ADVANCED TOPICS IN THEORETICAL PHYSICS II

Tutorial problem set 2, 18.09.2017.

(20 points in total)

Problems 4 and 5 are in-class exercises. Problem 6 is due at Monday, 25.09.2017.

■ PROBLEM 4 Entropy of coupled oscillators.

Consider two coupled simple harmonic oscillators (SHOs), whose Hamiltonian is given by,

$$\begin{aligned}
\hat{H} &= \hat{H}_0 + \hat{H}_{\rm int} \\
\hat{H}_0 &= \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{q}^2 + \frac{\hat{P}^2}{2M} + \frac{M\Omega^2}{2}\hat{Q}^2 \\
\hat{H}_{\rm int} &= \lambda_0 \hat{q}\hat{Q} + \lambda_1 \hat{p}\hat{Q} + \lambda_2 \hat{q}\hat{P} + \lambda_3 \hat{p}\hat{P}
\end{aligned} \tag{4.1}$$

where m, ω are the mass and frequency of the oscillator $(\hat{q}, \hat{p})$, M, Ω are the mass and frequency of the $(\hat{Q}, \hat{P})$ oscillator and λ_i $(i = 0, 1, 2, 3)$ are the coupling constants. Assume that initially at $t = t_0$ the quantum state is Gaussian, and that it is not entangled, *i.e.* the corresponding density operator is initially a direct product

$$\hat{\rho}(t_0) = \hat{\rho}_q(t_0)\hat{\rho}_Q(t_0) \ . \tag{4.2}$$

Furthermore, assume that the state is not squeezed,

$$\big\langle \{\hat{q}(t_0), \hat{p}(t_0)\} \big\rangle = 0 = \big\langle \{\hat{Q}(t_0), \hat{P}(t_0)\} \big\rangle \ , \tag{4.3}$$

and that the initial expectation values of position and momentum vanish, $\langle \hat{q}(t_0) \rangle = 0 = \langle \hat{Q}(t_0) \rangle$, $\langle \hat{p}(t_0) \rangle = 0 = \langle \hat{P}(t_0) \rangle$. Then, the density operator for this state has the following simple form,

$$\begin{aligned}
\hat{\rho}_q(t_0) &= \frac{1}{Z_q(t_0)} \exp\left[-\frac{1}{2\hbar}\Big(\alpha(t_0)\,\hat{q}^2 + \beta(t_0)\,\hat{p}^2\Big)\right] \ , \\
\hat{\rho}_Q(t_0) &= \frac{1}{Z_Q(t_0)} \exp\left[-\frac{1}{2\hbar}\Big(A(t_0)\hat{q}^2 + B(t_0)\hat{P}^2\Big)\right] \ .
\end{aligned} \tag{4.4}$$

Assume futher that the $(\hat{q}, \hat{p})$ oscillator is in a pure state, such that

$$\left(\frac{\hbar}{2}\Delta_q(t_0)\right)^2 = \langle \hat{q}^2(t_0) \rangle \langle \hat{p}^2(t_0) \rangle = \left(\frac{\hbar}{2}\right)^2 \tag{4.5}$$

and that

$$\left(\frac{\hbar}{2}\Delta_Q(t_0)\right)^2 = \langle \hat{Q}^2(t_0) \rangle \langle \hat{P}^2(t_0) \rangle = \hbar^2 \left(\overline{n}_Q(t_0) + \frac{1}{2}\right)^2 \ , \tag{4.6}$$

where $\overline{n}_Q(t_0) = 1/[\mathrm{e}^{\hbar\sigma_Q(t_0)} - 1]$, with $\bar{n}_Q(t_0) > 0$, $\sigma_Q(t_0) < \infty$, implying that the $(\hat{Q}, \hat{P})$ oscillator is initially in a thermal-like state with an equivalent temperature $T_Q(t_0)$ given by, $\sigma_Q(t_0) = \overline{\Omega}/(k_B T_Q(t_0))$.

(a) *(2 points)* Calculate the initial entropy of the state. What is the time dependence of the Gaussian von Neumann entropy of the whole system of the two interacting harmonic oscillators as given by (4.1)?

(b) *(2 points)* Consider for simplicity the case when $\lambda_0 = \lambda$ and $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

Derive the Heisenberg equations of motion for $\hat{q}$, $\hat{p}$, $\hat{Q}$, $\hat{P}$. These first order equations couple all 4 operators. Combine them into two second order equations that couple only $\hat{q}$ and $\hat{Q}$. Diagonalize (decouple) these two second order equations by rotating the position basis,

$$\hat{q} = A\hat{q}_1 + B\hat{q}_2 \ , \qquad \hat{Q} = C\hat{q}_1 + D\hat{q}_2 \ , \qquad AD - BC = 1 \tag{4.7}$$

i.e. put the equations in the form

$$\left[\frac{d^2}{dt^2} + \omega_1^2\right]\hat{q}_1 = 0 \ , \qquad \left[\frac{d^2}{dt^2} + \omega_2^2\right]\hat{q}_2 = 0 \ , \tag{4.8}$$

and calculate ω_1 and ω_2. Solve these two decoupled equations for $\hat{q}_1$ and $\hat{q}_2$ and impose properly initial conditons! From here determine what is the time evolution of operators $\hat{q}(t)$ and $\hat{Q}(t)$. Use the remaining two Heisenberg equations to find the time dependence of $\hat{p}(t)$ and $\hat{P}(t)$ operators.

(c) *(2 points)* Now that you have found the time evolution of operators in the Heisenberg picture, calculate the time evolution of entropy $S_q(t)$ associated with the $(\hat{q}, \hat{p})$ oscillator. By calculating $\langle\hat{q}^2(t)\rangle$, $\langle\hat{p}^2(t)\rangle$, and $\langle\{\hat{q}(t), \hat{p}(t)\}\rangle$ one can determine the function $\Delta_q(t)$ in terms of which the entropy associated to the given oscillator can be expressed. Can you explain the origin of the time dependence in this entropy?

■ PROBLEM 5 Two-point functions of a driven harmonic oscillator. *(7 points)*

Consider a driven harmonic oscillator whose Hamiltonian is,

$$\hat{H}(t) = \frac{\hat{P}^2}{2m} + \frac{m\omega^2}{2}\hat{Q}^2 + f(t)\hat{Q} + g(t)\hat{P} \ , \tag{5.1}$$

where $f(t)$ and $g(t)$ the time dependent driving 'forces' and m and ω are the mass and frequency of the oscillator.

(a) *(1 point)* Write down the Heisenberg equations for $\hat{Q}$ and $\hat{P}$. Redefine position and momentum operators in the following way

$$\hat{Q}(t) = \hat{q}(t) + \overline{Q}(t) \ , \qquad \hat{P}(t) = \hat{p}(t) + \overline{P}(t) \ , \tag{5.2}$$

where the new quantities are defined as $\overline{Q}(t) = \langle\hat{Q}(t)\rangle$ and $\overline{P}(t) = \langle\hat{P}(t)\rangle$, and operators $\hat{q}$ and $\hat{p}$ satisfy canonical commutation relations, $[\hat{q}(t), \hat{p}(t)] = i\hbar$, and by definition $\langle\hat{q}(t)\rangle = 0 = \langle\hat{p}(t)\rangle$. Write down equations that are satisfied by $\overline{Q}$ and $\overline{P}$, and by $\hat{q}$ and $\hat{p}$. Do $\overline{Q}$ and $\overline{P}$ obey classical equations of motion? If yes, why yes; if no, why not?

(b) *(1 points)* Assume for the sake of simplicity that $\overline{Q}(t_0) = 0$, $\overline{P}(t_0) = 0$, $f(t_0) = 0$, $q(t_0) = 0$. Solve the equations of motion in (a) to find $\hat{Q}$ and $\hat{P}$ by writing their formal solutions (in an integral form), *i.e.* show that

$$\begin{aligned}\hat{Q}(t) &= \hat{q}_0\cos(\omega t) + \frac{\hat{p}_0}{m\omega}\sin(\omega t) + \overline{Q}(t) \\ \hat{P}(t) &= \hat{p}_0\cos(\omega t) - m\omega\hat{q}_0\sin(\omega t) + \overline{P}(t)\end{aligned} \tag{5.3}$$

where $\hat{q}_0 = \hat{q}(t_0)$, $\hat{p}_0 = \hat{p}(t_0)$ and

$$\overline{Q}(t) = \int_{t_0}^{\infty} dt'\, G^{(0)}_{\rm ret}(t;t') \left[\dot{g}(t') - \frac{f(t')}{m}\right] \,, \quad \overline{P}(t) = m\frac{d\overline{Q}}{dt} - mg(t) \,. \tag{5.4}$$

where

$$G^{(0)}_{\rm ret}(t;t') = \Theta(t-t')\frac{\sin[\omega(t-t')]}{\omega} \tag{5.5}$$

is the retarded Green function of the free oscillator (for which $f = 0 = g$).

(c) *(2 points)* Solve explicitly for $\overline{Q}$ and $\overline{P}$ in the case when $f = f_0\sin(\Omega t)$ and $g = g_0\sin(\widetilde{\Omega} t)$. Sketch how in this case these states evolve on a $(Q, P/(m\omega))$ diagram for (i) $g_0 = 0$, $f_0 > 0$ and (ii) $f_0 = 0$, $g_0 > 0$. Discuss what are the solutions in the resonant cases when (i) $\Omega = \omega, g_0 = 0$ and (ii) $\widetilde{\Omega} = \omega, f_0 = 0$.

(d) *(2 points)* Use the solutions (5.3) to construct (for some initial Gaussian state with vanishing position and momentum!) the positive and negative frequency Wightman functions ($i\Delta^{\pm}$), and the corresponding spectral (causal, Pauli-Jordan) function ($i\Delta^{\rm C} = i\Delta_{\rm PJ} = i\mathcal{A}$) and the statistical (Hadamard) two point function ($F = i\Delta_{\rm H}$). Express your answer in terms of expectation values that define the initial Gaussian state, $\langle\hat{q}_0^2\rangle$, $\langle\hat{p}_0^2\rangle$, and $\langle\{\hat{q}_0,\hat{p}_0\}\rangle$. Show that the spectral function satisfies the spectral sum rule.

(e) *(1 point)* Write down the Feynman (time-ordered) propagator ($i\Delta^{\rm F}$) and the anti-Feynman (Dyson, anti-time ordered) propagator ($i\Delta^{\overline{\rm F}}$). You may give a formal answer in terms of the Wightman functions. Show that in general,

$$i\Delta^{\rm F} + i\Delta^{\overline{\rm F}} = i\Delta^{+} + i\Delta^{-} \tag{5.6}$$

such that not all two point functions are independent.

■ PROBLEM 6 Massless scalar field on de Sitter space. *(7 points + 2 bonus points)*

The covariant action of a real, minimally coupled scalar field Φ on a curved space-time background (described by a metric tensor $g_{\mu\nu}(x)$) is of the form,

$$S[\Phi] = \int d^4x\sqrt{-g}\,\left[-\frac{1}{2}g^{\mu\nu}(\partial_\mu\Phi)(\partial_\nu\Phi) - V(\Phi)\right] \,, \tag{6.1}$$

where $g = \det[g_{\mu\nu}]$, $g^{\mu\nu}$ is the metric inverse ($g^{\mu\nu}g_{\nu\rho} = \delta^\mu_\rho$) and V is a potential, which contains a mass term, and (self-)interactions. In the simple case of a massive scalar $V = (1/2)m^2\Phi^2$. For example, the Higgs field of the standard model is actually a two complex doublets (it contains four real scalar fields), it is symmetric under the exchange of any two components (an $O(4)$ symmetry), and it has a negative mass term (see Problem 2) and a (positive) quartic self-interaction.

The metric tensor of a flat, homogeneous, expanding cosmological space-time can be written as

$$g_{\mu\nu}(x) = a^2(\eta)\eta_{\mu\nu} \,, \qquad \eta_{\mu\nu} = {\rm diag}[-1,1,1,1] \,, \tag{6.2}$$

where a denotes a scale factor (calibrated by the size of objects) and η denotes conformal time, which is related to the proper (physical) time t of a comoving observer (this is an observer that is co-moving with the cosmological fluid that is causing the expansion) as, $ad\eta = dt$. In this problem we set $c = 1$, which means $x^0 = \eta$.

(a) *(1 point)* Show that, when written for the cosmological metric (6.2), the action (6.1) and the corresponding Hamiltonian can be written as,

$$S[\Phi] = \int dx^0 \int d^3x \left[\frac{1}{2}a^2(\Phi')^2 - \frac{1}{2}a^2(\vec{\nabla}\Phi)^2 - a^4 V(\Phi)\right] \tag{6.3}$$

$$H(\eta) = \int d^3x \left[\frac{\Pi^2}{2a^2} + \frac{1}{2}a^2(\vec{\nabla}\Phi)^2 + a^4 V(\Phi)\right] \tag{6.4}$$

where the canonically conjugate momentum field is

$$\Pi(\eta, \vec{x}\,) = \frac{\delta S}{\delta\Phi'(\eta, \vec{x})} = a^2\Phi'(\eta, \vec{x})\,, \tag{6.5}$$

where $\Phi' = \partial\Phi/\partial\eta = \partial_\eta\Phi$.

(b) *(1 point)* Promote the scalar field and its conjugate momentum field to operators and impose the canonical commutation relations,

$$[\hat{\Phi}(\eta, \vec{x}), \hat{\Pi}(\eta, \vec{x}')] = i\hbar\,\delta^3(\vec{x} - \vec{x}')\ , \quad [\hat{\Phi}(\eta, \vec{x}), \hat{\Phi}(\eta, \vec{x}')] = 0 = [\hat{\Pi}(\eta, \vec{x}), \hat{\Pi}(\eta, \vec{x}')]\ . \tag{6.6}$$

Show that the Heisenberg equations are

$$\hat{\Phi}' = \frac{\hat{\Pi}}{a^2}\ , \qquad \hat{\Pi}' = a^2\nabla^2\hat{\Phi} - a^4\left.\frac{dV}{d\Phi}\right|_{\hat{\Phi}}\ , \tag{6.7}$$

and that these can be reduced to one second order equation for $\hat{\Phi}$,

$$\hat{\Phi}'' + 2\frac{a'}{a}\hat{\Phi}' - \nabla^2\hat{\Phi} + a^2\frac{dV(\hat{\Phi})}{d\hat{\Phi}} = 0\,, \tag{6.8}$$

Show that this equation is in fact covariant, *i.e.* that it can be recast as,

$$\Box\hat{\Phi} - \left.\frac{dV}{d\Phi}\right|_{\hat{\Phi}} = 0\,, \tag{6.9}$$

where $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the d'Alembertian operator, ∇_μ is the covariant derivative operator, which when it acts on a scalar simplifies to, $\Box = (-g)^{-1/2}\partial_\mu[g^{\mu\nu}(-g)^{1/2}\partial_\nu]$.

(c) *(3 points)* We are interested in solving Eq. (6.8) (with a homogeneus state $|\Psi\rangle$ in mind). The first step is to perform a spatial Fourier transform,

$$\hat{\Phi}(\eta, \vec{x}\,) = \int \frac{d^3p}{(2\pi\hbar)^3}\, \mathrm{e}^{\imath\vec{p}\cdot\vec{x}/\hbar}\, \hat{\Phi}(\eta, \vec{p})\,, \tag{6.10}$$

where the Hermiticity of $\hat{\Phi}(\eta, \vec{x})$ implies, $\hat{\Phi}^\dagger(\eta, \vec{p}) = \hat{\Phi}(\eta, -\vec{p})$ (show this!). Show that this then implies that the following decomposition into the annihilation and creation operators $\hat{b}(\vec{p})$ and $\hat{b}^\dagger(\vec{p})$ is possible,

$$\hat{\Phi}(\eta, \vec{p}) = \varphi(\eta, p)\hat{b}(\vec{p}) + \varphi^*(\eta, p)\hat{b}^\dagger(-\vec{p})\,, \tag{6.11}$$

where $p = \|\vec{p}\,\|$, and $\hat{b}(\vec{p})$ defines the vacuum, $\hat{b}(\vec{p})|0\rangle = 0$, $\forall\vec{p}$. The commutation relations we impose on annihilation and creation operators are

$$\left[\hat{b}(\vec{p}), \hat{b}^\dagger(\vec{p}\,')\right] = (2\pi)^3\hbar\delta^3(\vec{p} - \vec{p}\,')\quad , \qquad \left[\hat{b}(\vec{p}), \hat{b}(\vec{p}\,')\right] = 0 = \left[\hat{b}^\dagger(\vec{p}), \hat{b}^\dagger(\vec{p}\,')\right]\ . \tag{6.12}$$

Creation operators $\hat{b}^\dagger(\vec{p})$ create a particle of momentum $\vec{p}$, $\hat{b}^\dagger(\vec{p})|0\rangle = |1_{\vec{p}}\rangle$, and $\varphi(\eta, p)$ and $\varphi^*(\eta, p)$ are the mode functions (why do they depend on $p = \|\vec{p}\|$ only?). In the following assume $\hbar = 1$ for simplicity.

Show that commutation relations (6.12) together with canonical commutation relations (6.6) imply a normalization of the Wronskian of the mode function,

$$\mathcal{W}[\varphi(\eta,p),\varphi^*(\eta,p)] \equiv \varphi(\eta,p)\frac{\partial}{\partial\eta}\varphi^*(\eta,p) - \varphi^*(\eta,p)\frac{\partial}{\partial\eta}\varphi(\eta,p) = \frac{i}{a^2}\,. \tag{6.13}$$

Show that, when $V = m^2\Phi^2/2$, the mode functions satisfy an ordinary differential equation,

$$\left[\frac{d^2}{d\eta^2} + 2\frac{a'}{a}\frac{d}{d\eta} + \omega_p^2(\eta)\right]\varphi(\eta,p) = \frac{1}{a}\left[\frac{d^2}{d\eta^2} + \omega_p^2(\eta) - \frac{a''}{a}\right][a\varphi(\eta,p)] = 0\,, \quad \omega_p^2 = p^2 + a^2m^2\,. \tag{6.14}$$

This equation tells us that the modes with different momenta $\vec{p}$ evolve independently, and hence, if the initial state is a product state (over the momenta $\vec{p}$), it will remain so under the evolution. We shall assume this to be the case. This means that we can solve this problem by considering it as a set of decoupled harmonic oscillators (with a time dependent frequency $\omega_p(\eta)$ and a time dependent (Hubble) dumping $2\mathcal{H} = 2a'/a > 0$ though). Because of the time dependence in (6.14), that equation cannot be solved for a general $a(\eta)$. Yet it can be solved in some simple cases, for example in the case of de Sitter space (which is exponentially expanding in physical time, $a(t) \propto e^{Ht}$), $a(\eta) = -1/(H\eta)$, where $H = a'/a^2$ is a (constant!) Hubble rate and $\eta < 0$. Show that on de Sitter the modes for a massless scalar ($m = 0$) obey the equation,

$$\left[\frac{d^2}{d\eta^2} + p^2 - \frac{2}{\eta^2}\right][a\varphi(\eta,p)] = 0\,, \tag{6.15}$$

and that the general (homogeneous) solutions to this equation can be written as,

$$\varphi(\eta,p)] = \frac{1}{a}\Big[\alpha(p)u(\eta,p) + \beta(p)u^*(\eta,p)\Big]\,, \qquad |\alpha(p)|^2 - |\beta(p)|^2 = 1\,, \tag{6.16}$$

where

$$u(\eta,p) = \frac{1}{\sqrt{2p}}\left[1 - \frac{i}{p\eta}\right]\mathrm{e}^{-ip\eta}\,, \qquad u^*(\eta,p) = \frac{1}{\sqrt{2p}}\left[1 + \frac{i}{p\eta}\right]\mathrm{e}^{ip\eta}\,, \tag{6.17}$$

are the fundamental solutions obeying the Wronskian, $W[u,u^*] = u(u^*)' - u^*u' = i$. An important choice is $\alpha(p) = 1, \beta(p) = 0$ $(\forall\vec{p})$, and it is known as the Bunch-Davies vacuum.

Hint: You can show this by inserting the field decomposition into the canonical commutation relation, and by requiring that it is satisfied.

(d) *(2 points)* Show that in the vacuum state $|\Psi\rangle \to |0\rangle$ (annihilated by $\hat{b}(\vec{p})$) the corresponding positive and negative frequency Wightman functions can be written as a mode sum (take $\hbar = 1$),

$$\imath\Delta^+(x;x') = \langle\hat{\Phi}(x)\hat{\Phi}(x')\rangle = \frac{1}{2\pi^2}\int_0^\infty dp\,p^2\frac{\sin[p\|\vec{x}-\vec{x}'\|]}{p\|\vec{x}-\vec{x}'\|}\varphi(\eta,p)\varphi^*(\eta',p) \tag{6.18}$$

$$\imath\Delta^-(x;x') = \langle\hat{\Phi}(x')\hat{\Phi}(x)\rangle = \frac{1}{2\pi^2}\int_0^\infty dp\,p^2\frac{\sin[p\|\vec{x}-\vec{x}'\|]}{p\|\vec{x}-\vec{x}'\|}\varphi(\eta',p)\varphi^*(\eta,p)\,. \tag{6.19}$$

Show further that, for the Bunch-Davies state, these integrals are logarithmically divergent in the infrared (IR), and thus the Bunch-Davies state is ill defined. What is the physical origin of this IR divergence? Do you have any idea how to get rid of it?

(e) *(2 bonus points)* Calculate the causal (Pauli-Jordan) two point function $i\Delta^{\rm C}(x;x')$ and the retarded propagator $\imath\Delta^{\rm R}(x;x') = \Theta(\eta-\eta')i\Delta^{\rm C}(x;x')$ for the Bunch-Davies state and show that it is IR finite. *Hint:* In doing the relevant integral, you can use *e.g.* `Mathematica`. But you will need to regulate your integrals in the ultraviolet, which can be done *i.e.* by adding a small imaginary contribution to $\eta-\eta'$, and then taking it to zero after the integration.