t;t') + D^{+-}(t)\imath\Delta^{-+}(t;t') &= \imath\hbar\delta(t-t') \\ D^{-+}(t)\imath\Delta^{++}(t;t') + D^{--}(t)\imath\Delta^{-+}(t;t') &= 0 \\ D^{++}(t)\imath\Delta^{+-}(t;t') + D^{+-}(t)\imath\Delta^{--}(t;t') &= 0 \\ D^{-+}(t)\imath\Delta^{+-}(t;t') + D^{--}(t)\imath\Delta^{--}(t;t') &= \imath\hbar\delta(t-t') \end{aligned} \tag{234}$$

where $D^{ab}(t)$ denote various components of $D(t;t')$ with $\delta(t-t')$ removed. Since the latter two equations can be obtained from the first two equations by (minus) the complex conjugation, they do not constitute independent equations. That simply means that we can solve the first two equations and infer the solutions to the second two equations, which significantly simplifies the analysis. The first two equations in (234) give us both the bulk evolution equations as well

as the boundary conditions for $\imath\Delta^{++}$ and $\imath\Delta^{-+}$. It is worth writing them in detail (see the analogous Eqs. (192) for the Feynman propagator of the in-out formalism),

$$\begin{aligned} m\big[-\partial_t^2-\omega^2\big]\imath\Delta^{++}(t;t') &= \imath\hbar\delta(t-t') \\ m\delta(t-t_0)\big[-\partial_t+\imath\omega\big]\imath\Delta^{++}(t;t') &= 0 \\ \frac{m}{2}\delta(t-t_*)\big[\partial_t\imath\Delta^{++}(t;t')-\partial_t\imath\Delta^{-+}(t;t')\big] &= 0 \\ \frac{2}{\epsilon}\delta(t-t_*)\big[-\imath\Delta^{++}(t;t')+\imath\Delta^{-+}(t;t')\big] &= 0 \end{aligned} \tag{235}$$

and

$$\begin{aligned} -m\big[-\partial_t^2-\omega^2\big]\imath\Delta^{-+}(t;t') &= 0 \\ -m\delta(t-t_0)\big[-\partial_t-\imath\omega\big]\imath\Delta^{-+}(t;t') &= 0 \\ \frac{m}{2}\delta(t-t_*)\big[-\partial_t\imath\Delta^{++}(t;t')+\partial_t\imath\Delta^{-+}(t;t')\big] &= 0 \\ \frac{2}{\epsilon}\delta(t-t_*)\big[\imath\Delta^{++}(t;t')-\imath\Delta^{-+}(t;t')\big] &= 0\,. \end{aligned} \tag{236}$$

The last two equations are the same as in (235) and hence redundant. The first equation in (235) tells us that the structure of the propagator $\imath\Delta^{++}$ is,

$$\imath\Delta^{++}(t;t') = \Theta(t-t')\imath\Delta^{+}(t;t') + \Theta(t-t')\imath\Delta^{-}(t;t')\,, \tag{237}$$

where $\imath\Delta^{\pm}(t;t')$ obey the homogeneous equation,

$$m\big[-\partial_t^2-\omega^2\big]\imath\Delta^{\pm}(t;t') = 0\,, \tag{238}$$

but are otherwise general functions of t and t', while $\imath\Delta^{-+}$ itself obeys the same homogeneous equation (236). Proceeding the same way as we did with the Feynman propagator in section IV A above (taking the first and second time derivatives), implies that $\imath\Delta^{\pm}(t;t')$ must satisfy the following equations (see Eqs. (199–200)),

$$\imath\Delta^{+}(t';t')-\imath\Delta^{-}(t';t') = 0\,, \qquad \delta(t-t')\Big[\partial_t\imath\Delta^{+}(t;t')-\partial_t\imath\Delta^{-}(t;t')+\frac{\imath\hbar}{m}\Big] = 0\,. \tag{239}$$

Next, upon inserting (237) into the second, third and fourth equation in (235) on obtains ($\Theta(t_0-t')=0, \Theta(t'-t_*)=0$),

$$\begin{aligned} \delta(t-t_0)\big[-\partial_t+\imath\omega\big]\imath\Delta^{-}(t;t') &= 0 \\ \delta(t-t_*)\big[\partial_t\imath\Delta^{+}(t;t')-\partial_t\imath\Delta^{-+}(t;t')\big] &= 0 \\ \delta(t-t_*)\big[-\imath\Delta^{+}(t;t')+\imath\Delta^{-+}(t;t')\big] &= 0 \end{aligned} \tag{240}$$

Together with the first two equations in (236), Eqs. (238), (239) and (240) comprise the nine equations that completely determine $\imath\Delta^{-+}(t;t')$, $\imath\Delta^{+}(t;t')$ and $\imath\Delta^{-}(t;t')$. Since all of these functions satisfy homogeneous equations of the form (238), we can write the general solutions as,

$$\imath\Delta^{-+}(t;t') = A^{-+}(t')\mathrm{e}^{-\imath\omega t} + B^{-+}(t')\mathrm{e}^{\imath\omega t}\,, \qquad \imath\Delta^{\pm}(t;t') = A^{\pm}(t')\mathrm{e}^{-\imath\omega t} + B^{\pm}(t')\mathrm{e}^{\imath\omega t}\,. \tag{241}$$

Up to this point we have used the three homogeneous equations, and thus we still have six equations for six general functions of t', which is encouraging. The second equation in (236) and the first equation in (240) imply that $B^{-+}(t') = 0$ and $A^{-}(t') = 0$, and hence we have,

$$\imath\Delta^{-+}(t;t') = A^{-+}(t')\mathrm{e}^{-\imath\omega t}\,, \qquad \imath\Delta^{-}(t;t') = B^{-}(t')\mathrm{e}^{\imath\omega t}\,. \tag{242}$$

Next, the latter two equations in (240) give,

$$-A^{+}(t')\mathrm{e}^{-\imath\omega t_*} + B^{+}(t')\mathrm{e}^{\imath\omega t_*} = -A^{-+}(t')\mathrm{e}^{-\imath\omega t_*}\,, \qquad A^{+}(t')\mathrm{e}^{-\imath\omega t_*} + B^{+}(t')\mathrm{e}^{\imath\omega t_*} = A^{-+}(t')\mathrm{e}^{-\imath\omega t_*}\,, \tag{243}$$

from which we conclude that

$$B^{+}(t') = 0\,, \qquad A^{+}(t') = A^{-+}(t')\,.$$

The remaining two equations (239) fix uniquely $A^{+} = A^{-+}$ and B^{-} to,

$$A^{+}(t') = A^{-+}(t') = \frac{\hbar}{2m\omega}\mathrm{e}^{\imath\omega t'}\,, \qquad B^{-}(t') = \frac{\hbar}{2m\omega}\mathrm{e}^{-\imath\omega t'}\,.$$

Taking everything together we finally get,

$$\imath\Delta^{+}(t;t') = \frac{\hbar}{2m\omega}\mathrm{e}^{-\imath\omega(t-t')} = \imath\Delta^{-+}(t;t')\,, \qquad \imath\Delta^{-}(t;t') = \frac{\hbar}{2m\omega}\mathrm{e}^{\imath\omega(t-t')}\,. \tag{244}$$

We have thus shown that (as announced) $\imath\Delta^{+}(t;t')$ and $\imath\Delta^{-}(t;t')$ are the positive and negative frequency Wightman functions of the SHO vacuum (201), respectively, and that the off-diagonal component of the Keldysh propagator, $\imath\Delta^{-+}(t;t')$, is the positive frequency Wightman function. As mentioned above, the remaining components of the Keldysh propagator (231) can be obtained from the hermiticity property of the Keldysh propagator (which, as mentioned above, can be inferred from the complex conjugation of the first two equations in (234)). The result is,

$$\begin{aligned}
\imath\Delta^{--}(t;t') &= \Theta(t-t')\imath\Delta^{-}(t;t') + \Theta(t'-t)\imath\Delta^{+}(t;t') = [\imath\Delta^{++}(t;t')]^{*} \\
\imath\Delta^{+-}(t;t') &= \imath\Delta^{-}(t;t') = [\imath\Delta^{-+}(t;t')]^{*}\,.
\end{aligned} \tag{245}$$

To summarize, we have proven that the Keldysh propagator (231) for a simple harmonic oscillator is of the form,

$$\imath\Delta_K(t;t') = \begin{pmatrix} \imath\Delta^{++} & \imath\Delta^{+-} \\ \imath\Delta^{-+} & \imath\Delta^{--} \end{pmatrix}(t;t') = \begin{pmatrix} \imath\Delta_F & \imath\Delta^{-} \\ \imath\Delta^{+} & \imath\Delta_{\bar{F}} \end{pmatrix}(t;t')\,. \tag{246}$$

Even though we have shown that this solution holds for an SHO in its ground state, this form is holds true for the general Keldysh propagator associated with a general initial state (Gaussian or non-Gaussian) and for actions that include interactions. This propagator is of fundamental importance for perturbative schemes in the in-in formalism, as it constitutes the fundamental building block for Feynman rules.

The Feynman rules of the in-in formalism follow from the generating functional $Z[J_+, J_-]$. They are the identical to those of usual interacting quantum mechanical systems (and more generally of quantum field theories), except:

- Each vertex V is assigned a plus $(+)$ or minus $(-)$ polarity, *i.e.* $V \to V_a$ $(a = \pm)$;
- a V_+ contributes as $(-\imath)\times$ [the coupling constant], while a V_- contributes as $(-\imath)\times$ [the coupling constant];
- two vertices $V_a(x;\cdots)$ and $V_b(x';\cdots)$ get connected by a $2-$point function $\imath\Delta^{ab}(x;x')$, where $a, b = \pm$ and x, x' are coordinate $4-$vectors;
- To calculate a complete (amputated) Feynman diagram, one ought to sum over all polarities of internal vertices such that for a diagram with N vertices there are 2^N times more diagrams than in the usual Feynman perturbation theory. If the diagram contains external legs, *i.e.* it is not amputated, polarity of the external legs is kept fixed and thus not summed over.

While the last rule may seem to be intimidating, if one is interested in one- or two-loop contributions (very rarely one attempts to calculate even higher loops in out-of-equilibrium problems), the number of diagrams is, in fact, manageable.

It is worth noting that using the Keldysh propagator (246) does not constitute the unique way to define the Feynman rules for the in-in formalism. For example, performing a (counter-clockwise) rotation of the coordinates,

$$\vec{q}(t) \to \vec{q}\,'(t) = R(\phi)\cdot\vec{q}, \qquad R(\phi) = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}, \qquad (\phi \in [0, 2\pi)) \tag{247}$$

transforms the Keldysh propagator,

$$\imath\Delta_K(t;t') = \mathrm{Tr}\left\{\hat{\rho}(t_0)T_{\mathcal{C}}\left[\begin{pmatrix}\hat{q}_+(t)\\ \hat{q}_-(t)\end{pmatrix}\otimes\begin{pmatrix}\hat{q}_+(t'),\ \hat{q}_-(t')\end{pmatrix}\right]\right\} = \mathrm{Tr}\left[\hat{\rho}(t_0)T_{\mathcal{C}}\left(\hat{\vec{q}}(t)\otimes\hat{\vec{q}}(t')^{\,T}\right)\right] \tag{248}$$

into

$$\imath\Delta_K^{\phi}(t;t') = R(\phi)\mathrm{Tr}\left[\hat{\rho}(t_0)T_{\mathcal{C}}\left(\hat{\vec{q}}(t)\otimes\hat{\vec{q}}(t')^{\,T}\right)\right]R^T(\phi)\,, \tag{249}$$

where $T_{\mathcal{C}}$ indicates time ordering along the closed time contour in figure 9. The transformation (247) does not change the path integral measure since the Jacobian of the transformation, $\det(\partial\vec{q}\,'/\partial\vec{q}) = \det[R(\phi)] = 1$. A particularly interesting (and useful) choice of R is a rotation by 45^o, in which case,

$$R(\phi=\pi/4) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\,, \tag{250}$$

in which case,

$$\vec{q}\to\frac{1}{\sqrt{2}}\begin{pmatrix}q_+ - q_-\\ q_+ + q_-\end{pmatrix} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix}\Delta q\\ 2\bar{q}\end{pmatrix} \tag{251}$$

and

$$\imath\Delta_K^{\pi/4}(t;t') = \begin{pmatrix}0 & \imath\Delta^a(t;t')\\ \imath\Delta^r(t;t') & 2F(t;t')\end{pmatrix}\,, \tag{252}$$

where $F(t;t')$ is the statistical two-point function (96), and

$$\begin{aligned}\imath\Delta^r(t;t') &= \imath\Delta^{++}(t;t') - \imath\Delta^{+-}(t;t') = -\imath\Delta^{--}(t;t') + \imath\Delta^{-+}(t;t')\\ \imath\Delta^a(t;t') &= \imath\Delta^{++}(t;t') - \imath\Delta^{-+}(t;t') = -\imath\Delta^{--}(t;t') + \imath\Delta^{+-}(t;t')\end{aligned} \tag{253}$$

are the retarded and advanced propagators, respectively, see also Eqs. (102) and (137). On the other hand, from Eqs. (251) and (249) we see that,

$$\begin{aligned}\imath\Delta_K^{\pi/4}(t;t') &= \frac{1}{2}\mathrm{Tr}\left\{\hat{\rho}(t_0)T_{\mathcal{C}}\left[\begin{pmatrix}\Delta\hat{q}(t)\\ 2\hat{\bar{q}}(t)\end{pmatrix}\otimes\Big(\Delta\hat{q}(t'),\ 2\hat{\bar{q}}(t')\Big)\right]\right\}\\ &= \begin{pmatrix}\frac{1}{2}\langle T_{\mathcal{C}}\left[\Delta\hat{q}(t)\Delta\hat{q}(t')\right]\rangle & \langle T_{\mathcal{C}}\left[\Delta\hat{q}(t)\hat{\bar{q}}(t')\right]\rangle\\ \langle T_{\mathcal{C}}\left[\hat{\bar{q}}(t)\Delta\hat{q}(t')\right]\rangle & 2\langle T_{\mathcal{C}}\left[\hat{\bar{q}}(t)\hat{\bar{q}}(t')\right]\rangle\end{pmatrix}\,.\end{aligned} \tag{254}$$

Comparing this with (252) we see that,

$$\begin{aligned}\imath\Delta^r(t;t') &= \langle T_{\mathcal{C}}\left[\hat{\bar{q}}(t)\Delta\hat{q}(t')\right]\rangle\,, \qquad \imath\Delta^a(t;t') = \langle T_{\mathcal{C}}\left[\Delta\hat{q}(t)\hat{\bar{q}}(t')\right]\rangle\\ F(t;t') &= \langle T_{\mathcal{C}}\left[\hat{\bar{q}}(t)\hat{\bar{q}}(t')\right]\rangle\,, \qquad 0 = \langle T_{\mathcal{C}}\left[\Delta\hat{q}(t)\Delta\hat{q}(t')\right]\rangle\,.\end{aligned} \tag{255}$$

Note that $\Delta\hat{q}$ (which measures the difference between the fluctuations on the upper and lower Schwinger contours) are not correlated at different times. [17] In this language, the retarded and advanced propagators correspond to the correlations between $\Delta\hat{q}$ and $\hat{\bar{q}}$, while F measures the fluctuations in the mean position, $\hat{\bar{q}}$, and for that reason it is often referred to as the correlator that provides the measure for (mainly) classical fluctuations. Next, Eq. (255) implies that the rotated Keldysh propagator (256) obeys the equation,

$$D(t)^{\pi/4}\cdot \imath\Delta_K^{\pi/4}(t;t') = \imath\hbar\delta(t-t')\mathbb{I}\,, \tag{256}$$

or when written in components,

$$\begin{pmatrix} \imath\Im[D^{++}(t)] & \Re[D^{++}(t)]+D^{+-}(t) \\ \Re[D^{++}(t)]-D^{+-}(t) & \imath\Im[D^{++}(t)] \end{pmatrix}\cdot\begin{pmatrix} 0 & \imath\Delta^a(t;t') \\ \imath\Delta^r(t;t') & 2F(t;t') \end{pmatrix} = \imath\hbar\delta(t-t')\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{257}$$

It can be checked that this (rotated) matrix equation uniquely defines the components of the rotated Keldysh propagator, and hence one can alternatively formulate a perturbative expansion by using this form of the Keldysh propagator. Of course, interactions now look different. For example, a quartic self-interaction becomes,

$$\sum_{\pm}\pm S_{\rm int}[q_\pm] = -\sum_{\pm}\pm\int_{t_0}^{t_*}dt\frac{\lambda}{4!}q_\pm^4 \to -\int_{t_0}^{t_*}dt\frac{\lambda}{3!}\left(\bar{q}\Delta q^3+\bar{q}^3\Delta q\right)\,. \tag{258}$$

It is also instructive to write the SHO free action (223) in the $(\bar{q},\Delta q)-$coordinates,

$$\sum_{\pm}\pm S_0[q_\pm] = \int_{t_0}^{t_*}dtm\left(\dot{\bar{q}}\Delta\dot{q}-\omega^2\bar{q}\Delta q\right)\,, \tag{259}$$

which thus looks off-diagonal. One can show that this action generates in Z_0 the rotated Keldysh propagator (256). Eqs. (252) and (258) tell us how to build the Feynman rules for this rotated basis: one introduces two types of lines: solid lines (that correspond to $\hat{\bar{q}}$) and dashed lines (that correspond to $\Delta\hat{q}$) such that the propagator $\imath\Delta^r$ ($\imath\Delta^a$) has can be drawn as a solid-dashed (dashed-solid) line and the Hadamard function is a solid line. There is no dashed propagator

[17] One can easily show that this is generally true by explicitly representing the correlator for fluctuations as follows,

$$\langle T_{\mathcal{C}}\left[\hat{\bar{q}}(t)\Delta\hat{q}(t')\right]\rangle = i\Delta_F(t;t')+i\Delta_D(t;t')-i\Delta^+(t;t')-i\Delta^-(t;t') = 0\,.$$

The last equality follows from the fundamental relations (60) and (64) that the Feynman and Dyson propagator must satisfy. This proves that quantum fluctuations are uncorrelated at different times which intuitively should be this way.

(because the propagator associated with fluctuations vanishes). From the interaction Lagrangian in (258) we see that from each vertex emanate either one dashed and three solid lines, or one solid and three dashed lines. In this rotated basis vertices have no polarities and thus there is no need for summation. We leave as an exercise to the reader to work out the Feynman rules in detail (see problem 12). Many practitioners of the in-in formalism find this rotated basis convenient to easily resolve the classical from quantum contributions (loosely speaking, the number of dashed lines in a diagram can be used as the measure of quantumness of a diagram). As an example of how this can be applied to study the (classical and quantum) one-loop effects on scalar field two-point functions on de Sitter space see Ref. [9].

V. RESUMMATION TECHNIQUES

It is often the case that standard perturbation theory fails in out-of-equlibrium problems even when interactions are endowed with a small expansion parameter. For that reason developing resummation techniques for non-equilibrium problems can be of utmost importance. The hope is that these resummation techniques restore perturbativity in the sense that the neglected corrections are suppressed by a small parameter with respect to those included. It is often not known whether that is true since rigorous proofs are not available. Nevertheless, it is useful to study them since, when compared with standard perturbation theory, resummation techniques are known to improve the results.

The most commonly used resummation techniques in literature are based on the one-particle irreducible (1PI) effective action, two-particle irreducible (2PI) effective action, and large N schemes. In this chapter we shall focus on the 1PI and 2PI resummation techniques. Sometimes nPI resummation schemes are also invoked, even though – due to their complexity – up to now they have not found a wide range of applications. Before we start developing resummation techiques in non-equilibrium setting, we shall discuss two examples which motive the need for resummation:

A. *Dressed harmonic oscillator*;

B. *Renormalization group improvement of an effective potential for conformal scalar.*

A. Dressed harmonic oscillator. To begin with, it is important to understand why in time-dependent problems a perturbative scheme can break down even when there exists a small perturbation parameter. To illustrate that explicitly let us consider a simple problem in which the free Hamiltonian is that of a SHO in Eq. (49) and the interaction term is small, *i.e.* it is suppressed as, $\hat{H}_{\rm int} \propto \epsilon$, where $\epsilon \ll 1$. In such situations it is often the case that the mass parameter of the theory gets renormalized such that it acquires a correction from the perturbation expansion of the type,

$$\delta_\epsilon \hat{H} = \frac{1}{2} m \delta_\epsilon \omega^2 \hat{q}^2 \,, \quad \delta_\epsilon \omega^2 = \epsilon \omega^2 \,, \tag{260}$$

where $0 < \epsilon \ll 1$ is the parameter we wish to perturb in. [18] Then the equations of motion for the oscillator get modified as (*c.f.* Eq. (51)),

$$\frac{d\hat{q}}{dt} = \frac{\hat{p}}{m} \,, \quad \frac{d\hat{p}}{dt} = -m\omega^2(1+\epsilon)\hat{q} \tag{261}$$

[18] For definiteness here we take $\epsilon > 0$; the case $-1 \ll \epsilon < 0$ is treated analogously.

whose exact solution is given by (*c.f.* Eq. (52))

$$\begin{aligned}\hat{q}(t) &= \hat{q}_0 \cos[\omega(1+\epsilon)t] + \frac{\hat{p}_0}{m\omega}\sin[\omega(1+\epsilon)t]\,,\\ \hat{p}(t) &= \hat{p}_0 \cos[\omega(1+\epsilon)t] - m\omega\hat{q}_0 \sin[\omega(1+\epsilon)t]\,.\end{aligned} \tag{262}$$

To get perturbative solutions of the problem (261) it is convenient to expand $\hat{q}$ and $\hat{p}$ in powers of ϵ as,

$$\hat{q}(t) = \hat{q}^{(0)}(t) + \epsilon\hat{q}^{(1)}(t) + \cdots + \epsilon^n\hat{q}^{(n)}(t) + \ldots\,, \quad \hat{p}(t) = \hat{p}^{(0)}(t) + \epsilon\hat{p}^{(1)}(t) + \ldots \tag{263}$$

where $\hat{q}^{(0)}$ and $\hat{p}^{(0)}$ are the solutions of the unperturbed oscillator (52) and $\hat{q}^{(n)}$ and $\hat{p}^{(n)}$ are the contributions of $\mathcal{O}(\epsilon^n)$ to $\hat{q}(t)$ and $\hat{p}(t)$. It is not hard to see that,

$$\begin{aligned}\hat{q}^{(1)}(t) &= \omega t\left[-\hat{q}_0 c\omega t\sin(\omega t) + \frac{\hat{p}_0}{m\omega}c\omega t\sin(\omega t)\right]\\ \hat{p}^{(1)}(t) &= \omega t\left[-\hat{p}_0\sin(\omega t) - m\omega\hat{q}_0\cos(\omega t)\right]\end{aligned} \tag{264}$$

Obviously, after a large enough time, $\{\hat{q}^{(0)} + \hat{q}^{(1)}, \hat{p}^{(0)} + \hat{p}^{(1)}\}$ stops being a good approximation to the exact solution (262). While the exact solution is oscillatory with a slightly different frequency, the approximate solution tries to reproduce the correct frequency by introducing an unphysical growing solution in time. This unphysical growth in time is known as *secular growth*, and obviously fails to reproduce the correct solution as soon as,

$$\epsilon\omega t \gtrsim 1\,. \tag{265}$$

The behavior is illustrated in figure 10.

What has really happened here? The time dependence in the solution generates a new dimensionless parameter $[\omega t]$ and standard perturbation theory fails to recognize its existence, so it blindly attempts to expand solutions in the dimensionless ϵ, even though now there are novel dimensionless parameters that can grow much larger than one and in this way invalidate standard perturbation expansion. The above problem is simple and hence the solution to it is obvious. What one gets by solving the Hamilton equations can be written as a double series,

$$\hat{q}(t) = \sum_{n=0}^{\infty}\sum_{n'=0}^{\infty}\hat{c}^{(q)}_{n,n'}[\omega t]^n\epsilon^{n'}\,, \qquad \hat{p}(t) = \sum_{n=0}^{\infty}\sum_{n'=0}^{\infty}\hat{c}^{(p)}_{n,n'}[\omega t]^n\epsilon^{n'}\,, \tag{266}$$

where for $n' \leq n$,

$$\hat{c}^{(q)}_{n,n'} = (-1)^n\binom{n}{n'}\left[\frac{\hat{q}_0}{(2n)!} + \frac{\hat{p}_0}{m\omega(2n+1)!}\right], \quad \hat{c}^{(q)}_{n,n'} = (-1)^n\binom{n}{n'}\left[\frac{\hat{p}_0}{(2n)!} - \frac{m\omega\hat{q}_0}{(2n+1)!}\right], \tag{267}$$

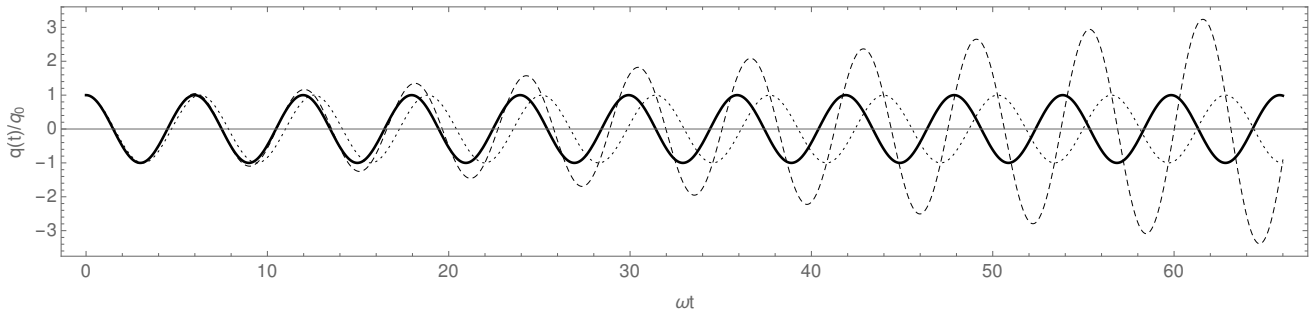

FIG. 10: The behavior of $\langle\hat{q}(t)\rangle$ for the harmonic oscillator with a renormalized frequency whose dynamics is governed by (261). We show the exact solution $\langle\hat{q}(t)\rangle/\langle\hat{q}_0\rangle$ in (262) (solid line), the leading order (classical) solution $\langle\hat{q}^{(0)}(t)\rangle/\langle\hat{q}_0\rangle$ (dotted) and the first order solution $\langle(\hat{q}^{(0)}(t)+\hat{q}^{(1)}(t))/\langle\hat{q}_0\rangle\rangle$ (dashed) for $\epsilon = 0.05$. Note that the amplitude of the first order perturbative solution exhibits a secular growth in time, which after $\omega t \gtrsim 1/\epsilon = 20$ exhibits a breakdown of perturbative expansion.

and $\hat{c}^{(q)}_{n,n'} = 0 = \hat{c}^{(p)}_{n,n'}$ otherwise (when $n' > n$). Obviously, equation (266) has now become a double sum in powers of two (small) parameters, $[\omega t]$ and ϵ and there is no reason whatsoever why it should be valid when any of the two become large, *i.e.* standard perturbation theory is guaranteed to give an approximately correct answer only if both $\omega t \ll 1$ and $\epsilon \ll 1$.

Usually we are interested in extending the validity of standard perturbation theory. The right procedure is then not to first some over n and then n' (as one does in standard perturbation theory), but instead to reorganize the double sum (266) such to first sum the series in ϵ (which is of the binomial type and gives $[\omega(1+\epsilon)]^n$) and then the second series, which eventually yields (262). While it is hard to see what the correct resummation procedure is from the double series form (266–267), it is easy to see it from the differential equation (261). Indeed, while in the solution (266) all orders of ϵ occur, in the differential equation ϵ figures only linearly. In other words, solving the differential equation exactly automatically reproduces the correct resummed result (262).

B. Renormalization group improvement of an effective potential for conformal scalar. [19] Before we embark in studying resummation in non-equilibrium problems, it is worth

[19] Even though this example is very useful, it is rather technical in nature and readers unfamiliar with the topic may skip it at the first reading.

considering another example of resummation which is known as *renormalization group* (RG). Let us consider a massless, self-interacting scalar field, whose classical action (on Minkowski space) reads,

$$S_0[\phi] = \int d^4x \left(-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{\lambda}{4!}\phi^4 \right) . \tag{268}$$

By making use of perturbation theory one can calculate the renormalized effective action [10] which, when expanded in loops, can be written as,

$$\Gamma[\bar{\phi}] = S_0[\bar{\phi}] + \Gamma^{(1)}[\bar{\phi}] + \dots , \tag{269}$$

where the one-loop contribution $\Gamma^{(1)}[\bar{\phi}]$ reads,

$$\begin{aligned} \Gamma^{(1)}[\bar{\phi}] &= -\int d^4x \left[V^{(1)}(\bar{\phi}) - \frac{1}{2}Z^{(1)}(\bar{\phi})\eta^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi} + \dots \right] \\ V^{(1)}(\bar{\phi}) &= \frac{\lambda^2}{256\pi^2}\bar{\phi}^4 \left[\ln\left(\frac{\lambda\bar{\phi}^2}{2\mu^2}\right) - \frac{25}{6} \right] \end{aligned} \tag{270}$$

where the dots signify higher derivative corrections, $V^{(1)}(\phi)$ is the one-loop effective potential, λ is the coupling constant evaluated at a scale μ (and whose value thus becomes scale dependent), $Z^{(1)}$ is the (one-loop) wave function renormalization factor and μ is an arbitrary scale introduced by the process of renormalization. [20] Note the occurrence of the log in (270),

$$L \equiv \frac{1}{2}\ln\left(\frac{\bar{\phi}^2}{\mu^2}\right) , \tag{271}$$

which arose as a consequence of breaking of scale invariance and which can be viewed as a *second dimensionless parameter*. Indeed, perturbative expansion is not anymore an expansion in powers λ, but it has become a double series which is for the effective potential of the form,

$$V(\bar{\phi}) = \frac{\lambda}{4!}\bar{\phi}^4 \sum_{n,n'} c_{n,n'}\lambda^n L^{n'} \tag{272}$$

with $c_{n,n'} = 0$ when $n' \geq n$,

$$c_{n,n-1} = \frac{3}{16\pi^2} , \tag{273}$$

etc.. These coefficients are known as the leading logarithmic order (LL)), $c_{n,n-2}$ are the next-to-leading logarithms (NLLO), *etc.* Observe that, when $\lambda L \gtrsim 1$, standard perturbation theory

[20] Strictly speaking the scale dependence in (270) is introduced by a scale dependent counter term which is then interpreted as the breaking of scale invariance by quantum effects. This breaking is generic and cannot be avoided.

breaks down. This breakdown can be restored by resumming the leading logarithms in (272), which is known as the LLO resummation. The result can be represented by a theory whose effective potential has a tree level form, but the coupling becomes scale and field dependent,

$$\lambda(\mu/\bar{\phi}) = \frac{\lambda_0}{1 - \frac{3}{16\pi^2}\lambda_0 \ln(\bar{\phi}/\mu)}\,, \qquad V^{(LLO)} = \frac{\lambda(\mu/\bar{\phi})}{4!}\bar{\phi}^4\,, \tag{274}$$

where $\lambda_0 = \lambda(1)$. By pushing this procedure further, *i.e.* by calculating the effective potential at NLLO, NNLLO, *etc.* (this is done by resumming the coefficients $c_{n,n-2}$, $c_{n,n-3}$ *etc.* in the series (272)) one restores perturbativity of the effective potential in the sense that whenever $\lambda \ll 1$, the effective potential approximates well the true effective action (of course, in the limit when derivative corrections can be neglected). Kenneth Wilson and others, who developed RG techniques, have realised that, thanks to the independence on n of $c_{n,n-1}$ in Eq. (273), one can resum the leading logarithms by solving the following differential equation,

$$\mu\frac{d}{d\mu}\lambda = \beta(\lambda) \tag{275}$$

with the boundary condition, $\lambda(\mu_0) = \lambda_0$ and where the (one-loop) β-function equals,

$$\beta^{(1)}(\lambda) = -\mu\frac{d}{d\mu}\frac{d^4V^{(1)}}{d\bar{\phi}^4} = \frac{3}{16\pi^2}\lambda^2\,. \tag{276}$$

The above procedure generalizes straightforwardly to higher loop orders and thus also to N^{n}LLO ($n \geq 1$) resummation, whereby $V^{(1)}$ should be replaced by the suitable higher loop effective potential. Further advancement in understanding of how the effective action Γ behaves under the change of scale resulted in the Cällan-Symanzik equation,

$$\left(\mu\partial_\mu + \beta\partial_\lambda - \frac{1}{2}\gamma\int d^4x\bar{\phi}(x)\frac{\delta}{\delta\bar{\phi}(x)}\right)\Gamma[\bar{\phi},\lambda,\mu] = 0\,, \tag{277}$$

where γ is the anomalous dimension of the field (which is zero at the classical level but becomes generally non-vanishing when loop corrections are included) defined by (see Eq. (270)),

$$\gamma = \mu\frac{\partial}{\partial\mu}Z[\bar{\phi},\mu]\,. \tag{278}$$

The anomalous dimension tells us how one should rescale the field when one changes μ such that the gradient term in the effective action remains its canonical form, $\int d^4x(-1/2)\eta^{\mu\nu}\partial_\mu\bar{\phi}(\mu)\bar{\partial}_\nu\phi(\mu)$. The solution of (277) defines the RG improved effective action. The physical interpretation of the dependence of Γ on μ described by (277) can be summed up as follows. $\Gamma = \Gamma[\lambda,\bar{\phi},\mu]$ can be defined at an arbitrary scale μ and, for every point in configuration space $\{\lambda,\bar{\phi}\}$ at a scale μ,

Eq. (277) assigns another point $\{\lambda', \bar{\phi}'\}$ at some other scale μ' at which Γ assumes the identical value, *i.e.* $\Gamma[\lambda, \bar{\phi}, \mu] = \Gamma[\lambda', \bar{\phi}', \mu']$. In this sense effective actions associated with different values of μ are physically equivalent and therefore no physical process can depend on the value of μ on picks. One can therefore choose μ to be any physical scale that is convenient for the problem at hand. [21] The curves that link these points are called characteristic curves (and are usually parameterized by the parameter t which is defined by $\mu(t) = \mu(0)\mathrm{e}^t$) and the flow generated in this way is known as characteristic flow. We use The Cällan-Symanzik equation (277) as a useful analogy in the following subsection, where we discuss how to improve applicability of perturbative expansion in time dependent problems (see Ref. [11] where a 'reverse' example of bacterial growth is given to illustrate (277)).

The example of RG resummation shows that understanding the structure of the double series (exemplified by the independence on n of the coefficients $c_{n,n-1}$ in (273)) can lead to powerful resummation techniques which can be used to significantly improve the range of applicability of standard perturbative expansion.

In what follows, and building on the above examples, we illustrate how one can extend the range of applicability of standard perturbation theory in non-equilibrium settings.

A. 1PI effective action

The commonly used resummation techniques in the in-in formalism are the one-particle irreducible, or 1PI, effective action. Analogously as in the in-out formalism, the 1PI in-in effective action is obtained by a Legendre transform of the logarithm of the in-in generating functional.

For those who are not familiar with the concepts involving in building the 1PI effective action in what follows we review the basics, but adapt them to the in-in formalism. Recall that the

[21] When dealing with practical problems we typically know Γ only approximately, and these approximants do dependent on μ, which makes a judicious choice of μ important.

partition function $Z[J_+, J_-]$ generates n-point correlators of the form, [22]

$$\langle \bar{T}\big[\hat{q}(t_1)\dots\hat{q}(t_{n_-})\big] T\big[\hat{q}(t_1)\dots\hat{q}(t_{n_+})\big]\rangle = \Big[\frac{-\delta}{\delta J_-(t_1)}\Big]\dots\Big[-\frac{\delta}{\delta J_-(t_{n_-})}\Big]\Big[\frac{\delta}{\delta J_+(t_1)}\Big]\dots\Big[\frac{\delta}{\delta J_+(t_{n_+})}\Big] Z[J_+, J_-]\Big|_{J_+=J_-=0} . \quad (279)$$

While all Feynman diagrams contribute to the partition function, it is useful to introduce the generator of connected Feynman diagrams,

$$W[J_+, J_-] = -i\hbar \ln\big(Z[J_+, J_-]\big) . \quad (280)$$

While the partition function Z generates moments, W generates cumulants. Cumulants have the property that they are all zero beyond the second one for Gaussian distributions. In weakly interacting quantum theories the vacuum is usually well approximated by a Gaussian state and higher order cumulant – which measure the strength of interactions – are typically small. We refer to any textbook on QFT for a more detailed discussion on the difference between Z and W.

We want to derive an equation of motion that governs the evolution of $\langle\hat{q}(t)\rangle$, including all the quantum corrections to it. These arise from the fact that $\langle\hat{q}^n\rangle \neq \langle\hat{q}\rangle^n$. Preferably this equation would follow from an action principle, much in the same way as the classical equation does.

The starting point are the following two equations,

$$\frac{\delta W}{\delta J_+(t)} = \overline{q}_+[J_+, J_-; t] , \qquad \frac{\delta W}{\delta J_-(t)} = -\overline{q}_-[J_+, J_-; t] . \quad (281)$$

We want to define W as a functional of $\overline{q}_+$ and $\overline{q}_-$ instead of J_+ and J_-. So take the relations (281) to define solutions

$$J_+(t) = \widetilde{J}_+[\overline{q}_+, \overline{q}_-; t] , \qquad J_-(t) = \widetilde{J}_-[\overline{q}_+, \overline{q}_-; t] . \quad (282)$$

Plug these back into W to define $\widetilde{W}$,

$$\widetilde{W}[\overline{q}_+, \overline{q}_-] \equiv W[\widetilde{J}_+, \widetilde{J}_-] . \quad (283)$$

[22] The correlators in (279) are not all possible correlators one can write down. In general there can be many T- and $\bar{T}$-products, in whichever order. There is a generalization of the Schwinger-Keldysh formalism [12], that allows for their computation. It boils down to the closed-time-path contour making multiple trips to the turning point and back, each trip representing one product of the type $\bar{T}[\dots]T[\dots]$. While possible, this formulation is rather baroque and not often used and for that reason we do not further pursue it here.

Taking a functional derivative of $\widetilde{W}$ one obtains,

$$\begin{aligned}\frac{\delta\widetilde{W}}{\delta\overline{q}_a(t)} &= \int dt' \sum_{c=\pm} \frac{\delta W}{\delta J_c(t')} \frac{\delta\widetilde{J}_+(t')}{\delta\overline{q}_a(t)} \\ &= \int dt' \sum_{c=\pm} \overline{q}_c(t') c \frac{\delta\widetilde{J}_c(t')}{\delta\overline{q}_a(t)}\,, \end{aligned} \tag{284}$$

where, notational convenience, we have introduced $a, b, c, \cdots = \pm$. Rearranging this relation so that everything is on one side gives

$$\begin{aligned} 0 &= \frac{\delta\widetilde{W}}{\delta\overline{q}_a(t)} - \int dt' \sum_{c} \overline{q}_c(t') c \frac{\delta\widetilde{J}_c(t')}{\delta\overline{q}_a(t)} \\ &= \frac{\delta}{\delta\overline{q}_+(t)} \left[\widetilde{W} - \int dt' \left(\widetilde{J}_+ \overline{q}_+ - \widetilde{J}_- \overline{q}_- \right) \right] + \widetilde{J}_+(t) - \widetilde{J}_-(t)\,. \end{aligned} \tag{285}$$

Here we define the *1PI effective action*,

$$\Gamma[\overline{q}_+, \overline{q}_-] \equiv \widetilde{W}[\overline{q}_+, \overline{q}_-] - \int dt \left(\widetilde{J}_+ \overline{q}_+ - \widetilde{J}_- \overline{q}_- \right) = W[\widetilde{J}_+, \widetilde{J}_-] - \int dt \left(\widetilde{J}_+ \overline{q}_+ - \widetilde{J}_- \overline{q}_- \right). \tag{286}$$

This is in fact a double Legendre transform of W. Note that everything on the RHS is a function of q_+ and q_-, which is due to the inversion of (281) that formally resulted in (282).

Taking a derivative of the effective action now yields

$$\frac{\delta\Gamma}{\delta\overline{q}_a(t)} = -a\widetilde{J}_a(t)\,. \tag{287}$$

This equation of motion describes the sought-for evolution equation for $\bar{q}_\pm$ in presence of external classical sources (to which $\hat{q}$ couples linearly). In absence of such sources, the currents $J_\pm$ are used just as a mathematical tool and at the end of the derivation $J_\pm$ ought to be set to zero. In that case evolution equation for $\bar{q}_\pm$ simplifies to,

$$\frac{\delta\Gamma}{\delta\overline{q}_\pm} = 0\,. \tag{288}$$

This relation explains the origin of the name the effective action. Since $\Gamma[\bar{q}_\pm]$ is an off-shell object, it is defined for all possible (on- and off-shell) trajectories $\bar{q}_\pm(t)$ satisfying boundary conditions, $\bar{q}_\pm(t_0) = \bar{q}_0^\pm$ and $\bar{q}_\pm(t_*) = \bar{q}_*$. Therefore, higher functional derivatives of Γ are also meaningful (remember that we have just reorganized the entire generating functional!) and the limit $J_\pm = 0$ can be taken only after all the functional derivatives of interest have been taken.

At first sight the above construction of the effective action might seem simple. In most of physical situation of interest it is not simple at all. The most difficult part is inverting (281), which most often cannot be done exactly and where most of approximations come in.

Before we proceed to considering examples, in order to get a better insight into what the effective action can be used for, let us take another functional derivative of (287),

$$\frac{\delta^2\Gamma}{\delta\overline{q}_c(\bar{t})\delta\overline{q}_a(t)} = -a\frac{\delta\widetilde{J}_a(t)}{\delta\overline{q}_c(\bar{t})}\,,\,. \tag{289}$$

To get to the physical significance of these relations it is useful to multiply them with a functional derivative of (281),

$$\frac{\delta^2 W}{\delta J_b(t')\delta J_a(t)} = ab\imath\Delta^{ab}(t;t')\,, \tag{290}$$

which defines the full (dressed) two-point function, to obtain

$$\begin{aligned}\int_{t_0}^{t_*} dt_1 \sum_{c=\pm} \imath\Delta^{ac}(t;t_1)\frac{\delta^2\Gamma}{\delta\overline{q}_c(t_1)\delta\overline{q}_a(t)} &= -\int_{t_0}^{t_*} dt_1 \sum_c \frac{\delta\bar{q}_a(t)}{\delta J_c(t_1)}\frac{\delta J_c(t_1)}{\delta\bar{q}_b(t')} = -\frac{\delta\bar{q}_a(t)}{\delta\bar{q}_b(t')} \\ &= -\delta^{ab}\delta(t-t')\,. \end{aligned} \tag{291}$$

This identity shows that the two-point proper (1PI) vertex, $-\Gamma^{(2)}_{ab}(t;t')$ is the mathematical inverse of the full (Keldysh) propagator $\imath\Delta^{ab}(t;t')$, *i.e.*

$$-\Gamma^{(2)}\odot\imath\Delta = \mathbb{1}\!\!\!\!\perp\,, \tag{292}$$

where $\odot$ signifies the operator multiplication, whereby integration over time and summation over all internal indices is implied. Repeated differentiations of the last identity generates a tower of identities for higher proper n-point proper vertices, which show that they formally represent the inverse of connected n-point functions. We encourage the reader to derive the identity that relates the proper 1PI vertex $\Gamma^{(3)}$ to the connected 3-point function $\langle q_a(t)q_b(t')q_c(t'')\rangle$. A nice diagrammatic proof of what the 1PI formalism is can be found in [13].

The above discussion explains why the effective action is said to be the generator of proper $n-$point vertices (which can be represented by a collection of all 1PI diagrams) and the cumulants (connected $n-$point functions) can be constructed by piecing together the 1PI diagrams.

One of the results of this analysis is that the 1PI effective action can be represented as,

$$\Gamma\big[\{\bar{q}_a\}\big] = \sum_{a=\pm} aS[q_a] + \frac{\imath\hbar}{2}\mathrm{Tr}\left[\ln\left(D^{ab}(t;t';\{\bar{q}_a\})\right)\right] + \Gamma_{\geq 2}\big[\{\bar{q}_a\}\big] \tag{293}$$

where the first term is the classical contribution, the second term is the one-loop (bubble) diagram contribution and the last term $\Gamma_{\geq 2}$ contains all two and higher loops 1PI diagrams. The operator $D^{ab}(t;t';\{\bar{q}_a\}) = \delta^2\big[\sum_c cS\big]/\delta q_b(t')\delta q_a(t)$ is the operator (226) that is formally the inverse of the Keldysh propagator (232) but now with the contribution from the interactions also included. The

trace operation Tr in (293) involves taking $t' \to t$ and $b \to a$ and in addition both an integration over time and summation over $a = \pm$.

In order to better appreciate what $\Gamma_{\geq 2}$ in practice entails, in what follows we consider several examples.

1. *Two oscillators with a cubic interaction*

Let us now consider two (simple harmonic) oscillators q and x whose action is,

$$S_0 = \int_{t_0}^{t_*} dt \left(\frac{M\dot{q}^2}{2} - \frac{M\Omega^2 q^2}{2} + \frac{m\dot{x}^2}{2} - \frac{m\omega^2 x^2}{2} \right) . \tag{294}$$

These oscillators interact *via* the following *cubic* interaction,

$$S_{\rm int}[q,x] = \int_{t_0}^{t_*} dt \left(-\frac{hqx^2}{2} \right) . \tag{295}$$

If we are interested in how this interaction affects the dynamics of q oscillator, one can 'integrate out' the $x-$oscillator, which results in an effective action for $\bar{q}$. The integration ought to be treated in perturbation theory, and at the one-loop (and higher loop) order non-trivial $n-$point vertices for q are generated, such that the effective action can be written as,

$$\begin{aligned} \Gamma\big[\{\bar{q}_a\}\big] \;=\; & \sum_a a S_0[\bar{q}_a] - \frac{1}{2}\sum_{ab}\int_{t_0}^{t_*} dt \int_{t_0}^{t_*} dt' \bar{q}_a(t) M^{ab}(t;t') \bar{q}_b(t') \\ & -\frac{1}{3!}\sum_{abc}\int_{t_0}^{t_*} dt \int_{t_0}^{t_*} dt' \int_{t_0}^{t_*} dt'' \bar{q}_a(t)\bar{q}_b(t')\bar{q}_c(t'') V^{(3)}_{abc}(t;t';t'') + \mathcal{O}(\bar{q}_a^4) \, . \end{aligned} \tag{296}$$

Notice that, even though cubic, quartic, and higher order self-interactions are absent at the tree level, they are generated perturbatively at the one- and higher loop orders.

For example, the one-loop diagram that generates the self-mass $M^{ab}(t;t')$ in (296) is shown in figure ONE LOOP and it contributes as,

$$\begin{aligned} -\imath M^{ab}(t;t') \;&=\; \frac{1}{2}\hbar \left(-\frac{\imath a h}{\hbar} \right) \big[\imath\Delta^{ab}(t;t')\big]^2 \left(-\frac{\imath b h}{\hbar} \right) \\ &=\; -\frac{h^2}{2\hbar} ab \big[\imath\Delta^{ab}(t;t')\big]^2 \, , \quad (a,b=\pm) \end{aligned} \tag{297}$$

where the factor $1/2$ is the symmetry factor of the diagram and $\big(-\imath a h/\hbar\big)$ is the tree-level $3-$point vertex function. The reader should write down the analogous one-loop formula for $V^{(3)}$ in (296).

CURRENT END OF LECTURE NOTES!

Since we are interested in the effect of the fluctuations of x on q, we shall assume that the mean field x vanishes. Furthermore, the terms linear in q multiply the equation of motion for q and therefore yield a vanishing contribution to the effective action. Hence, the structure of the effective action can be assume to be of the form,

$$\Gamma_1[\bar{q}_+,\bar{q}_-] = \sum_{\pm} \pm S_0[\bar{q}_\pm] - \frac{1}{2}\int_{t_0}^{t_*} dt \int_{t_0}^{t_*} dt' \sum_{a,b=\pm} ab\bar{q}_a(t) M^{ab}(t;t')\bar{q}_b(t') + \mathcal{O}(\bar{q}_\pm^3)\,. \tag{298}$$

The leading (one-loop) contribution to the self-mass $M^{ab}(t;t')$ can be obtained by performing Wick contractions on the non-coincident x_a's in (**??**). The result is,

$$\imath M^{++}(t;t') = \frac{h^2}{2\hbar}\left[\langle\hat{x}_+(t)\hat{x}_+(t')\right]^2 = \frac{h^2}{2\hbar}\left[\imath\Delta^{++}(t;t')\right]^2\,, \tag{299}$$

where we used, $\exp\left(\imath\Gamma_1/\hbar\right) \simeq 1+\imath\Gamma_1/\hbar$ in Eq. (**??**). The other elements of the self-mass Keldysh matrix are, [23]

$$M_K(t;t') = \begin{pmatrix} \imath M^{++}(t;t') & \imath M^{+-}(t;t') \\ \imath M^{-+}(t;t') & \imath M^{--}(t;t') \end{pmatrix}\,, \qquad \imath M^{ab}(t;t') = \frac{h^2}{2\hbar}\left[\imath\Delta^{ab}(t;t')\right]^2\,, \quad (a,b,=\pm)\,. \tag{300}$$

Now, let us assume that the x oscillator is in a mixed Gaussian state characterised by the following Gaussian two-point functions (cf. Eqs. (76)),

$$\imath\Delta^{+-}(t;t) = \imath\Delta^{-}(t;t') = \frac{\hbar}{2m\omega}\Big[n(\omega)\mathrm{e}^{-\imath\omega\tau} + [1+n(\omega)]\mathrm{e}^{\imath\omega\tau} + \sinh[2r_0]\cos[2\phi(\bar{t})]\Big] \tag{301}$$

where $\tau = t-t')$, $\bar{t} = (t+t')/2$ and r_0 and $\phi(\bar{t}) = \phi_0 - \omega\bar{t}$ are the squeeze factor and phase, respectively, and $n(\omega)$ is the particle number, which in a thermal state,

$$n(\omega) = \frac{1}{\mathrm{e}^{\beta\omega}-1}\,, \qquad \beta = \frac{1}{k_B T}\,. \tag{302}$$

The other elements of the Keldysh propagator can be obtained from,

$$\begin{aligned} \imath\Delta^{-+}(t;t') &= \Theta(t-t')\imath\Delta^{-+}(t;t') + \Theta(t'-t)\imath\Delta^{+-}(t;t') \\ \imath\Delta^{-+}(t;t') &= [\imath\Delta^{+-}(t;t)]^*\,, \qquad \imath\Delta^{--}(t;t') = [\imath\Delta^{++}(t;t)]^*\,. \end{aligned} \tag{303}$$

[23] Sometimes, a different sign convention is used for M^{ab}, where a factor $ab \to (\sigma^3)^{ab}$ is present in the last equation in (300). This factor, of course, then reappears in the definition of the effective action (298), canceling the ab there. That definition is inconvenient, because the relations between the self-masses then differ from those among the two-point functions. For example, the retarded self-mass would be then, $M^r = M^{++} + M^{+-}$, which ought to be compared with Eq. 253).

Upon inserting these into (300), after a brief calculation one obtains,

$$\begin{aligned} \imath M^{\pm\mp}(t;t') &= \frac{\hbar h^2}{2m^2\omega^2}\left[n^2(\omega)\mathrm{e}^{\mp 2\imath\omega\tau} + [1+n(\omega)]^2\mathrm{e}^{\pm 2\imath\omega\tau} + 2n(\omega)[1+n(\omega)]\right] \\ \imath M^{\mp\pm}(t;t') &= \Theta(t-t')\imath M^{\mp\pm}(t;t') + \Theta(t'-t)\imath M^{\pm\mp}(t;t')\,, \end{aligned} \tag{304}$$

where, for simplicity, we assumed no squeezing, $r_0 = 0$. From these one easily obtains the retarded and advanced self-masses as,

$$\begin{aligned} \imath M^r(t;t') &= \imath M^{++}(t;t') - \imath M^{+-}(t;t') = -\imath M^{--}(t;t') + \imath M^{-+}(t;t') = -\Theta(t-t')\imath M^c(t'-t) \\ \imath M^a(t;t') &= \imath M^{++}(t;t') - \imath M^{-+}(t;t') = -\imath M^{--}(t;t') + \imath M^{+-}(t;t') = \Theta(t'-t)\imath M^c(t'-t), \end{aligned} \tag{305}$$

where $\imath M^c$ is the causal (Pauli-Jordan) self-mass (*cf.* Eq. (96)),

$$\imath M^c(t;t') = \imath M^{+-}(t;t') - \imath M^{-+}(t;t') = \frac{\imath\hbar h^2}{m^2\omega^2}[1+2n(\omega)]\sin(2\omega\tau)\,. \tag{306}$$

Our sign convention is such that the relations among the self-masses are identical to those among the two point functions (102–104). Analogously, one can define the statistical (Hadamard) (see Refs. (96–97)) as

$$\begin{aligned} M^F(t;t') &= \frac{1}{2}\Big[\imath M^{+-}(t;t') + \imath M^{-+}(t;t')\Big] \\ &= \frac{\hbar h^2}{2m^2\omega^2}\left[[1+2n(\omega)+n^2(\omega)]\cos(2\omega\tau) + 2n(\omega)[1+n(\omega)]\right]\,. \end{aligned} \tag{307}$$

Of course, these self-masses constitute a very simple case of a more general problem where one ought to calculate self-energies in a quantum field theory at some loop order (self-masses are the special case of self-energies, which occur generically in various field theories). But the basic calculational principles are are the same as outlined in the simple calculation we have just completed.

B. Self-interacting scalar theory

it is straightforward to generalise this equation to the scalar field theory,

$$(\Box^2 - m^2)\bar{\phi}(x) + \int_{t_0}^{t} d^4x' M^c(x;x')\bar{\phi}(x') + \mathcal{O}(\bar{\phi}^2) = 0\,. \tag{308}$$

One can show that this equation is generally manifestly causal, in that a field $\bar{\phi}(x)$ is influenced by its value within the past light-cone. This follows from the form of M^c. For example, on flat space one can show that the scalar field Pauli-Jordan self-mass, $M^c(x;x') \propto \Theta(t-t' - \|\vec{x}-\vec{x}'\|)$, which implies that the integral in (308) vanishes outside the past light-cone.

1. Fermions with Yukawa interactions

2. Scalar electrodynamics

3. Electrodynamics coupled to fermions (QED)

C. 2PI effective action

Cornwall, Jackiw and Tomboulis [15]

VI. APPLICATIONS

- NOT A SEPARATE SECTION - PUT IT IN SECTION **??**?

In this section we study in detail some simple non-equilibrium systems and show how to solve them by making use of the in-in formalism.

1. A simple one-loop calculation

VII. CLASSICAL LIMIT

Here we discuss the general classicality criteria that a quantum system must meet in order for the classical evolution to give approximately correct answers. A useful tool in this direction is the Wigner function, which in quantum mechanics and QFT is defined in quite different ways. We also discuss how to derive the classical kinetic (Boltzmann) equations and how to consistently derive the quantum corrections to the classical evolution. There are several ways in which quantum correction can emerge in the system, and we outline the typical sources of quantum corrections.

Appendix A: Coherent States

For coherent states, please consult Refs. [1, 2].

Appendix B: Squeezed States

A good review on squeezed states is presented in [3, 4]. The simple example of an inverted harmonic oscillator is discussed in [5].

Appendix C: General Gaussian States

A discussion on the general Gaussian states and their von Neumann entropy can be found in Refs. [6, 7].

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