PROJECT F

The Spinning Black Hole

Black holes are macroscopic objects with masses varying from a few solar masses to millions of solar masses. To the extent they may be considered as stationary and isolated, to that extent, they are all, every single one of them, described exactly by the Kerr solution. This is the only instance we have of an exact description of a macroscopic object. Macroscopic objects, as we see them all around us, are governed by a variety of forces, derived from a variety of approximations to a variety of physical theories. In contrast, the only elements in the construction of black holes are our basic concepts of space and time. They are, thus, almost by definition, the most perfect macroscopic objects there are in the universe. And since the general theory of relativity provides a single unique two-parameter family of solutions for their description, they are the simplest objects as well.

—S. Chandrasekhar

1 Introduction

In this project we explore some of the properties of spacetime near a spinning black hole. Analogous properties describe spacetime external to the surface of the spinning Earth, Sun, or other spinning uncharged heavenly body. For a black hole these properties are truly remarkable. Near enough to a spinning black hole—even outside its horizon—you cannot resist being swept along tangentially in the direction of rotation. You can have a negative total energy. From outside the horizon you can, in principle, harness the rotational energy of the black hole.

Do spinning black holes exist? The primary question is: Do black holes exist? If the answer is yes, then spinning black holes are inevitable, since astronomical bodies most often rotate. As evidence, consider the most compact stellar object short of a black hole, the neutron star. Detection of radio and X-ray pulses from some spinning neutron stars (called **pulsars**) tells us that many neutron stars rotate, some of them very rapidly. These are impressive structures, with more mass than our Sun, some of them spinning once every few milliseconds. Conclusion: If black holes exist, then spinning black holes exist.

General relativity predicts that when an isolated spinning star collapses to a black hole, gravitational radiation quickly (in a few seconds of far-away time!) smooths any irregularities in rotation. Thereafter the metric exterior to the horizon of the spinning black hole will be the Kerr metric used in this project.

However, the typical spinning black hole is not isolated; it is surrounded by other matter that is attracted to it. The inward-swirling mass of this **accretion disk** may affect spacetime in its vicinity, distorting the metric away from that of the isolated spinning black hole that we analyze here.

2 Angular Momentum of the Black Hole

An isolated spinning uncharged black hole is completely specified by just two quantities: its mass M and its angular momentum. In Chapter 4 (page 4-3) we defined the angular momentum per unit mass for a particle orbiting a nonspinning black hole as $L/m = r^2 d\phi/d\tau$. In this expression, the angle ϕ has no units and proper time τ has the unit meter. Therefore L/m has the unit meter. To avoid confusion, the angular momentum of a spinning black hole of mass M is given the symbol J and its angular momentum per unit mass is written J/M. The ratio J/M appears so often in the analysis that it is given its own symbol: $a = J/M$. We call the constant "a" the **angular momentum parameter**. Just as the angular momentum L/m of a stone orbiting a non-rotating black hole has the unit meter, so does the angular momentum parameter $a = J/M$ have the unit meter. In what follows it is usually sufficient to treat the angular momentum parameter a as a positive scalar quantity.

Newman and others found the metric for a spinning black hole with net electric charge (see equation [51] and references at the end of this project). The most general steady-state black hole has mass, angular momentum, and electric charge. However, we have no evidence that astronomical bodies carry sufficient net electric charge (which would ordinarily be rapidly neutralized) to affect the metric. If actual black holes are uncharged, then the Kerr metric describes the most general stable isolated black hole likely to exist in nature.

3 The Kerr Metric in the Equatorial Plane

For simplicity we are going to study spacetime and particle motion in the **equatorial plane** of a symmetric spinning black hole of angular momentum J and mass M. The equatorial plane is the plane through the center of the spinning black hole and perpendicular to the spin axis.

Here is the **Kerr metric** in the equatorial plane, expressed in what are called **Boyer-Lindquist coordinates**. The angular momentum parameter a appears in a few unaccustomed places.

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 + \frac{4Ma}{r}dt\,d\phi - \frac{dr^2}{1 - \frac{2M}{r} + \frac{a^2}{r^2}} - \left(1 + \frac{a^2}{r^2} + \frac{2Ma^2}{r^3}\right)r^2 d\phi^2 \quad [1]$$

For the *nonrotating* black hole examined in Chapters 2 through 5, the Schwarzschild metric describing spacetime on a plane is the same for *any* plane that cuts through the center of the black hole, since the Schwarzschild black hole is spherically symmetric. The situation is quite different for the spinning Kerr black hole; the metric [1] is correct *only* for the plane passing through the center of the black hole and perpendicular to its axis of rotation. We choose the equatorial plane because it leads to the simplest and most interesting results.

The time t in equation [1] is the "far-away time" registered on clocks far from the center of attraction, just as for the Schwarzschild metric. In contrast, for $a > 0$ the Boyer-Lindquist r-coordinate does *not* have the simple geometrical meaning that it had for the Schwarzschild metric. More on the meaning of r in Sections 4 and 9. The metric [1] provides a *complete* description of spacetime in the equatorial plane outside the horizon of a spinning uncharged black hole. No additional information is needed to answer every possible question about its (nonquantum) properties and (with the Principle of Extremal Aging) about orbits of free particles and light pulses in the equatorial plane.

You say that the Kerr metric provides a complete nonquantum *description of the spinning black hole. Why this reservation? What more do we need to know to apply general relativity to quantum phenomena?*

In answer, listen to Stephen Hawking as he discusses the "singularity" of spacetime at the beginning of the Universe. A similar comment applies to the singularity inside any black hole.

The general theory of relativity is what is called a classical theory. That is, it does not take into account the fact that particles do not have precisely defined positions and velocities but are "smeared out" over a small region by the uncertainty principle of quantum mechanics that does not allow us to measure simultaneously both the position and the velocity. This does not matter in normal situations, because the radius of curvature of space-time is very large compared to the uncertainty in the position of a particle. However, the singularity theorems indicate that space-time will be highly distorted, with a small radius of curvature at the beginning of the present expansion phase of the universe [or at the center of a black hole]. In this situation, the uncertainty principle will be very important. Thus, general relativity brings about its own downfall by predicting singularities. In order to discuss the beginning of the universe [or the center of a black hole], we need a theory that combines general relativity with quantum mechanics.

—Stephen Hawking

Suggestion: As you go along, check the units of all equations, the equations in the project and also your own derived equations. An equation can be wrong if the units are right, but the equation cannot be right if the units are wrong!

Do Spinning Black Holes Power Quasars?

In contrast to dead solitary black holes, the most powerful steady source of energy we know or conceive or see in all the universe may be powered by a spinning black hole of many millions of solar masses, gulping down enormous amounts of matter swirling around it Maarten Schmidt, working at the Palomar Mountain Observatory in 1956, was the first to uncover evidence for these **quasi-stellar objects**, or **quasars**, starlike sources of light located not billions of kilometers but billions of light-years away Despite being far smaller than any galaxy, the typical quasar manages to put out more than a hundred times as much energy as our entire Milky Way with its hundred billion stars Quasars—unsurpassed in brilliance and remoteness—can justly be called lighthouses of the heavens

Observation and theory have come together to explain in broad outline how a quasar operates A spinning black hole of some hundreds of millions of solar masses, itself perhaps built by accretion, accretes more mass from its surroundings The incoming gas, and stars converted to gas, does not fall in directly, any more than the water rushes directly down the bathtub drain when the plug is pulled This gas, as it goes round and round, slowly makes its way inward to regions of ever-stronger gravity In the process it is compressed and heated and finally breaks up into positive ions and electrons, which emit copious amounts of radiation at many wavelengths The in-falling matter brings with it some weak magnetic fields, which are also compressed and powerfully strengthened These magnetic fields link the swirling electrons and ions into a gigantic accretion disk Matter little by little makes its way to the inner boundary of this accretion disk and then, in a great swoop, falls across the horizon into the black hole During that last swoop, hold on the particle is relinquished. Therefore, the chance is lost to extract as energy the full 100 percent of the mass of each in-falling bit of matter However, magnetic fields do hold on to the ions effectively enough and long enough to extract, as radiant energy, several percent of the mass In contrast, neither nuclear fission nor nuclear fusion is able to obtain a conversion efficiency of more than a fraction of 1 percent Of all methods to convert bulk matter into energy, no one has ever seen evidence for a more effective process than accretion into a spinning black hole, and no one has ever been able to come up with a more feasible scheme to explain the action of quasars See Section 11 for more details

QUERY 1 **Equatorial-plane Kerr metric in the limit of zero angular momentum.** Show that for zero angular momentum ($a = J/M = 0$), the Kerr metric, equation [1], reduces to the Schwarzschild metric (equation [A] in Selected Formulas at the end of this book).

QUERY 2 **Motion stays in plane.** Make an argument from symmetry that a free object that begins to orbit a spinning black hole in the equatorial plane will stay in the equatorial plane.

The Kerr metric has four central new features that distinguish it from the Schwarzschild metric.

The first new feature of the Kerr metric is a new r-value for the horizon. In the Schwarzschild metric, the coefficient of dr^2 is $1/(1 - 2M/r)$. This coefficient increases without limit at the Schwarzschild horizon, $r_H = 2M$. For the Kerr metric, in contrast, the horizon—the point of no return—has an r-value that depends on the value of the angular momentum parameter a. *(Note:* A true proof that a horizon exists requires the demonstration that worldlines can run through it only in the inward direction, not outward. See Project B, pages B-14–15. Our choice of the horizon at the place where the coefficient of dr^2 blows up is an intuitive, but yet correct, choice.)

QUERY 3 **Radial coordinate of the horizon.** Show that for the spinning black hole, the coefficient of dr^2 increases without limit at the r-value:

$$r_H = M \pm (M^2 - a^2)^{1/2} \qquad [2]$$

Look first at the case with the plus sign. What value does r_H have when $a = 0$? For a spinning black hole, is the value of r_H greater or less than the corresponding r-value for the Schwarzschild horizon?

Unless stated otherwise, when we say "the horizon" we refer to equation [2] with the plus sign.

Research note: Choosing the minus sign in equation [2] leads to a second horizon that is *inside* the outer, plus-sign horizon. This inner horizon is called the **Cauchy horizon**. Theoretical research shows that spacetime is stable (correctly described by the Kerr metric) immediately inside the outer horizon and most of the way down to the inner (Cauchy) horizon. However, near the Cauchy horizon, spacetime becomes unstable and therefore is *not* described by the Kerr metric. At the Cauchy horizon is located the so-called *mass-inflation singularity* described in the box on page B-5. The presence of the mass-inflation singularity at the Cauchy horizon bodes ill for a diver wishing to experience in person the region between the outer horizon and the center of a rotating black hole. It is delightful to read in a serious theoretical research paper a sentence such as the following: "Such . . . results strongly suggest (though they do not prove) that inside a black hole formed in a generic collapse, an observer falling toward the inner [Cauchy] horizon should be engulfed in a wall of (classically) infinite density immediately after seeing the entire future history of the outer universe pass before his eyes in a flash." (Poisson and Israel)

4 The Kerr Metric for Extreme Angular Momentum

In this project we want to uncover the central features of the spinning black hole with minimum formalism. The equations become simpler for the case of a black hole that is spinning at the maximum possible rate.

QUERY 4 **Maximum value of the angular momentum.** How "live" can a black hole be? That is, how large is it possible to make its angular momentum parameter $a = J/M$? Show that the largest value of the angular momentum parameter, a, consistent with a real value of r_H is $a = M$. This maximum value of the angular momentum parameter a is equivalent to angular momentum $J = M^2$. What happens to the (inner) Cauchy horizon in this case?

A black hole spinning at the maximum rate derived in Query 4 is called an **extreme Kerr black hole**. How fast are existing black holes likely to spin; how "live" are they likely to be? Listen to Misner, Thorne, and Wheeler (page 885): "Most objects (massive stars; galactic nuclei; . . .) that can collapse to form black holes have so much angular momentum that the holes they produce should be 'very live' (the angular momentum parameter $a = J/M$ nearly equal to M; J nearly equal to M^2)."

QUERY 5	**Maximum angular momentum of Sun?** A recent estimate of the angular momentum of Sun is 1.91×10^{41} kilogram meters^2 per second (see the references). What is the value of the angular momentum parameter $a = J/M$ for Sun, in meters? (Hint: Divide the numerical value above by M_{kg}, the mass of Sun in kilograms, to obtain an intermediate result in units of meter^2/second. What conversion factor do you then use to obtain the result in meters?) What fraction a/M is this of the maximum possible value permitted by the Kerr metric?

The metric for the equatorial plane of the extreme-spin black hole results if we set $a = M$ in equation [1], which then becomes

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 + \frac{4M^2}{r}dt\,d\phi - \frac{dr^2}{\left(1 - \frac{M}{r}\right)^2} - R^2 d\phi^2 \qquad \text{[3. extreme Kerr]}$$

Note how the denominator of the dr^2 term in the Kerr metric differs in two ways from the dr^2 term in the Schwarzschild metric: here the denominator is squared and also contains M/r instead of $2M/r$.

Equation [3] has been simplified by defining

$$R^2 \equiv r^2 + M^2 + \frac{2M^3}{r} \qquad \text{[4. extreme Kerr]}$$

The form $R^2 d\phi^2$ of the last term on the right side of equation [3] tells us that R is the **reduced circumference** for extreme Kerr spacetime. That is, the value of R is determined by measuring the circumference of a stationary ring in the equatorial plane concentric to the black hole and dividing this circumference by 2π. This means that r is *not* the reduced circumference but has a value derived from equation [4]. Finding an explicit expression for r in terms of R requires us to solve an equation in the third power of r, which leads to an algebraic mess. Rather than solving such an equation, we carry along expressions containing both R and r. Note from equation [4] that R is not equal to r even for large values of r, although the *percentage* difference between R and r does decrease as r increases.

QUERY 6	**Limiting values of R.** What is r_H, the value of r at the horizon? What is R_H, the value of R at the horizon? Find the *approximate* range of r-values for which the value of R differs from the value of r by less than one part in a million.

QUERY 7	**More general R_a.** Consider the more general case of arbitrary angular momentum parameter a given in equation [1]. What is the expression for R^2 (call it R_a^2) in this case? What is the value of R_a in the limiting case of the nonspinning black hole?

Now move beyond the new r-value for the horizon—the first new feature of the Kerr metric—to the **second new feature of the Kerr metric,** which is the presence of the product $dtd\phi$ of two different spacetime coordinates, called a cross **product**. The cross product implies that coordinates ϕ and t are intimately related. In the following section we show that the Kerr metric predicts **frame dragging**. What does "frame dragging" mean? Near any center of attraction, radial rocket thrust is required to keep a stationary observer at a fixed radius. Near a spinning black hole, an additional *tangential* rocket thrust is required to prevent orbiting, that is to keep the fixed stars in steady position overhead. One might say that spacetime is swept around by the rotating black hole: spacetime itself on the move!

Unless otherwise noted, everything that follows applies to the equatorial plane around an extreme Kerr black hole.

5 The Static Limit

The third new feature of the Kerr metric is the presence of a so-called **static limit**. The horizon of a rotating black hole lies at an r-value *less* than $2M$ (equation [2] with the plus sign). The horizon is where the metric coefficient of dr^2 blows up. In contrast, for the equatorial plane, the coefficient of dt^2, namely, $(1 - 2M/r)$, goes to zero at $r = 2M$, just as it does in the Schwarzschild metric for a nonrotating black hole. The r-value $r = 2M$ in the equatorial plane at which the coefficient of the dt^2 term goes to zero is called the **static limit**. A comparison of equations [3] and [1] shows that the expression for the static limit in the equatorial plane is the same whatever the value of the angular momentum parameter a, namely

$$r_S = 2M \qquad [5]$$

The static limit gets its name from the prediction that for radii smaller than r_S (but greater than that of the horizon r_H) an observer cannot remain at rest, cannot stay *static*. The space between the static limit and the horizon is called the **ergosphere**. Inside the ergosphere you are inexorably dragged along in the direction of rotation of the black hole. No matter how powerful your rockets, you cannot stand at one fixed angle ϕ. For you the fixed stars cannot remain at rest overhead. In principle, a small amount of frame dragging is detectable near any spinning astronomical object. An experimental Earth satellite (Gravity Probe B), now under construction at Stanford University, will measure the extremely small frame-dragging effects predicted near the spinning Earth. Inside the static limit of a rotat-

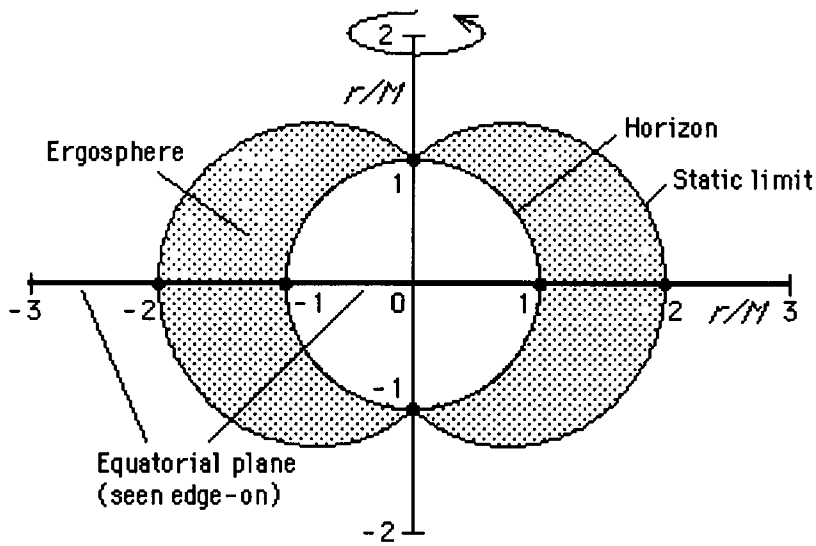

Figure 1 *Computer plot of the cross-section of an extreme black hole showing the static limit and horizon using the Kerr bookkeeper (Boyer-Lindquist) coordinate r (not R) From inside the horizon no object can escape, even one traveling at the speed of light Between the horizon and the static limit lies the* ***ergosphere,*** *shaded in the figure Within this ergosphere everything—even light—is swept along by the rotation of the black hole Inside the ergosphere, too, a stone can have a negative total energy (Section 10)*

ing black hole, in contrast, the frame dragging is irresistible, as will be described on the following page.

The Kerr metric for three space dimensions—not discussed in this book—reveals that the horizon has a constant r-value in all directions (is a sphere) while the static limit has cusps at the poles. Figure 1 shows this result. This figure is drawn in Kerr bookkeeper (Boyer-Lindquist) coordinates, which present only one possible way to view these structures. Other coordinate systems, representing a more "intrinsic" geometry, stretch the horizon in the horizontal direction, giving it the approximate shape of a hamburger bun.

QUERY 8 **Reduced circumference of the static limit.** For the extreme black hole, find an expression for R_S, the reduced circumference of the static limit, in the equatorial plane.

QUERY 9 **Viewing the spinning black hole from above.** Draw a cross-section of the extreme black hole in the equatorial plane. That is, show what it would look like to display the static limit and horizon in bookkeeper coordinates on a plane cut through the horizontal axis of Figure 1, as if looking downward along the vertical axis in that figure. Label the static limit, horizon, and ergosphere and put in expressions for their radii.

Now look more closely at the nature of the static limit in the equatorial plane. Examine the Kerr metric for the case of light moving initially in the ϕ direction ($dr = 0$). (Only the initial motion in the equatorial plane will be in this tangential direction; later the beam may be deflected radially inward or outward.) Because this is light, the proper time is zero between adjacent events on its path: $d\tau = 0$. Make these substitutions in the metric [3], divide through by dt^2, and rearrange to obtain

$$R^2\left(\frac{d\phi}{dt}\right)^2 - \frac{4M^2}{r}\left(\frac{d\phi}{dt}\right) - \left(1 - \frac{2M}{r}\right) = 0 \qquad \text{[6. light]}$$

Equation [6] is quadratic in the angular velocity $d\phi/dt$.

QUERY 10 **Tangential motion of light.** Solve equation [6] for $d\phi/dt$. Show that the result has two possible values:

$$\frac{d\phi}{dt} = \frac{2M^2}{rR^2} \pm \frac{2M^2}{rR^2}\left[1 + \frac{r^2R^2}{4M^4}\left(1 - \frac{2M}{r}\right)\right]^{1/2} \qquad \text{[7. light]}$$

Look closely at this expression at the static limit, namely, where $r = 2M$ and $R^2 = 6M^2$. The two solutions are

$$\frac{d\phi}{dt} = 0 \qquad \text{and} \qquad \frac{d\phi}{dt} = \frac{4M^2}{rR^2} = \frac{1}{3M} \qquad \text{[8. light]}$$

To paraphrase Schutz (see references), the second solution in [8] represents light sent off in the same direction as the hole is rotating. The first solution says that the other light flash—the one sent "backward"—does not move at all as recorded by the far-away bookkeeper. The dragging of orbits has become so strong that this light cannot move in the direction opposite to the rotation! Clearly, any material particle, which must move slower than light, will therefore have to rotate with the hole, even if it has an angular momentum arbitrarily large in the sense opposite to that of hole rotation.

QUERY 11 **Light dragging in the ergosphere.** Show that inside the ergosphere (r such that $r_H < r < r_S$), light launched in *either* tangential direction in the equatorial plane moves in the direction of rotation of the black hole as recorded by the far-away bookkeeper. That is, show that the initial tangential angular velocity $d\phi/dt$ is always positive.

The static limit creates a difficulty of principle in measuring the reduced circumference R, defined by equation [4] on page F-6. According to that definition, one measures R by laying off the total distance—the circumference—around a stationary ring in the equatorial plane concentric to the black hole and dividing the circumference by 2π to find the value of R. But inside the static limit no such ring can remain stationary; it is inevitably swept along in a tangential direction, no matter how powerful the rockets we use to try to keep it stationary. Thus, for the present, we have no practical definition for R inside the static limit. We will overcome this difficulty in principle in Section 9.

For completeness, we mention here the fourth new feature of the Kerr metric, which is analyzed further in Sections 10 and 11.

The fourth new feature of the Kerr metric is available energy. No net energy can be extracted from a nonspinning black hole (except for the quantum "Hawking radiation," page 2-4, which is entirely negligible for star-mass black holes). For this reason, the nonspinning black hole carries the name *dead*. In contrast, energy of rotation is available from a spinning black hole, which therefore deserves its name *live*. See Section 12.

6 Radial and Tangential Motion of Light

QUERY 12 **Radial motion of light.** For light ($d\tau = 0$) moving in the radial direction ($d\phi = 0$), show from the metric that

$$\frac{dr}{dt} = \pm\left(1 - \frac{M}{r}\right)\left(1 - \frac{2M}{r}\right)^{1/2} \qquad \text{[9. light, } d\phi = 0\text{]}$$

Show that this radial speed goes to zero at the static limit and is imaginary (therefore unreal) inside the ergosphere. *Meaning:* No purely radial motion is possible inside the ergosphere. See Figure 2.

For light ($d\tau = 0$) moving in the tangential direction ($dr = 0$), we call the tangential velocity $Rd\phi/dt$ as recorded by the Kerr bookkeeper. From equation [7], this tangential velocity is given by

$$R\frac{d\phi}{dt} = \frac{2M^2}{rR} \pm \frac{2M^2}{rR}\left[1 + \frac{r^2R^2}{4M^4}\left(1 - \frac{2M}{r}\right)\right]^{1/2} \qquad \text{[10. light, } dr = 0\text{]}$$

The second term on the right side of [10] can be simplified by substituting for R^2 in the numerator from equation [4]. (Trust us or work it out for yourself!) Equation [10] becomes

$$R\frac{d\phi}{dt} = \frac{2M^2}{rR} \pm \frac{r - M}{R} \qquad \text{[11. light, } dr = 0\text{]}$$

QUERY 13	**Light dragging at the horizon.** What happens to the light dragging at the horizon (r_H given by equation [2] with the plus sign and a = *M,* and R_H derived in Query 6)? Show that at the horizon the initial tangential rotation $d\phi/dt$ for light has a single value whichever way the pulse is launched. Show that the bookkeeper velocity $Rd\phi/dt$ for this light at the horizon has the value shown in Figure 2.

The radial and tangential velocities of light in equations [9] and [11] are *bookkeeper velocities,* reckoned by the Kerr bookkeeper using the coordinates r and ϕ and the far-away time t. Nobody measures the Kerr bookkeeper velocities directly, just as nobody measured directly bookkeeper velocities near a non-spinning black hole (Chapters 3 through 5).

Figure 2 shows the radial and tangential bookkeeper velocities of light for the extreme Kerr metric. Note again that these plots show the *initial* velocity of a light flash launched in the various directions. After launch, a radially moving light flash may be dragged sideways or a tangentially moving flash may be deflected inward.

QUERY 14	**Locked-in motion?** (Optional) Kip Thorne says, "I guarantee that, if you send a robot probe down near the horizon of a spinning hole, blast as it may it will never be able to move forward or backward [in either tangential direction] at any speed other than the hole's own spin speed. . . ." What evidence do equation [11] and Figure 2 give for this conclusion? What is "the hole's own spin speed"? (See Kip S. Thorne, *Black Holes and Time Warps,* W. W. Norton & Co., New York, 1994, page 57.)

7 Wholesale Results, Extreme Kerr Black Hole

Now suppose that you have never heard of the Kerr metric and someone presents you with the "anonymous" metric [3] (which we know to be the metric for the extreme Kerr black hole) plus the definition of R:

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 + \frac{4M^2}{r}dtd\phi - \frac{dr^2}{\left(1 - \frac{M}{r}\right)^2} - R^2d\phi^2 \qquad [3]$$

$$R^2 \equiv r^2 + M^2 + \frac{2M^3}{r} \qquad [4]$$

You say to yourself, "This equation is just a crazy kind of mixed-up Schwarzschild-like metric, with a nutty denominator for the dr^2 term, a cross-term in $dtd\phi$, and R^2 instead of r^2 as a coefficient for $d\phi^2$. Still, it's a metric. So let's try deriving expressions for angular momentum, energy, and so forth for a particle moving in a region described by this metric in analogy to similar derivations for the Schwarzschild metric." So saying,

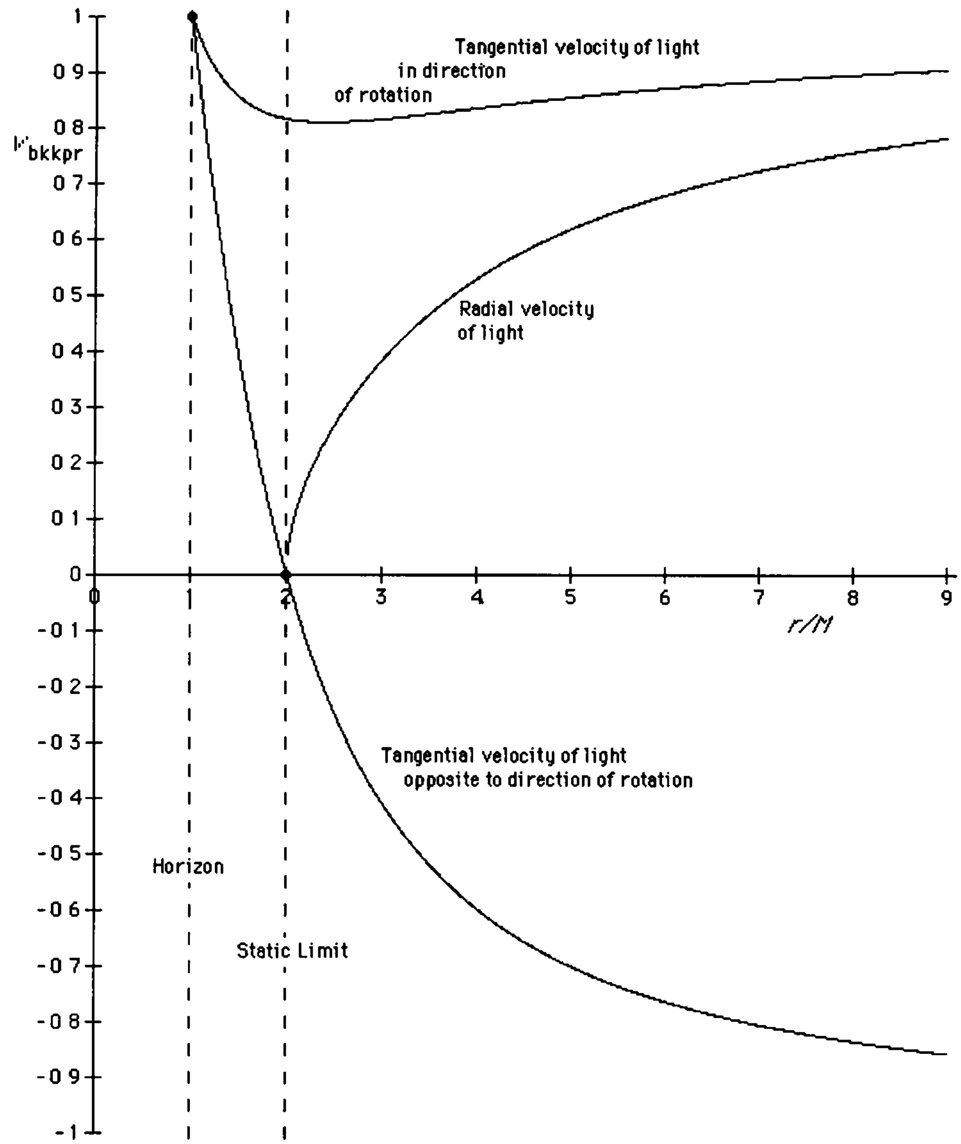

Figure 2 *Computer plot of bookkeeper radial and tangential velocities of light near an extreme Kerr black hole (a = J/M = M) Note that as r/M becomes large, the different bookkeeper velocities all approach plus or minus unity Note also that purely radial motion of light is not possible inside the static limit Important These are* initial *velocities of light just after launch in the given direction After launch, the light will generally change direction For the case of a nonrotating black hole, see Figures 6 and 7, pages B-1B–19*

Table 1 Comparison of results of nonspinning and extreme-spin black holes

Quantity	Nonspinning Schwarzschild black hole	Extreme-spin Kerr black hole ("shell" = stationary ring outside static limit)
Define r and R	Reduced circumference = $r \equiv \dfrac{\text{(circumference of shell)}}{2\pi}$ [12]	Reduced circumference R given by: $R^2 \equiv r^2 + M^2 + \dfrac{2M^3}{r}$ [13]
Shell time vs. far-away time: (gravitational red shift)	$dt_{\text{shell}} = \left(1 - \dfrac{2M}{r}\right)^{1/2} dt$ [14]	$dt_{\text{shell}} = \left(1 - \dfrac{2M}{r}\right)^{1/2} dt$ [15. stationary]
dr_{shell} vs. dr	$dr_{\text{shell}} = \left(1 - \dfrac{2M}{r}\right)^{-1/2} dr$ [16]	$dr_{\text{shell}} = \left(1 - \dfrac{M}{r}\right)^{-1} dr$ [17. stationary]
Energy (constant of the motion)	$\dfrac{E}{m} = \left(1 - \dfrac{2M}{r}\right)\dfrac{dt}{d\tau}$ [18]	$\dfrac{E}{m} = \left(1 - \dfrac{2M}{r}\right)\dfrac{dt}{d\tau} + \dfrac{2M^2}{r}\dfrac{d\phi}{d\tau}$ [19]
Angular momentum (constant of the motion)	$\dfrac{L}{m} = r^2\dfrac{d\phi}{d\tau}$ [20]	$\dfrac{L}{m} = R^2\dfrac{d\phi}{d\tau} - \dfrac{2M^2}{r}\dfrac{dt}{d\tau}$ [21]

you use the Principle of Extremal Aging and other methods of Chapters 2 through 5 to derive expressions similar to results in those chapters and enter them in the last column of Table 1.

Notes: (1) We limit ourselves to the equatorial plane. (2) Outside the static limit we can still set up stationary spherical shells (which we have limited to stationary *rings* in the equatorial plane), but we must use continual tangential rocket blasts to keep these rings from rotating in the tangential direction.

QUERY 15 **Energy and angular momentum as constants of the motion.** Derive Table 1, entries [19] and [21] for energy and angular momentum of a free object moving in the equatorial plane of an extreme Kerr black hole.

8 Plunging: The "Straight-In Spiral"

For the nonrotating black hole the simplest motion was radial plunge (Chapter 3). What is the simplest motion near a spinning black hole? By analogy, let us examine motion starting from infinity and proceeding with zero angular momentum.

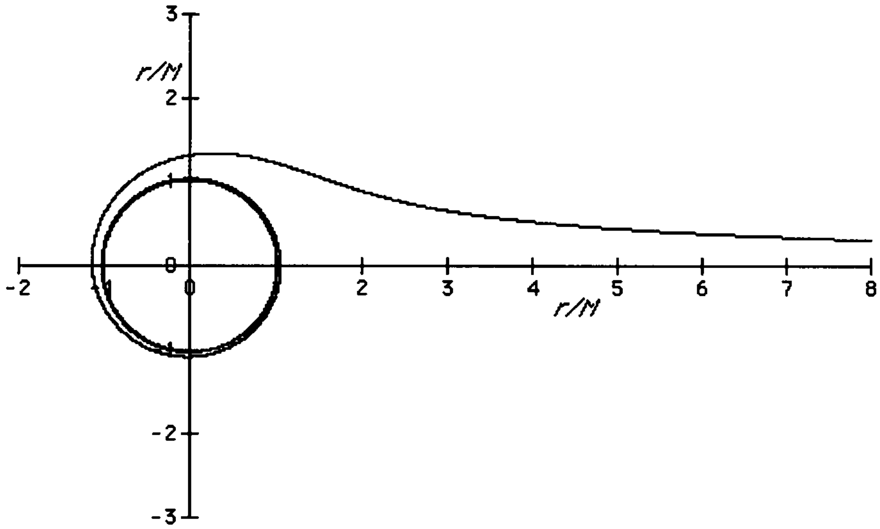

__Figure 3__ Computer plot. Kerr map (Kerr bookkeeper plot) of the trajectory in space of a stone dropped from rest far from a black hole (therefore with zero angular momentum). According to the far-away bookkeeper, the stone spirals in to the horizon at r = M and circulates there forever

QUERY 16 **No angular momentum. But angular motion!** Set angular momentum [21] equal to zero and verify the following equation:

$$\frac{d\phi}{dt} = \frac{2M^2}{rR^2} \qquad [22.\ L = 0]$$

Equation [22] gives the remarkable result that a particle with zero angular momentum nevertheless circulates around the black hole! This result is evidence for our interpretation that the black hole drags nearby spacetime around with it. Figure 3 shows the trajectory of an inward plunger with zero angular momentum, as calculated in what follows.

Let's see if we can set up the equations to follow a stone that starts at rest far from a rotating black hole and moves inward with zero angular momentum. At remote distance, in flat spacetime, the stone has energy $E/m = 1$. It keeps the same energy as it falls inward. From equation [19] in Table 1,

$$\frac{E}{m} = 1 = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} + \frac{2M^2}{r}\frac{d\phi}{d\tau} \qquad [23]$$

Equations [22] and [23] are two equations in the four unknowns dr, dt, $d\tau$, and $d\phi$. A third equation is the metric [3] for the extreme-spin black hole. With these three independent equations, we can eliminate three of the four unknowns to find a relation between any two remaining differentials. We choose the quantities dr and $d\phi$, because we want to draw the trajectory, the Kerr map. Don't bother doing the algebra—it is a mess. After substituting equation [4] for R^2 into the result, one obtains the relation between dr and $d\phi$:

$$\begin{aligned} dr &= \frac{r-M}{r}\left[\frac{r^5}{2M^3} - \frac{r^4}{M^2} + \frac{r^3}{M} - r^2 + \frac{Mr}{2}\right]^{1/2} d\phi \\ &= \frac{(r-M)^2}{r}\left[\frac{r^3}{2M^3} + \frac{r}{2M}\right]^{1/2} d\phi \end{aligned} \qquad [24.\ L = 0]$$

The computer has no difficulty integrating and plotting this equation, as shown in Figure 3. Since we used the Kerr bookkeeper angular velocity [22], the resulting picture is that of the Kerr bookkeeper. For her, the zero-angular-momentum stone spirals around the black hole and settles down in a tight circular path at $r = M$, there to circle forever.

QUERY 17 **Final circle according to the bookkeeper.** Verify that dr goes to zero (that is, r does not change) once this stone reaches the horizon.

Remember that for the nonspinning black hole an object plunging inward slows down as it approaches the horizon, according to the records of the Schwarzschild bookkeeper. For both spinning and nonspinning black holes, the in-falling stone with $L = 0$ never crosses the horizon when clocked in far-away time.

QUERY 18 **Bookkeeper speed in the "final circle."** At the horizon, what is the numerical value of the tangential speed $Rd\phi/dt$ of the stone dropped from rest at infinity, as measured by the Kerr bookkeeper?

The observer who has fallen from rest at infinity has quite a different perception of the trip inward! For her there is no pause at the horizon; she has a quick, smooth trip to the center (assuming that the Kerr metric holds all the way to the center!). An algebra orgy similar to the previous one gives a relation between dr and $d\tau$, where $d\tau$ is the wristwatch time increment of the in-faller:

$$\begin{aligned} \left(\frac{dr}{d\tau}\right)^2 &= \frac{2Mr^3 - 4M^2r^2 + 4M^3r - 4M^4 + \dfrac{2M^5}{r}}{r^2(r-M)^2} \\ &= \frac{2M}{r}\left(1 + \frac{M^2}{r^2}\right) \end{aligned} \qquad [25.\ L = 0]$$

Figure 4 compares the magnitude of the square root of this expression with the magnitude of the velocity of the stone dropped from rest at a great distance in the Schwarzschild case (equation [32], page 3-22):

$$\frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2} \qquad [26.\ L = 0 \text{ Schwarzschild}]$$

Both equations [25] and [26] show radial components of speed greater than unity in the region of small radius. The resulting speed is even more impressive when one adds the tangential motion forced on the diver descending into the spinning black hole (Figure 2 and 3). Does such motion violate the "cosmic speed limit" of unity for light? A similar question is debated for the Schwarzschild black hole in Section 3 of Project B, Inside the Black Hole, pages B-6–12.

Research note: When applied inside the horizon, equation [25] assumes that the Kerr metric correctly describes spacetime all the way to the center of the extreme Kerr black hole. This may not be the case. See the box Egg-beater Spacetime? on page B-5.

9 Ring Riders

Equation [22] in Section 8 describes the angular rotation rate ω of an in-falling stone that has zero angular momentum:

$$\frac{d\phi}{dt} \equiv \omega = \frac{2M^2}{rR^2} \qquad [27.\ L = 0]$$

In some way, ω in this equation describes the angular rate at which space is "swept along" by the nearby spinning black hole. What happens if we "go with the flow," moving tangentially at angular rate ω given by this equation? Will we cease to feel a tangential force? What happens to us at the static limit?

To pursue these ideas, we envision a set of nested rings in the equatorial plane and concentric to the black hole (Figure 5). Each of these rings revolves at an angular rate given by equation [27] as reckoned by the Kerr bookkeeper. Rings at different values of r rotate at different angular rates.

The result of this construction is a set of observers in the equatorial plane whom we call **ring riders**. A ring rider is an observer who stands at rest on one of the zero angular momentum rotating rings. In times past, ring riders were known as **locally nonrotating observers**, but now the customary name is **zero angular momentum observers** or **ZAMOs**. Each ring rider, like each shell observer in Schwarzschild geometry, is subject to a gravitational acceleration directed toward the center of the black hole, but experiences no frame dragging in the tangential direction (because he rides along with the rotating ring). In both cases the radially inward gravitational acceleration becomes infinite at the horizon, destroying any possible circumferential ring structures at or inside the horizon. According

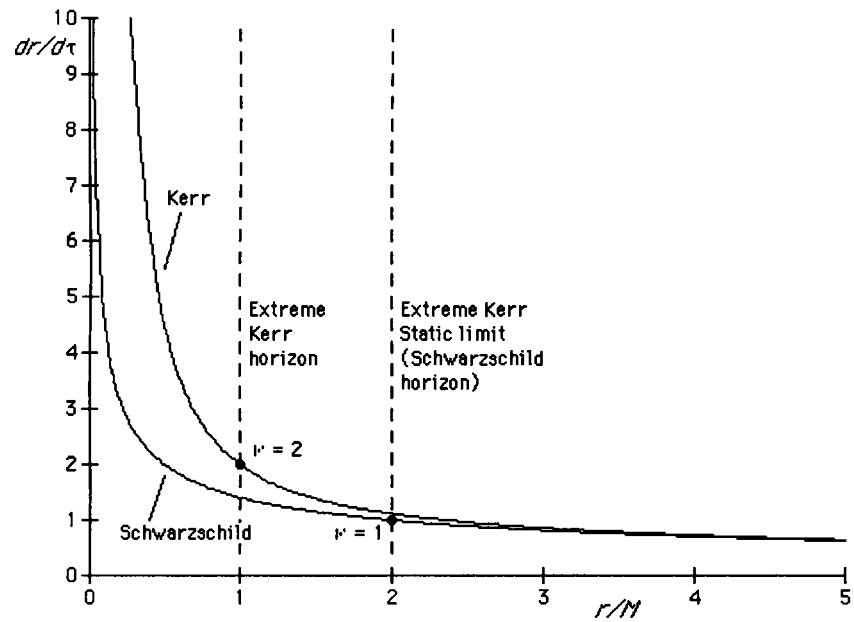

Figure 4 *Comparison of radial components of plunge velocities experienced by different in-fallers who drop from rest (so with L = 0) at a great distance from Schwarzschild and extreme Kerr black holes.*

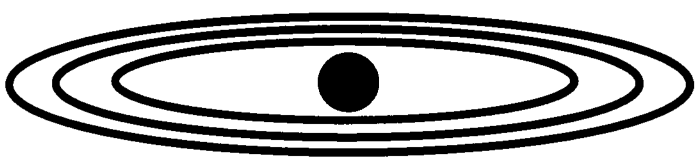

Figure 5 *Kerr map (perspective plot) of rings surrounding a rotating black hole The rings rotate in the same direction as the black hole but at angular rates that differ from ring to ring*

to ring rider measurements, light has speed unity, the same speed in both tangential directions, as we shall see.

QUERY 19 **Ring slippage.** Will the inner rings rotate with larger or smaller angular velocity than the rings farther out? Justify your choice.

QUERY 20 **Ring speed according to the bookkeeper.** What are the units of ω in equation [27]? What is the numerical value of the bookkeeper speed $R\omega$ for each of the rings $r = 100M$, $r = 10M$, $r = 2M$, and $r = M$? Express the answers as a fraction of the speed of light.

QUERY 21 **Does rain fall vertically?** Present an argument that a stone dropped from rest starting at a great radial distance falls vertically past the rider on every ring. *Guess:* Is the same true if the stone is *flung* radially inward from a great distance? *Guess:* What about light?