Problemset 1 for Cosmology (ns-tp430m)

Problems are to be handed in on Thu Feb 13. (18 points plus 4 bonus points in total.)

1. The geodesic equation. *(6 points)*

(a) *(4 points)* By making use of the action principle, derive the geodesic equation for the 4-velocity of a point particle, $u^\mu = dx^\mu/d\lambda$ from the following general relativistic action for a point particle,

$$S_{\text{point particle}} = mc\int ds = mc\int d\lambda\Big(g_{\mu\nu}(x)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\Big)^{1/2}, \tag{1}$$

where λ is an affine parameter characterizing the geodesic. In showing this you may use $\nabla_\alpha g_{\mu\nu} = 0$ (see Problem 2c below). What is the special relativistic limit of the action (1)?

(b) *(2 points)* Show that the geodesic equation is invariant under affine transformations,

$$\lambda \to \tilde{\lambda} = a\lambda + b\,, \tag{2}$$

where a, b, are real constants. By making use of a general reparametrization of the geodesic parameter, $\lambda \to \tilde{\lambda}(\lambda)$, derive the geodesic equation that uses $\tilde{\lambda}$ as the parameter. Show that (2) represents the most general transformation that leaves the geodesic equation invariant.

In this course we shall use the metric convention, $\mathrm{sign}[g_{\mu\nu}] = (+,-,-,-)$, we shall assume a symmetric, metric compatible (Levi-Civitá or Christoffel) connection, $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{(\mu\nu)}$, and for a while we shall keep the speed of light c and the Newton constant G_N explicitly in the equations.

2. General covariance and tensors. *(8 points)*

(a) *(2 points)* Show that

$$\int d^4x\sqrt{-g} \tag{3}$$

represents a generally covariant measure. The symbol $g = \det[g_{\alpha\beta}]$ denotes the determinant of the metric tensor. You may find the following definition of the determinant of a 2-indexed tensor $t_{\mu\nu}$ useful,

$$\epsilon_{\mu\nu\rho\sigma}\det[t_{\alpha\beta}] = \epsilon_{\alpha\beta\gamma\delta}t^\alpha_{\ \mu}t^\beta_{\ \nu}t^\gamma_{\ \rho}t^\delta_{\ \sigma}\,, \tag{4}$$

where $\epsilon_{\mu\nu\rho\sigma}$ represents a totally antisymmetric Levi-Civitá ϵ-symbol in 3+1 dimensions, such that $\epsilon_{0123} = 1$, and it is antisymmetric under the exchange of any two indices. The ϵ-symbol vanishes whenever any two indices are identical.

Show that $\sqrt{-g}\epsilon_{\mu\nu\rho\sigma}$ transforms as a four-indexed covariant tensor, while $\epsilon^{\mu\nu\rho\sigma}/\sqrt{-g}$ transforms as a contravariant tensor.

(b) *(2 points)* Show that the covariant derivative of a two indexed contravariant tensor field $T^{\mu\nu}$ reads

$$\nabla_\alpha T^{\mu\nu} \equiv T^{\mu\nu}_{\ \ \ ;\alpha} = T^{\mu\nu}_{\ \ \ ,\alpha} + \Gamma^\mu_{\rho\alpha}T^{\rho\nu} + \Gamma^\nu_{\rho\alpha}T^{\mu\rho}\,, \tag{5}$$

where $_{,\alpha} \equiv \partial/\partial x^\alpha$.

Hint: Make use of *e.g.* Eq. (39) in Part I of lecture notes,

$$\nabla_\gamma B_{\alpha\beta} \equiv B_{\alpha\beta;\gamma} = B_{\alpha\beta,\gamma} - \Gamma^\mu_{\alpha\gamma} B_{\mu\beta} - \Gamma^\mu_{\beta\gamma} B_{\mu\alpha}\,. \tag{6}$$

(c) *(2 points)* Show that the covariant derivative of the metric tensor vanishes (assume the Levi-Civitá connection),

$$\nabla_\alpha g_{\mu\nu} = 0\,. \tag{7}$$

(d) *(2 points)* Show that the Einstein curvature tensor $G_{\mu\nu} = \mathcal{R}_{\mu\nu} - (1/2)\mathcal{R} g_{\mu\nu}$ satisfies the following Bianchi identity

$$\nabla^\nu G_{\mu\nu} = 0\,, \tag{8}$$

where $\mathcal{R}_{\mu\nu}$ denotes the Ricci curvature tensor and $\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$ is the Ricci curvature scalar.

Hint: Show first that the following cyclic derivative property for the Riemann curvature tensor,

$$\nabla_\gamma \mathcal{R}_{\mu\nu\alpha\beta} + \nabla_\alpha \mathcal{R}_{\mu\nu\beta\gamma} + \nabla_\beta \mathcal{R}_{\mu\nu\gamma\alpha} = 0\,, \tag{9}$$

and then contract the appropriate indices.

3. Einstein's equation. *(4 points + 4 bonus points)*

(a) (*4 points*) Starting from the action,

$$S = S_{\rm HE} + S_{\rm M}\,, \tag{10}$$

where $S_{\rm HE}$ is the Hilbert-Einstein action,

$$S_{\rm HE} = -\frac{c^4}{16\pi G_N}\int d^4x\sqrt{-g}\Big(\mathcal{R} + \frac{\Lambda}{c^2}\Big)\,, \tag{11}$$

$\mathcal{R}$ is the Ricci scalar, Λ the cosmological constant, and $S_{\rm M}$ is a matter action, derive the Einstein's equation. Recall that the stress-energy tensor, $T_{\mu\nu} = [2(-g)^{-1/2}](\delta S_{\rm M}/\delta g^{\mu\nu})$.

(b*) (*4 bonus points*) As in part **(a)**, but instead of $S_{\rm HE}$ use of the scalar-tensor gravitational action (a generalized Jordan-Fierz-Brans-Dicke action),

$$S_{\rm STe} = \int d^4x\sqrt{-g}\Big[-F(\Phi)\mathcal{R} + \frac{\Lambda}{c^2} + \frac{1}{2}(\partial_\mu\Phi)(\partial_\mu\Phi)g^{\mu\nu} - V(\Phi)\Big]\,, \tag{12}$$

where Φ is a (gravitational) scalar field, $V(\Phi)$ is a potential and $F(\Phi)$ is a function of Φ, which has a Taylor expansion around $\Phi \sim 0$,

$$F(\Phi) = \frac{c^4}{16\pi G_N} + \frac{1}{2}\xi\mathcal{R}\Phi^2 + \mathcal{O}(\Phi^4)\,, \tag{13}$$

where ξ is a constant (non-minimal) coupling. If existed, such a field could lead to strange effects, such as making the gravitational force ('Newton constant') dependent on the local matter density, a new 'fifth force', etc.

NB: Bonus points are given to more challenging problems. They are counted as normal points, unless your number of points exceeds 100% in an individual homework.