Introduction to MHD theory

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based on

PRINCIPLES OF MAGNETOHYDRODYNAMICS

by J.P. Goedbloed & S. Poedts (Cambridge University Press, 2004)

&

ADVANCED MAGNETOHYDRODYNAMICS

by J.P. Goedbloed, R. Keppens & S. Poedts (Cambridge University Press, 2009)

Contents

Other literature

Introductory plasma physics:

- F.C. Chen, *Introduction to Plasma Physics and Controlled Fusion* (1984).
- J.A. Bittencourt, *Fundamentals of Plasma Physics* (1986).

Magnetohydrodynamics:

- J.P. Freidberg, *Ideal Magnetohydrodynamics* (1987).
- J.P. Goedbloed and S. Poedts, *Principles of Magnetohydrodynamics* (2004).
  http://www.cambridge.org/uk/catalogue/catalogue.asp?isbn=0521626072
  www.rijnh.nl/users/goedbloed *(ErrataPrMHD.pdf)*
- M. Goossens, *An introduction to plasma astrophysics and magnetohydrodynamics* (2003)

Plasma astrophysics:

- E.R. Priest, *Solar Magnetohydrodynamics* (1984).
- C.J. Schrijver and C. Zwaan, *Solar and Stellar Magnetic Activity* (2000).
- A.R. Choudhuri, *The Physics of Fluids and Plasmas, intro for Astrophysicists* (1998).

Brief introduction

Three theoretical models:

- **Theory of motion of single charged particles** in given magnetic and electric fields; [book: Sec. 2.2]

- **Kinetic theory of a collection of such particles,** describing plasmas *microscopically* by means of particle distribution functions $f_{e,i}(\mathbf{r}, \mathbf{v}, t)$; [book: Sec. 2.3]

- **Fluid theory (magnetohydrodynamics),** describing plasmas in terms of averaged *macroscopic* functions of $\mathbf{r}$ and t . [book: Sec. 2.4]

Within each of these descriptions, we will give an example illustrating the plasma property relevant for our subject, viz. plasma confinement by magnetic fields.

Plasma

- Most common (90%) state of matter in the universe.
- On earth exceptional, but obtained in laboratory thermonuclear fusion experiments at high temperatures ($T \sim 10^8\,\mathrm{K}$).
- Crude definition: *Plasma is a completely ionised gas, consisting of freely moving positively charged nuclei and negatively charged electrons.*

Applications

- Magnetic plasma confinement for (future) energy production by Controlled Thermonuclear Reactions.
- Dynamics of astrophysical plasmas (solar corona, planetary magnetospheres, pulsars, accretion disks, jets, etc.).
- Common ground: *Plasma interacting with a magnetic field.*

The Standard View of Nature

Nuclear forces

$\Downarrow$

quarks / leptons

nuclei (+) / electrons (−) 10^{-15} m

Electrostatic forces

$\Downarrow$

atoms / molecules 10^{-9} m

(ordinary matter: electrically neutral)

⋮

Gravity

$\Downarrow$

stars / solar system $10^{9}/10^{13}$ m

galaxies / clusters $10^{20}/10^{23}$ m

universe 10^{26} m

However, ...

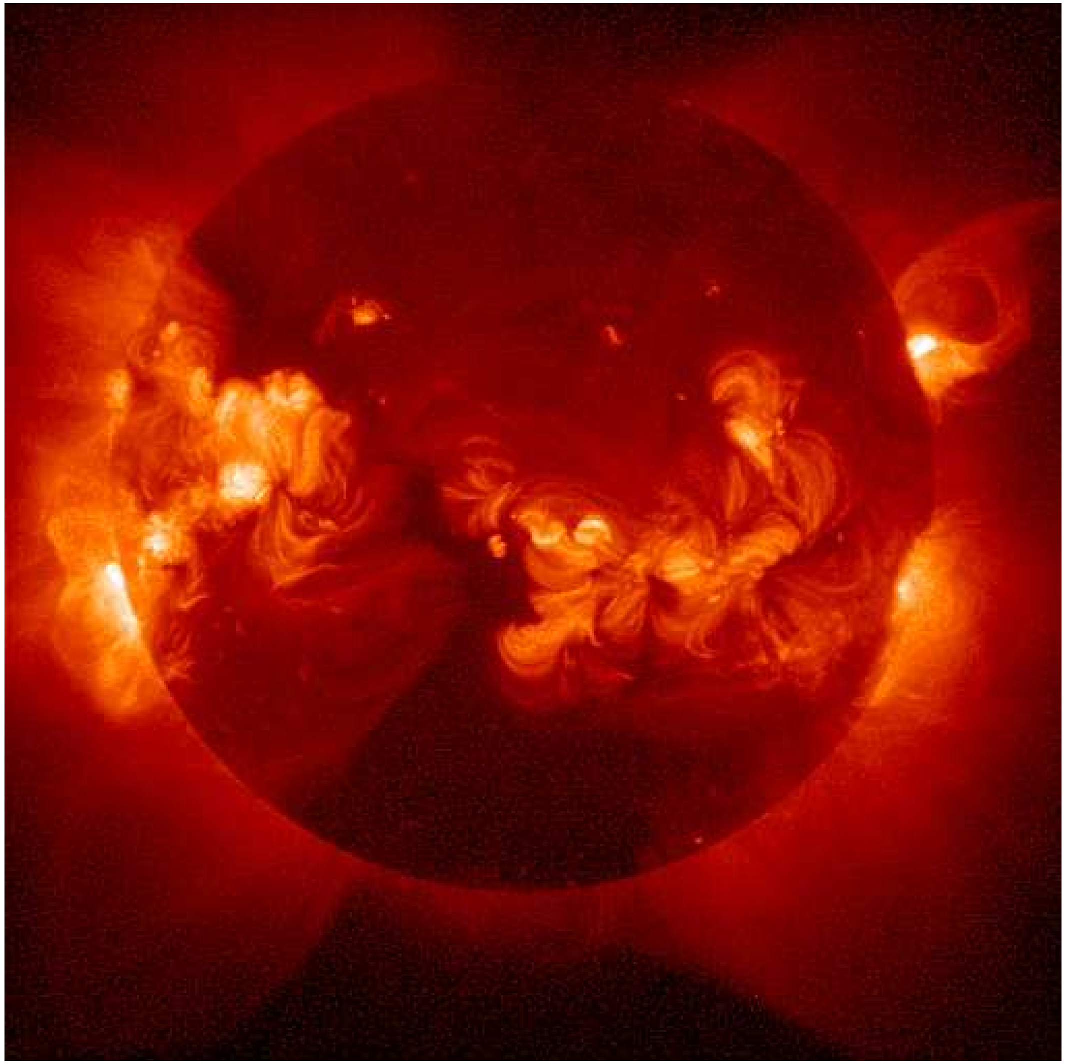
The universe does not consist of ordinary matter

- $> 90\%$ *is plasma:*
  electrically neutral, where the nuclei and electrons are not tied in atoms but
  freely move as fluids.
- The large scale result is *Magnetic fields*
  (example: interaction solar wind – magnetosphere).

Geometry

- Spherical symmetry of atomic physics and gravity (central forces) not present on the plasma scale:
  $\nabla \cdot \mathbf{B} = 0$ *is not compatible with spherical symmetry*
  (example: solar flares).

Example: The Sun



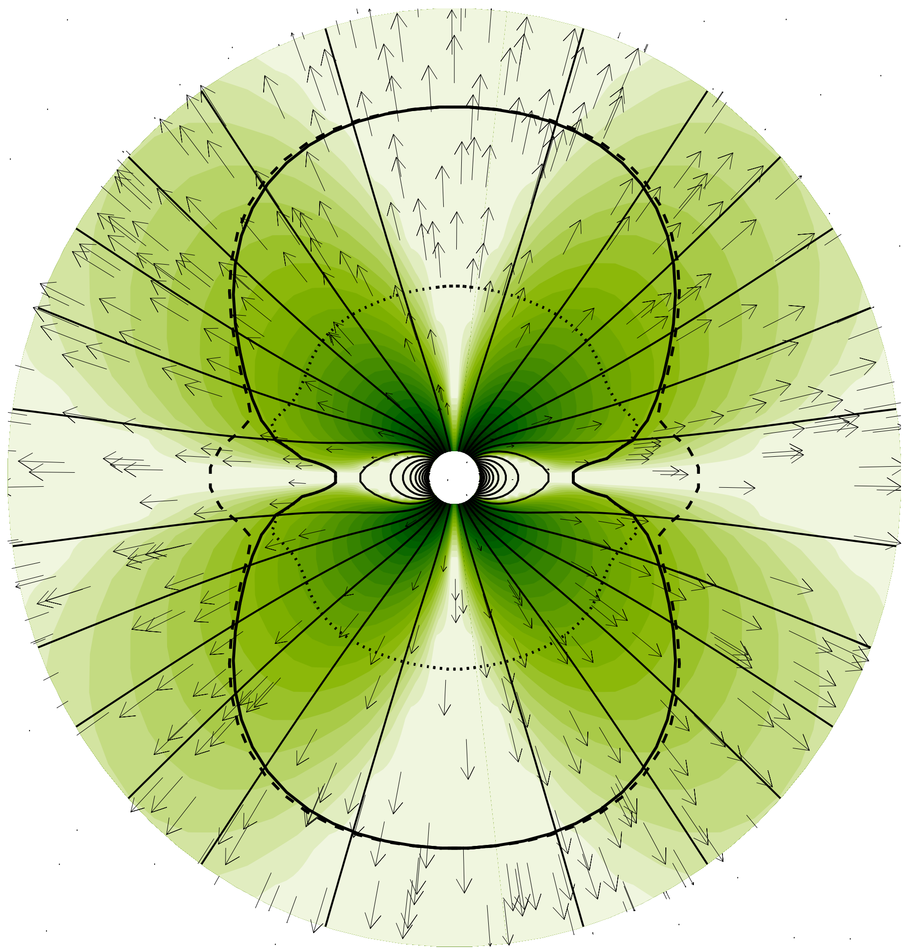
a magnetized plasma!

(sunatallwavelengths.mpeg)

Example: Coronal loops (cont'd)

[from recent observations with TRACE spacecraft]

Example: Stellar wind outflow (simulation)



- Axisymmetric magnetized wind with a 'wind' and a 'dead' zone

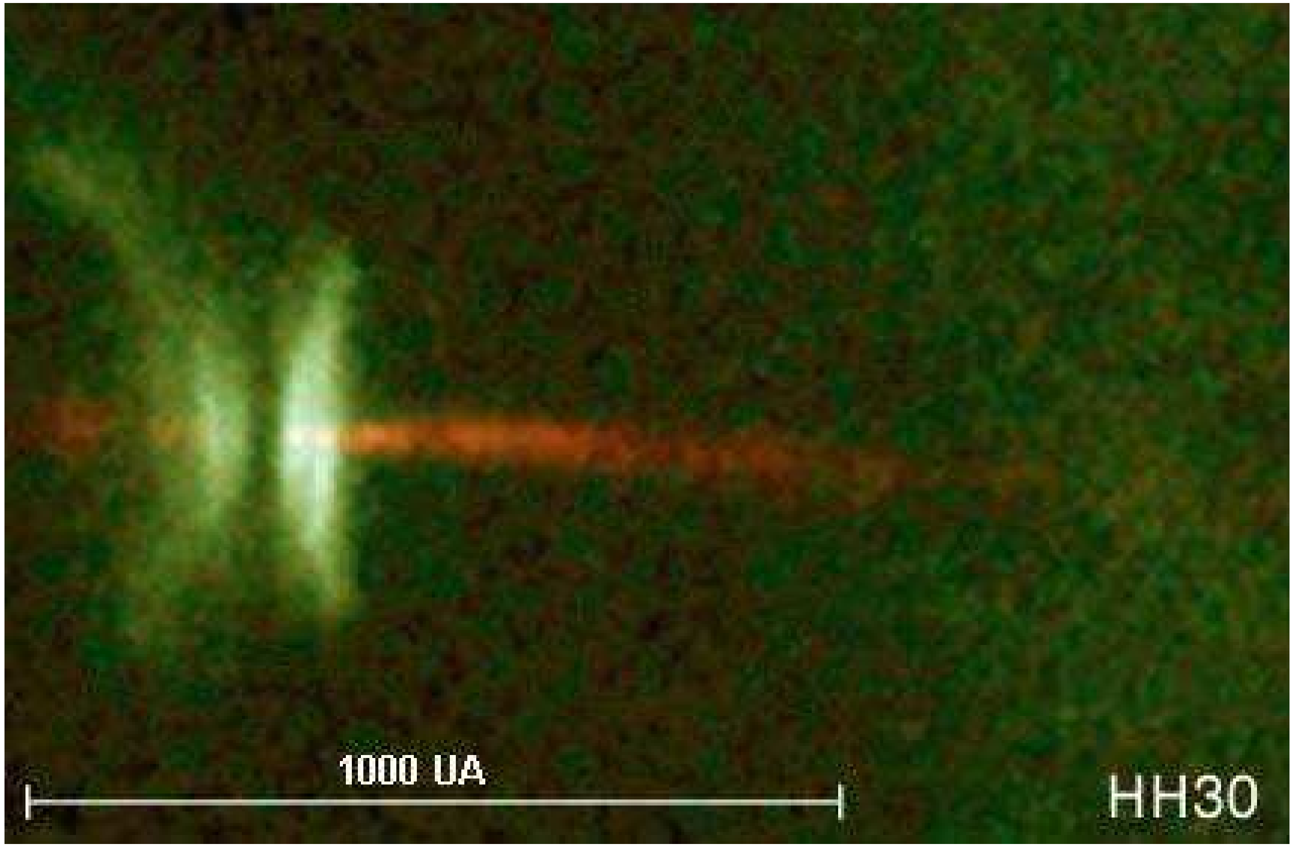
  [Keppens & Goedbloed, Ap. J. **530**, 1036 (2000)]

Example: Polar lights

Beauty of the polar lights (a1smallweb.mov)

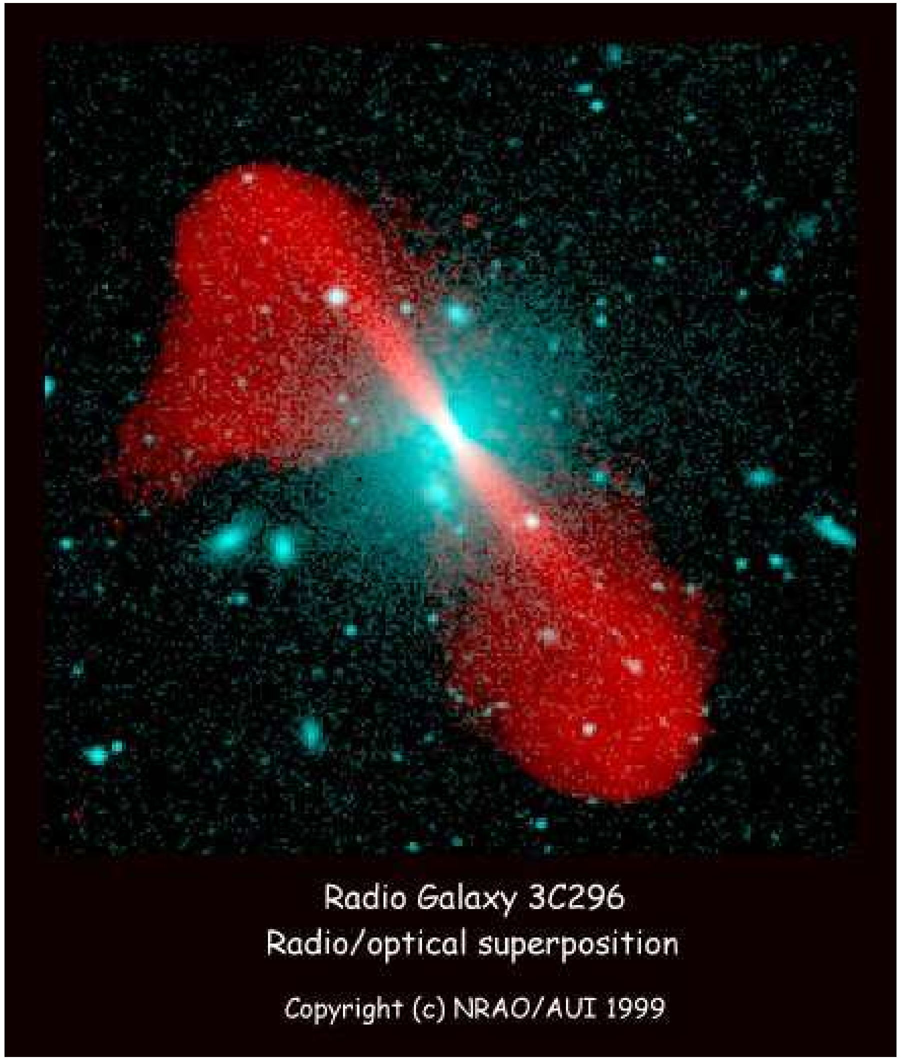
Solar wind powering auroral displays (fuvmovie.mpeg)

Example: Accretion disk and jets (YSO)



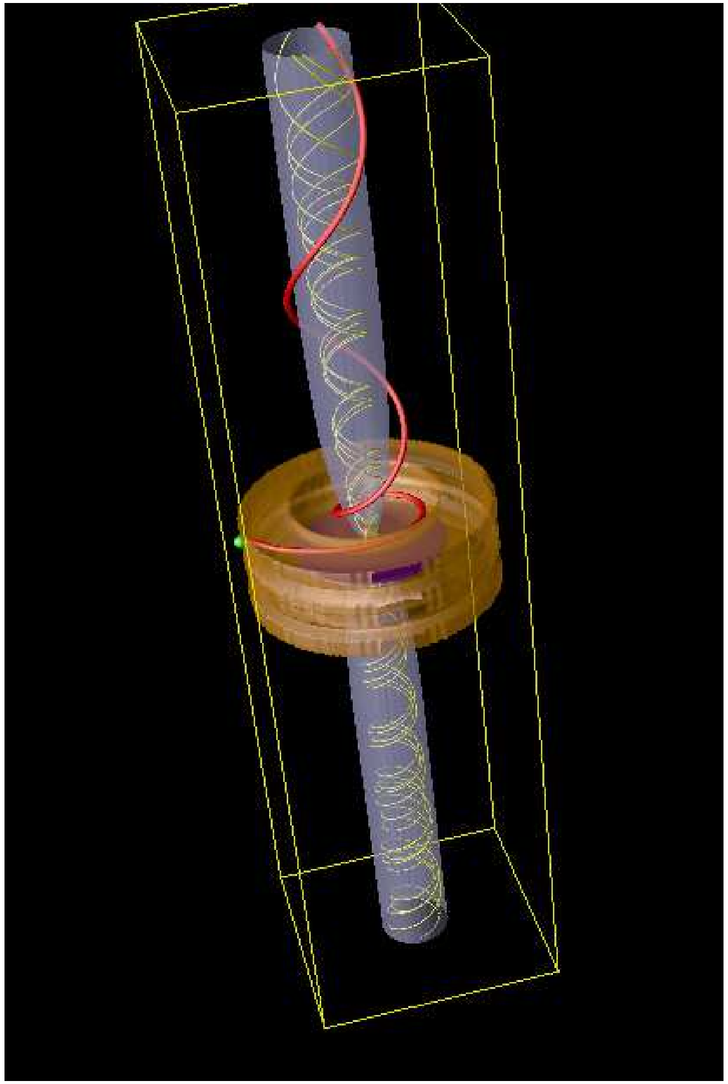
Young stellar object ($M_* \sim 1 M_\odot$): accretion disk 'seen' edge-on as dark strip, jets colored red.

Example: Accretion disk and jets (AGN)



Active galactic nucleus ($M_* \sim 10^8 M_\odot$): optical emission (blue) centered on disk, radio emission (red) shows the jets.

Example: Accretion disk and jets (simulation)



Stationary end state from the simulation of a Magnetized Accretion Ejection Structure: disk density surfaces (brown), jet magnetic surface (grey), helical field lines (yellow), accretion-ejection particle trajectory (red).
[Casse & Keppens, Ap. J. **601**, 90 (2004)]

Crude definition:

Plasma is an ionized gas.

Rate of ionization: $\frac{n_i}{n_n} = \left(\frac{2\pi m_e k}{h^2}\right)^{3/2} \frac{T^{3/2}}{n_i} e^{-U_i/kT}$ (Saha equation)

– air: $T = 300\,\mathrm{K}$, $n_n = 3 \times 10^{25}\,\mathrm{m}^{-3}$, $U_i = 14.5\,\mathrm{eV}$ $\Rightarrow$ $n_i/n_n \approx 2 \times 10^{-122}$ (!)

– H in tokamak: $T = 10^8\,\mathrm{K}$, $n_i = 10^{20}\,\mathrm{m}^{-3}$, $U_i = 13.6\,\mathrm{eV}$ $\Rightarrow$ $n_i/n_n \approx 2.4 \times 10^{13}$

– solar corona: $T = 10^6\,\mathrm{K}$, $n_i = 10^{12}\,\mathrm{m}^{-3}$, $U_i = 13.6\,\mathrm{eV}$ $\Rightarrow$ $n_i/n_n \approx 2.4 \times 10^{18}$

Microscopic definition:

Plasma is a quasi-neutral gas of charged and neutral particles which exhibits collective behaviour (cf. Chen).

(a) Long-range collective interactions dominate over binary collisions with neutrals

(b) Length scales large enough that quasi-neutrality ($n_e \approx Z n_i$) holds

(c) Sufficiently many particles in a Debye sphere (statistics)

Collective behavior

Conditions:

(a) $\tau \ll \tau_n \equiv \dfrac{1}{n_n \sigma v_{\rm th}}$

tokamak: $\tau \ll 2.4 \times 10^6\,\mathrm{s}$

corona: $\tau \ll 2 \times 10^{20}\,\mathrm{s}\,;$

(b) $\lambda \gg \lambda_D \equiv \sqrt{\dfrac{\epsilon_0 kT}{e^2 n}}$

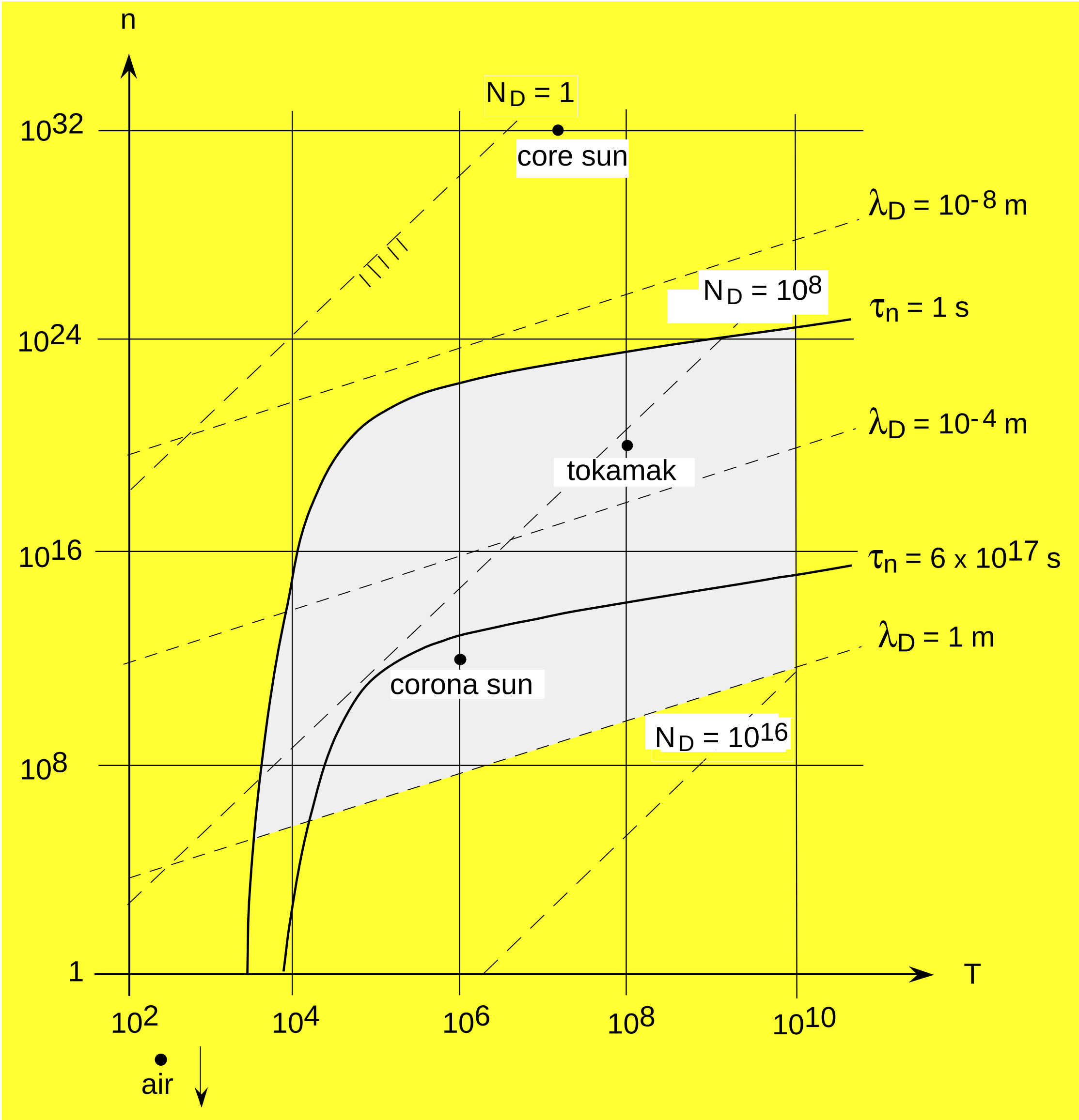
tokamak: $\lambda_D = 7{\times}10^{-5}\,\mathrm{m}$

corona: $\lambda_D = 0.07\,\mathrm{m}\,;$

(c) $N_D \equiv \frac{4}{3}\pi\lambda_D^3\, n \gg 1$

tokamak: $N_D = 1.4 \times 10^8$

corona: $N_D = 1.4 \times 10^9\,.$



So far, only the electric field appeared. (LOCAL)

Macroscopic definition:

For a valid macroscopic model of magnetized plasma dynamical configurations, size, duration, density, and magnetic field strength should be large enough to establish fluid behavior and to average out the microscopic phenomena (i.e. collective plasma oscillations and cyclotron motions of electrons and ions).

Now, the magnetic field enters: (GLOBAL !)

(a) $\tau \gg \Omega_i^{-1} \sim B^{-1}$ (time scale longer than inverse cyclotron frequency);

(b) $\lambda \gg R_i \sim B^{-1}$ (length scale larger than cyclotron radius).

$\Rightarrow$ MHD $\equiv$ magnetohydrodynamics

Elements of plasma physics

Three theoretical models:

- **Theory of motion of single charged particles** in given magnetic and electric fields; [book: Sec. 2.2]
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- **Fluid theory (magnetohydrodynamics),** describing plasmas in terms of averaged *macroscopic* functions of $\mathbf{r}$ and t . [book: Sec. 2.4]

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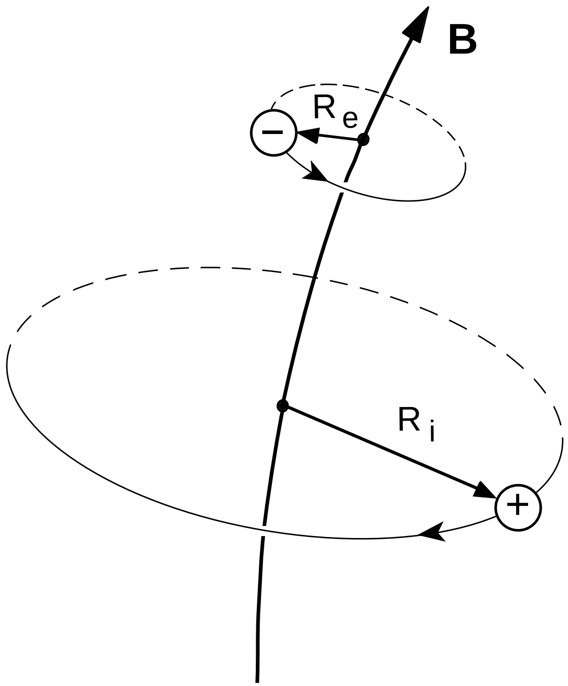
Cyclotron motion



- Equation of motion of charged particle in given $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$:

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})\,. \qquad (1)$$

- Apply to constant $\mathbf{B} = B\mathbf{e}_z$ and $\mathbf{E} = 0$:

$$\begin{aligned} \ddot{x} - (qB/m)\,\dot{y} &= 0\,, \\ \ddot{y} + (qB/m)\,\dot{x} &= 0\,. \end{aligned} \qquad (2)$$

$\Rightarrow$ periodic motion about a fixed point $x = x_c$, $y = y_c$ *(the guiding centre)*.

This yields periodic motion in a magnetic field, with *gyro- (cyclotron) frequency*

$$\Omega \equiv \frac{|q|B}{m} \qquad (3)$$

and *cyclotron (gyro-)radius*

$$R \equiv \frac{v_\perp}{\Omega} \approx \frac{\sqrt{2mkT}}{|q|B}\,. \qquad (4)$$

$\Rightarrow$ Effectively, charged particles stick to the field lines.

Cyclotron motion (cont'd)

Orders of magnitude

- Typical gyro-frequencies, e.g. for tokamak plasma ($B = 3\,\mathrm{T}$):

$$\Omega_e = 5.3 \times 10^{11}\,\mathrm{rad\,s^{-1}} \quad (\text{frequency of } 84\,\mathrm{GHz}),$$

$$\Omega_i = 2.9 \times 10^{8}\,\mathrm{rad\,s^{-1}} \quad (\text{frequency of } 46\,\mathrm{MHz}).$$

- Gyro-radii, with $v_\perp = v_{\mathrm{th}} \equiv \sqrt{2kT/m}$ for $T_e = T_i = 1.16 \times 10^8\,\mathrm{K}$:

$$v_{\mathrm{th},e} = 5.9 \times 10^{7}\,\mathrm{m\,s^{-1}} \quad \Rightarrow \quad R_e = 1.1 \times 10^{-4}\,\mathrm{m} \approx 0.1\,\mathrm{mm},$$

$$v_{\mathrm{th},i} = 1.4 \times 10^{6}\,\mathrm{m\,s^{-1}} \quad \Rightarrow \quad R_i = 4.9 \times 10^{-3}\,\mathrm{m} \approx 5\,\mathrm{mm}.$$

$\Rightarrow$ Tokamak time scales $(\sim 1\,\mathrm{s})$ and dimensions $(\sim 1\,\mathrm{m})$ *justify averaging.*

Since the gyro-frequencies essentially depend on B alone

$\Rightarrow$ excellent diagnostic to *determine the magnetic field strength!*

Drifts

- Single particle motion in constant $\mathbf{E}\ (= E\mathbf{e}_y) \perp$ constant $\mathbf{B}\ (= B\mathbf{e}_z)$.
- Transverse equations of motion:

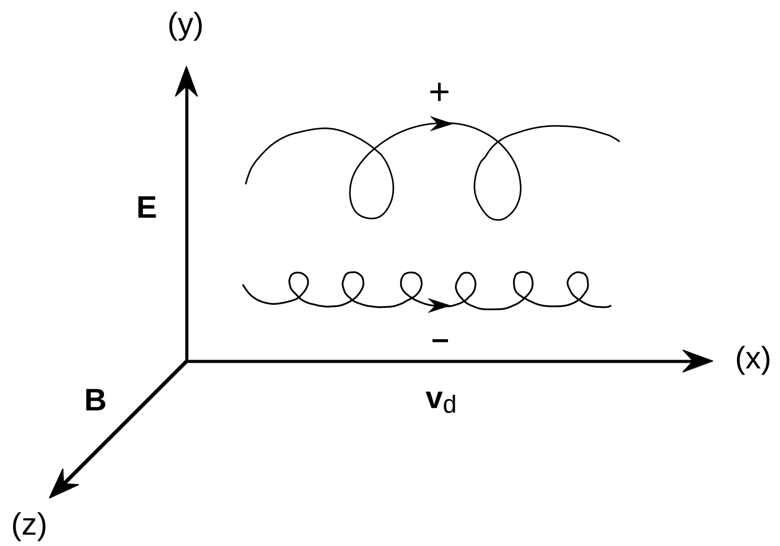
$$\begin{aligned} \ddot{x} - \frac{qB}{m}\dot{y} &= 0\,, \\ \ddot{y} + \frac{qB}{m}(\dot{x} - E/B) &= 0\,, \end{aligned} \tag{5}$$

replacing $\dot{x} \rightarrow \dot{x} - E/B \ \Rightarrow$ gyration superposed with constant drift in x-direction.

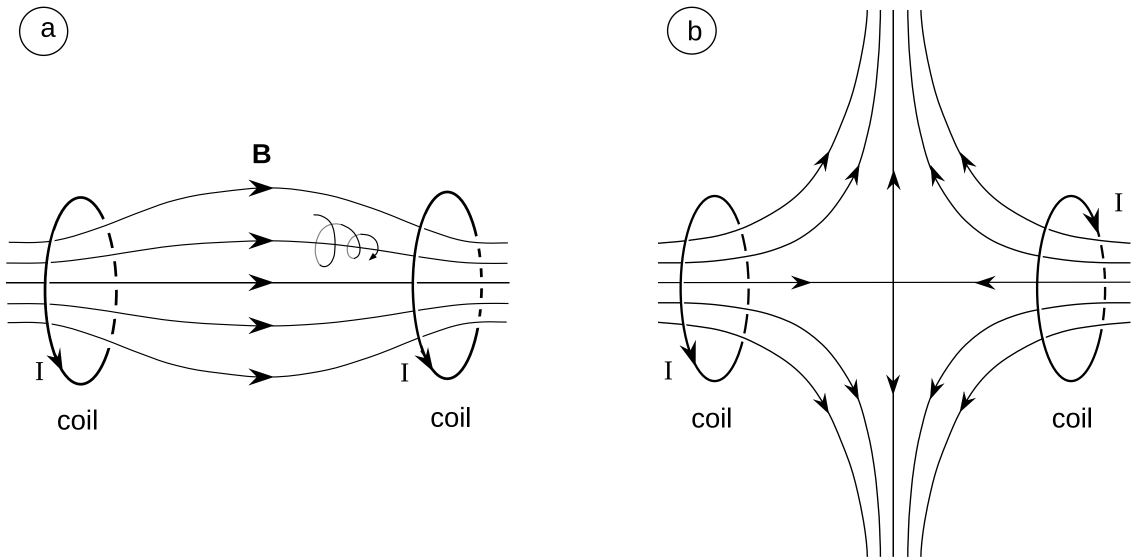
- Hence, $\perp$ electric field gives $\mathbf{E} \times \mathbf{B}$ drift :

$$\mathbf{v}_\mathrm{d} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}\,, \tag{6}$$

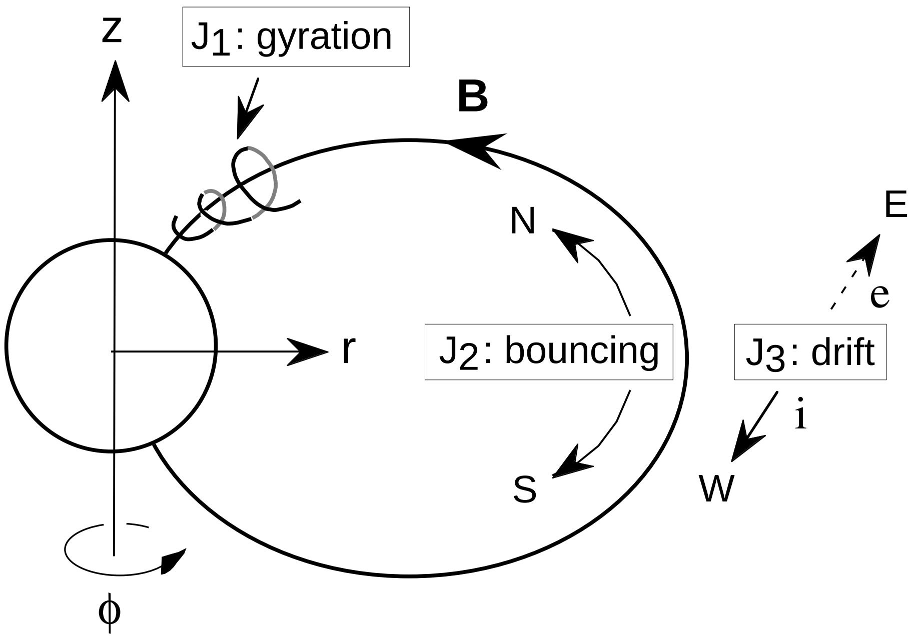
independent of the charge, so that electrons and ions drift in same direction!



Mirror effect



- Particles entering region of higher $|B|$ are reflected back into region of smaller $|B|$ where gyro-radius is larger and $v_\perp$ smaller $\Rightarrow$ (a) mirror, (b) cusp.



Example: Charged particles trapped in the magnetosphere (Van Allen belts).

Distribution functions

- A plasma consists of a very large number of interacting charged particles $\Rightarrow$ *kinetic plasma theory* derives the equations describing the *collective behavior* of the many charged particles by applying the methods of statistical mechanics.

- The physical information of a plasma consisting of electrons and ions is expressed in terms of *distribution functions* $f_\alpha(\mathbf{r}, \mathbf{v}, t)$, where $\alpha = \mathrm{e}, \mathrm{i}$. They represent the density of particles of type α in the *phase space* of position and velocity coordinates. The probable number of particles α in the 6D volume element centered at $(\mathbf{r}, \mathbf{v})$ is given by $f_\alpha(\mathbf{r}, \mathbf{v}, t)\, d^3r\, d^3v$. The motion of the swarm of phase space points is described by the total time derivative of f_α:

$$\begin{aligned} \frac{\mathrm{d}f_\alpha}{\mathrm{d}t} &\equiv \frac{\partial f_\alpha}{\partial t} + \frac{\partial f_\alpha}{\partial \mathbf{r}} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} + \frac{\partial f_\alpha}{\partial \mathbf{v}} \cdot \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \\ &= \frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \frac{\partial f_\alpha}{\partial \mathbf{r}} + \frac{q_\alpha}{m_\alpha}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} \,. \end{aligned} \tag{7}$$

Boltzmann equation

- Interactions (collisions) between the particles determine this time derivative:

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\cdot\frac{\partial f_\alpha}{\partial \mathbf{r}} + \frac{q_\alpha}{m_\alpha}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f_\alpha}{\partial \mathbf{v}} = C_\alpha \equiv \left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{coll}} . \tag{8}$$

- Here, $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ are the sum of the external fields *and* the averaged internal fields due to the long-range inter-particle interactions. C_α represents *the rate of change of the distribution function due to the short-range inter-particle collisions*. In a plasma, these are the cumulative effect of many small-angle velocity changes effectively resulting in large-angle scattering. The first task of kinetic theory is to justify this distinction between long-range interactions and binary collisions, and to derive expressions for the collision term.

- One such expression is the *Landau collision integral* (1936). Neglect of the collisions (surprisingly often justified!) leads to the *Vlasov equation* (1938).

Completing the system

- *Combine the Boltzmann equation*, determining $f_\alpha(\mathbf{r},\mathbf{v},t)$, *with Maxwell's equations*, determining $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$. In the latter, charge density $\tau(\mathbf{r},t)$ and current density $\mathbf{j}(\mathbf{r},t)$ appear as source terms. They are related to the particle densities $n_\alpha(\mathbf{r},t)$ and the average velocities $\mathbf{u}_\alpha(\mathbf{r},t)$:

$$\tau(\mathbf{r},t) \equiv \sum q_\alpha n_\alpha\,, \qquad n_\alpha(\mathbf{r},t) \equiv \int f_\alpha(\mathbf{r},\mathbf{v},t)\,d^3v\,, \tag{9}$$

$$\mathbf{j}(\mathbf{r},t) \equiv \sum q_\alpha n_\alpha \mathbf{u}_\alpha\,, \quad \mathbf{u}_\alpha(\mathbf{r},t) \equiv \frac{1}{n_\alpha(\mathbf{r},t)} \int \mathbf{v}\, f_\alpha(\mathbf{r},\mathbf{v},t)\,d^3v. \tag{10}$$

  This completes the microscopic equations.

- Solving such kinetic equations in seven dimensions (with the details of the single particle motions entering the collision integrals!) is a formidable problem
  $\Rightarrow$ look for *macroscopic reduction!*

Moment reduction

- Systematic procedure to obtain macroscopic equations, no longer involving velocity space details, is to expand in *finite number of moments of the Boltzmann equation*, by multiplying with powers of $\mathbf{v}$ and integrating over velocity space:

$$\int d^3v \cdots \,, \quad \int d^3v\, \mathbf{v} \cdots \,, \quad \int d^3v\, v^2 \cdots \Big|_{\text{truncate}} . \tag{11}$$

- E.g., the *zeroth moment* of the Boltzmann equation contains the terms:

$$\int \frac{\partial f_\alpha}{\partial t}\, d^3v = \frac{\partial n_\alpha}{\partial t} , \qquad \int \mathbf{v} \cdot \frac{\partial f_\alpha}{\partial \mathbf{r}}\, d^3v = \nabla \cdot (n_\alpha \mathbf{u}_\alpha) ,$$

$$\int \frac{q_\alpha}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}}\, d^3v = 0 , \qquad \int C_\alpha\, d^3v = 0 .$$

Adding them yields the *continuity equation* for particles of species α:

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{u}_\alpha) = 0 . \tag{12}$$

Moment reduction (cont'd)

- The *first moment* of the Boltzmann equation yields the *momentum equation:*

$$\frac{\partial}{\partial t}(n_\alpha m_\alpha \mathbf{u}_\alpha) + \nabla \cdot (n_\alpha m_\alpha \langle \mathbf{v}\mathbf{v} \rangle_\alpha) - q_\alpha n_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) = \int C_{\alpha\beta}\, m_\alpha \mathbf{v}\, d^3 v\,. \quad (13)$$

- The *scalar second moment* of Boltzmann Eq. yields the *energy equation:*

$$\frac{\partial}{\partial t}(n_\alpha \tfrac{1}{2} m_\alpha \langle v^2 \rangle_\alpha) + \nabla \cdot (n_\alpha \tfrac{1}{2} m_\alpha \langle v^2 \mathbf{v} \rangle_\alpha) - q_\alpha n_\alpha \mathbf{E} \cdot \mathbf{u}_\alpha = \int C_{\alpha\beta} \tfrac{1}{2} m_\alpha v^2\, d^3 v\,. \quad (14)$$

- This chain of moment equations can be continued indefinitely. Each moment introduces a new unknown whose temporal evolution is described by the next moment of the Boltzmann equation. The infinite chain must be truncated to be useful. *In fluid theories truncation is just after the above five moments:* continuity (scalar), momentum (vector), and energy equation (scalar).

From kinetic theory to fluid description

- **(a) Collisionality:** Lowest moments of Boltzmann equation with transport closure gives system of *two-fluid equations* in terms of the ten variables $n_{e,i}$, $\mathbf{u}_{e,i}$, $T_{e,i}$. To establish the two fluids, the electrons and ions must undergo *frequent collisions*:

$$\tau_{\rm H} \gg \tau_i \;\; [\; \gg \tau_e \,] \,. \tag{15}$$

- **(b) Macroscopic scales:** Since the two-fluid equations still involve small length and time scales ($\lambda_{\rm D}$, $R_{e,i}$, ω_{pe}^{-1}, $\Omega_{e,i}^{-1}$), the essential step towards the *MHD* description is to consider *large length and time scales*:

$$\lambda_{\rm MHD} \sim a \gg R_i \,, \qquad \tau_{\rm MHD} \sim a/v_{\rm A} \gg \Omega_i^{-1} \,. \tag{16}$$

  The larger the magnetic field strength, the more easy these conditions are satisfied. On these scales, the plasma is considered as a *single conducting fluid*.

- **(c) Ideal fluids:** Third step is to consider plasma dynamics on time scales *faster than the slow dissipation* causing the resistive decay of the magnetic field:

$$\tau_{\rm MHD} \ll \tau_{\rm R} \sim a^2/\eta \,. \tag{17}$$

  This condition is well satisfied for the small size of fusion machines, and very easily for the sizes of astrophysical plasmas $\Rightarrow$ model of *ideal MHD*.

In summary:

Kinetic theory

$\Downarrow$

frequent collisions

$\Downarrow$

Two-fluid theory

$\Downarrow$

large scales

$\Downarrow$

Diss. MHD $\Rightarrow$ *slow dissipation* $\Rightarrow$ **Ideal MHD**

Resistive MHD equations

- Define one-fluid variables that are linear combinations of the two-fluid variables:

$$\rho \equiv n_e m_e + n_i m_i \,, \quad \textit{(total mass density)} \tag{18}$$

$$\tau \equiv -e\,(n_e - Z n_i)\,, \quad \textit{(charge density)} \tag{19}$$

$$\mathbf{v} \equiv (n_e m_e \mathbf{u}_e + n_i m_i \mathbf{u}_i)/\rho\,, \quad \textit{(center of mass velocity)} \tag{20}$$

$$\mathbf{j} \equiv -e\,(n_e \mathbf{u}_e - Z n_i \mathbf{u}_i)\,, \quad \textit{(current density)} \tag{21}$$

$$p \equiv p_e + p_i \,. \quad \textit{(pressure)} \tag{22}$$

- Operate on pairs of the two-fluid equations
- Evolution expressions for τ and $\mathbf{j}$ disappear by exploiting:

$$|n_e - Z n_i| \ll n_e \,, \quad \textit{(quasi charge-neutrality)} \tag{23}$$

$$|\mathbf{u}_i - \mathbf{u}_e| \ll v \,, \quad \textit{(small relative velocity of ions \& electrons)} \tag{24}$$

$$v \ll c \,. \quad \textit{(non-relativistic speeds)} \tag{25}$$

Resistive MHD equations (cont'd)

Combining one-fluid moment equations thus obtained with pre-Maxwell equations (dropping displacement current and Poisson's equation) results in *resistive MHD equations:*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0\,, \qquad \textit{(continuity)} \tag{26}$$

$$\rho\,(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}) + \nabla p - \mathbf{j}\times\mathbf{B} = 0\,, \qquad \textit{(momentum)} \tag{27}$$

$$\frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p + \gamma p \nabla\cdot\mathbf{v} = (\gamma - 1)\eta|\mathbf{j}|^2\,, \qquad \textit{(internal energy)} \tag{28}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla\times\mathbf{E} = 0\,, \qquad \textit{(Faraday)} \tag{29}$$

where

$$\mathbf{j} = \mu_0^{-1}\,\nabla\times\mathbf{B}\,, \qquad \textit{(Ampère)} \tag{30}$$

$$\mathbf{E}' \equiv \mathbf{E} + \mathbf{v}\times\mathbf{B} = \eta\,\mathbf{j}\,, \qquad \textit{(Ohm)} \tag{31}$$

and

$$\nabla\cdot\mathbf{B} = 0 \qquad \textit{(no magnetic monopoles)} \tag{32}$$

is initial condition on Faraday's law.

Ideal MHD equations

- Substitution of $\mathbf{j}$ and $\mathbf{E}$ in Faraday's law yields *the induction equation*:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \mu_0^{-1} \nabla \times (\eta \nabla \times \mathbf{B}) \,, \tag{33}$$

where the resistive diffusion term is negligible when the *magnetic Reynolds number*

$$R_m \equiv \frac{\mu_0 l_0 v_0}{\eta} \gg 1 \,. \tag{34}$$

- Neglect of resistivity and substitution of $\mathbf{j}$ and $\mathbf{E}$ leads to *the ideal MHD equations:*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \,, \tag{35}$$

$$\rho \, (\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}) + \nabla p - \mu_0^{-1} (\nabla \times \mathbf{B}) \times \mathbf{B} = 0 \,, \tag{36}$$

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0 \,, \tag{37}$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0 \,, \qquad \nabla \cdot \mathbf{B} = 0 \,, \tag{38}$$

which will occupy us for most of this course.

The MHD model

Overview

- **The ideal MHD equations:** postulating the basic equations, scale independence, what is a physical model?; [book: Sec. 4.1]
- **Magnetic flux:** flux tubes, global magnetic flux conservation; [book: Sec. 4.2]
- **Conservation laws:** conservation form of the equations, global conservation laws, local conservation laws – conservation of magnetic flux; [book: Sec. 4.3]
- **Discontinuities:** shocks and jump conditions, boundary conditions for interface plasmas; [book: Sec. 4.5]
- **Model problems:** laboratory models I–III, astrophysical models IV–VI. [book: Sec. 4.6]

Postulating the basic equations

Equations of magnetohydrodynamics can be introduced by

- *averaging the kinetic equations* by moment expansion and closure through transport theory (book: Chaps. 2 and 3);
- just *posing them as postulates* for a hypothetical medium called 'plasma' and use physical arguments and mathematical criteria to justify the result (Chaps. 4, . . .).

[There is nothing suspicious about posing the basic equations. That is what is actually done with all basic equations in physics.]

In the second approach, since *the MHD equations describe the motion of a conducting fluid interacting with a magnetic field*, we need to *combine Maxwell's equations with the equations of gas dynamics* and provide *equations describing the interaction*.

- **Maxwell's equations** describe evolution of electric field $\mathbf{E}(\mathbf{r},t)$ and magnetic field $\mathbf{B}(\mathbf{r},t)$ in response to current density $\mathbf{j}(\mathbf{r},t)$ and space charge $\tau(\mathbf{r},t)$:

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}\,, \qquad \textit{(Faraday)} \qquad (1)$$

$$\nabla\times\mathbf{B} = \mu_0\mathbf{j} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\,, \qquad c \equiv (\epsilon_0\mu_0)^{-1/2}\,, \qquad \textit{(`Ampère')} \qquad (2)$$

$$\nabla\cdot\mathbf{E} = \frac{\tau}{\epsilon_0}\,, \qquad \textit{(Poisson)} \qquad (3)$$

$$\nabla\cdot\mathbf{B} = 0\,. \qquad \textit{(no monopoles)} \qquad (4)$$

- **Gas dynamics equations** describe evolution of density $\rho(\mathbf{r},t)$ and pressure $p(\mathbf{r},t)$:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\mathbf{v} \equiv \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0\,, \qquad \textit{(mass conservation)} \qquad (5)$$

$$\frac{\mathrm{D}p}{\mathrm{D}t} + \gamma p\nabla\cdot\mathbf{v} \equiv \frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p + \gamma p\nabla\cdot\mathbf{v} = 0\,, \qquad \textit{(entropy conservation)} \qquad (6)$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla$$

is the *Lagrangian time-derivative* (moving with the fluid).

- **Coupling between system described by $\{\mathrm{E},\mathrm{B}\}$ and system described by $\{\rho, p\}$** comes about through equations involving the velocity $\mathbf{v}(\mathbf{r}, t)$ of the fluid:
  'Newton's' *equation of motion for a fluid element* describes the acceleration of a fluid element by pressure gradient, gravity, and electromagnetic contributions,

$$\rho\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \mathbf{F} \equiv -\nabla p + \rho\mathbf{g} + \mathbf{j}\times\mathbf{B} + \tau\mathbf{E}\,; \qquad \textit{(momentum conservation)} \qquad (7)$$

  'Ohm's' law (for a perfectly conducting moving fluid) expresses that *the electric field $\mathbf{E}'$ in a co-moving frame vanishes*,

$$\mathbf{E}' \equiv \mathbf{E} + \mathbf{v}\times\mathbf{B} = 0\,. \qquad \textit{('Ohm')} \qquad (8)$$

- Equations (1)–(8) are complete, but inconsistent for *non-relativistic velocities*:

$$v \ll c\,. \qquad (9)$$

  $\Rightarrow$ We need to consider **pre-Maxwell equations**.

Consequences of pre-Maxwell

1. *Maxwell's displacement current negligible* [$\mathcal{O}(v^2/c^2)$] for non-relativistic velocities:

$$\frac{1}{c^2}\left|\frac{\partial \mathbf{E}}{\partial t}\right| \sim \frac{v^2}{c^2}\frac{B}{l_0} \quad \ll \quad \mu_0|\mathbf{j}| \approx |\nabla \times \mathbf{B}| \sim \frac{B}{l_0} \qquad \text{[using Eq. (8)]}\,,$$

indicating length scales by l_0 and time scales by t_0, so that $v \sim l_0/t_0$.

$\Rightarrow$ Recover original Ampère's law:

$$\mathbf{j} = \frac{1}{\mu_0}\nabla \times \mathbf{B}\,. \tag{10}$$

2. *Electrostatic acceleration is also negligible* [$\mathcal{O}(v^2/c^2)$]:

$$\tau|\mathbf{E}| \sim \frac{v^2}{c^2}\frac{B^2}{\mu_0 l_0} \quad \ll \quad |\mathbf{j} \times \mathbf{B}| \sim \frac{B^2}{\mu_0 l_0} \qquad \text{[using Eqs. (3), (8), (10)]}\,.$$

$\Rightarrow$ Space charge effects may be ignored and Poisson's law (3) can be dropped.

3. *Electric field then becomes a secondary quantity,* determined from Eq. (8):

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B}\,. \tag{11}$$

$\Rightarrow$ For non-relativistic MHD, $|\mathbf{E}| \sim |\mathbf{v}||\mathbf{B}|$, i.e. $\mathcal{O}(v/c)$ smaller than for EM waves.

Basic equations of ideal MHD

- Exploiting these approximations, and eliminating $\mathbf{E}$ and $\mathbf{j}$ through Eqs. (10) and (11), the basic equations of ideal MHD are recovered *in their most compact form*:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0\,, \tag{12}$$

$$\rho\Big(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\Big) + \nabla p - \rho \mathbf{g} - \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} = 0\,, \tag{13}$$

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0\,, \tag{14}$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0\,, \qquad \nabla \cdot \mathbf{B} = 0\,. \tag{15}$$

$\Rightarrow$ *Set of eight nonlinear partial differential equations (PDEs) for the eight variables $\rho(\mathbf{r},t)$, $\mathbf{v}(\mathbf{r},t)$, $p(\mathbf{r},t)$, and $\mathbf{B}(\mathbf{r},t)$*.

- The magnetic field equation (15)(b) is to be considered as a initial condition: once satisfied, it remains satisfied for all later times by virtue of Eq. (15)(a).

Thermodynamic variables

- Alternative thermodynamical variables (replacing ρ and p):
  e – internal energy per unit mass ($\sim$ temperature T) and s – entropy per unit mass.
  Defined by the ideal gas relations, with $p = (n_e + n_i)kT$:

$$e \equiv \frac{1}{\gamma - 1}\frac{p}{\rho} \approx C_v T \,, \qquad C_v \approx \frac{(1+Z)k}{(\gamma-1)m_i} \,,$$
$$s \equiv C_v \ln S + \text{const} \,, \qquad S \equiv p\rho^{-\gamma} \,. \tag{16}$$

- From Eqs. (12) and (14), we obtain an evolution equation for the internal energy,

$$\frac{\mathrm{D}e}{\mathrm{D}t} + (\gamma - 1)\, e \nabla \cdot \mathbf{v} = 0 \,, \tag{17}$$

  and an equation expressing that *the entropy convected by the fluid is constant* (i.e. adiabatic processes: thermal conduction and heat flow are negligible),

$$\frac{\mathrm{D}s}{\mathrm{D}t} = 0 \,, \quad \text{or} \quad \frac{\mathrm{D}S}{\mathrm{D}t} \equiv \frac{\mathrm{D}}{\mathrm{D}t}(p\rho^{-\gamma}) = 0 \,. \tag{18}$$

  This demonstrates that Eq. (14) actually expresses entropy conservation.

Gravity

- In many astrophysical systems, the external gravitational field of a compact object (represented by point mass M_* situated at position $\mathbf{r} = \mathbf{r}_*$ far outside the plasma) is more important than the internal gravitational field. The Poisson equation

$$\nabla^2 \Phi_{\rm gr} = 4\pi G[M_* \delta(\mathbf{r} - \mathbf{r}_*) + \rho(\mathbf{r})] \tag{19}$$

then has a solution with negligible internal gravitational acceleration (2nd term):

$$\mathbf{g}(\mathbf{r}) = -\nabla \Phi_{\rm gr}(\mathbf{r}) = -GM_* \frac{\mathbf{r} - \mathbf{r}_*}{|\mathbf{r} - \mathbf{r}_*|^3} - G \int \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \, d^3 r' \,. \tag{20}$$

- Estimate gravitational forces $\mathbf{F}_g^{\rm ex/in} \equiv \rho \mathbf{g}^{\rm ex/in}$ compared to Lorentz force $\mathbf{F}_B \equiv \mathbf{j} \times \mathbf{B}$:

  1) *Tokamak* (with radius a of the plasma tube and M_*, R_* referring to the Earth):

$$\begin{aligned}
|\mathbf{F}_B| \equiv |\mathbf{j} \times \mathbf{B}| &\sim \frac{B^2}{\mu_0 a} &= 7.2 \times 10^{6} \quad \mathrm{kg\, m^{-2}\, s^{-2}} \,, \\
|\mathbf{F}_g^{\rm ex}| \equiv |\rho \mathbf{g}^{\rm ex}| &\sim \rho G \frac{M_*}{R_*^2} &= 1.7 \times 10^{-6} \quad \mathrm{kg\, m^{-2}\, s^{-2}} \,, \\
|\mathbf{F}_g^{\rm in}| \equiv |\rho \mathbf{g}^{\rm in}| &\sim \rho^2 G a &= 1.9 \times 10^{-24} \, \mathrm{kg\, m^{-2}\, s^{-2}} \,.
\end{aligned} \tag{21}$$

Scale independence

- The MHD equations (12)–(15) can be made dimensionless by means of a choice for *the units of length, mass, and time*, based on typical magnitudes l_0 for length scale, ρ_0 for plasma density, and B_0 for magnetic field at some representative position. The unit of time then follows by exploiting *the Alfvén speed*:

$$v_0 \equiv v_{A,0} \equiv \frac{B_0}{\sqrt{\mu_0 \rho_0}} \quad \Rightarrow \quad t_0 \equiv \frac{l_0}{v_0} \,. \tag{22}$$

- By means of this basic triplet l_0, B_0, t_0 (and derived quantities ρ_0 and v_0), we create dimensionless independent variables and associated differential operators:

$$\bar{l} \equiv l/l_0 \,, \quad \bar{t} \equiv t/t_0 \qquad \Rightarrow \qquad \bar{\nabla} \equiv l_0 \nabla \,, \quad \partial/\partial\bar{t} \equiv t_0\, \partial/\partial t \,, \tag{23}$$

and dimensionless dependent variables:

$$\bar{\rho} \equiv \rho/\rho_0 \,, \quad \bar{\mathbf{v}} \equiv \mathbf{v}/v_0 \,, \quad \bar{p} \equiv p/(\rho_0 v_0^2) \,, \quad \bar{\mathbf{B}} \equiv \mathbf{B}/B_0 \,, \quad \bar{\mathbf{g}} \equiv (l_0/v_0^2)\, \mathbf{g} \,. \tag{24}$$

- Barred equations are now identical to unbarred ones (except that μ_0 is eliminated).

  $\Rightarrow$ *Ideal MHD equations independent of size of the plasma (l_0), magnitude of the magnetic field (B_0), and density (ρ_0), i.e. time scale (t_0).*

Scales of actual plasmas

	l_0 (m)	B_0 (T)	t_0 (s)
tokamak	20	3	3×10^{-6}
magnetosphere Earth	4×10^{7}	3×10^{-5}	6
solar coronal loop	10^{8}	3×10^{-2}	15
magnetosphere neutron star	10^{6}	10^{8} *	10^{-2}
accretion disc YSO	1.5×10^{9}	10^{-4}	7×10^{5}
accretion disc AGN	4×10^{18}	10^{-4}	2×10^{12}
galactic plasma	10^{21} $(= 10^{5}$ ly$)$	10^{-8}	10^{15} $(= 3 \times 10^{7}$ y$)$

* Some recently discovered pulsars, called magnetars, have record magnetic fields of 10^{11} T : the plasma Universe is ever expanding!

- **Remark**: value $\bar{p}_0$ automatically becomes important: $\beta \equiv \dfrac{2\mu_0 p_0}{B_0^2} = 2\bar{p}_0$

$\Rightarrow$ often: $\beta \ll 1$ $\Rightarrow$ *pressure terms frequently neglected*

A crucial question:

Do the MHD equations (12)–(15) provide a complete model for plasma dynamics?

Answer: **NO!**

Two most essential elements of a scientific model are still missing, viz.

1. What is the *physical problem* we want to solve?
2. How does this translate into *conditions on the solutions of the PDEs?*

This brings in the space and time constraints of the *boundary conditions* and *initial data*. Initial data just amount to prescribing arbitrary functions

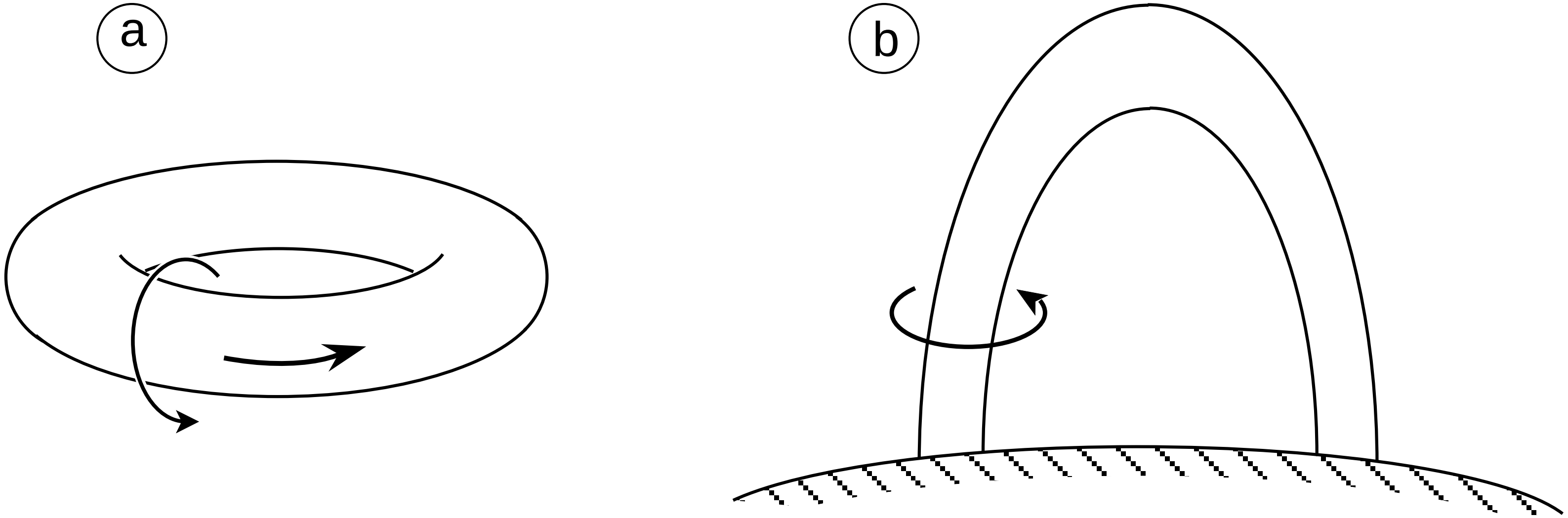
$$\rho_i(\mathbf{r})\,[\,\equiv \rho(\mathbf{r}, t\!=\!0)\,]\,, \quad \mathbf{v}_i(\mathbf{r})\,, \quad p_i(\mathbf{r})\,, \quad \mathbf{B}_i(\mathbf{r}) \quad \text{on domain of interest}\,. \tag{25}$$

Boundary conditions is a much more involved issue since it implies specification of a **magnetic confinement geometry**.

$\Rightarrow$ *magnetic flux tubes* (Sec.4.2), *conservation laws* (Sec.4.3), *discontinuities* (Sec.4.4), *formulation of model problems for laboratory and astrophysical plasmas* (Sec.4.5).

Flux tubes

- Magnetic flux tubes are the basic magnetic structures that determine which *boundary conditions* may be posed on the MHD equations.



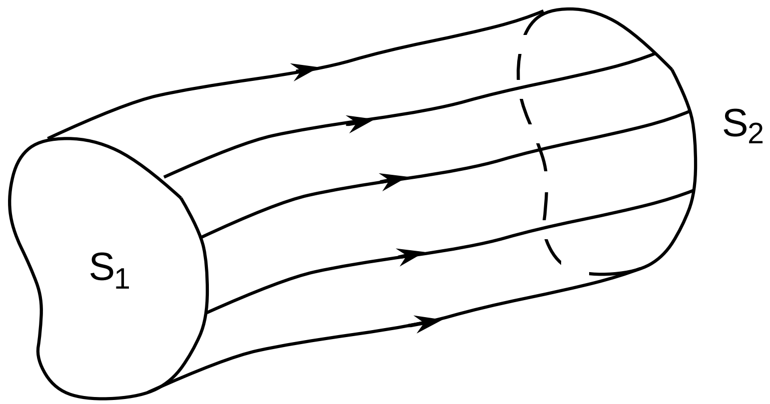
- Two different kinds of flux tubes:
  (a) closed onto itself, like in thermonuclear *tokamak* confinement machines,
  (b) connecting onto a medium of vastly different physical characteristics so that the flux tube may be considered as finite and separated from the other medium by suitable jump conditions, like in *coronal flux tubes*.

Flux tubes (cont'd)

- Magnetic fields confining plasmas are essentially *tubular structures*: The magnetic field equation

$$\nabla \cdot \mathbf{B} = 0 \tag{26}$$

is not compatible with spherical symmetry. Instead, magnetic flux tubes become the essential constituents.



- Gauss' theorem:

$$\iiint_V \nabla \cdot \mathbf{B}\, d\tau = \oint\!\!\oint \mathbf{B} \cdot \mathbf{n}\, d\sigma = -\iint_{S_1} \mathbf{B}_1 \cdot \mathbf{n}_1\, d\sigma_1 + \iint_{S_2} \mathbf{B}_2 \cdot \mathbf{n}_2\, d\sigma_2 = 0\,,$$

Magnetic flux of all field lines through surface element $d\sigma_1$ is the same as through arbitrary other element $d\sigma_2$ intersecting that field line bundle.

$$\Rightarrow \qquad \Psi \equiv \iint_S \mathbf{B} \cdot \mathbf{n}\, d\sigma \quad \text{is well defined} \tag{27}$$

(does not depend on how S is taken). Also true for smaller subdividing flux tubes!

Conservative form of the MHD equations

- general form of a (scalar) conservation law:

$$\boxed{\frac{\partial u}{\partial t} + \nabla \cdot (f(u)) = 0}$$

u: conserved quantity (actually $\int_V u \,\mathrm{d}V$, not u)

$f(u)$: rate of flow (or 'flux')

$\Rightarrow$ expresses the fact that $\int_V u \,\mathrm{d}V$ can only change due to a flux $f(u)$ through the surface of the volume V

$\Rightarrow$ used to obtain *local and global conservation laws* and *shock conditions* and also in *numerical techniques (FVM)*

- **Remark**: this is the *differential form* of the conservation law $\Rightarrow$ derived from the *integral form* (even more general):

ASSUMING u and $f(u)$ are DIFFERENTIABLE!

Derivation of differential from integral form

- e.g. scalar *integral form* in 1D:

$$\overbrace{\int_{x_1}^{x_2} \underbrace{[u(x,t_2) - u(x,t_1)]}_{\int_{t_1}^{t_2} \frac{\partial u}{\partial t}\,\mathrm{d}t} \,\mathrm{d}x}^{\text{total change of } u \text{ in } [x_1,x_2]} = \overbrace{\int_{t_1}^{t_2} \underbrace{[f(x_1,t) - f(x_2,t)]}_{-\int_{x_1}^{x_2} \frac{\partial f}{\partial x}\,\mathrm{d}x} \,\mathrm{d}t}^{\text{total flux of } u \text{ (through boundaries)}}$$

$$\Downarrow$$

(provided u and f are *differentiable*!)

$$\int_{x_1}^{x_2} \int_{t_1}^{t_2} \left[\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} \right] \mathrm{d}x\mathrm{d}t = 0$$

- must hold for all x_1, x_2, t_1, and t_2 $\quad \Rightarrow \quad \boxed{\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0}$

Conservation form of the MHD equations

- Next step: systematic approach to *local conservation properties.*
- The MHD equations can be brought in conservation form:

$$\frac{\partial}{\partial t}(\cdots) + \nabla\cdot(\cdots) = 0\,. \tag{28}$$

This yields: *conservation laws*, *jump conditions*, *and powerful numerical algorithms!*

- By intricate vector algebra, one obtains the *conservation form of the ideal MHD equations* (suppressing gravity): $\Downarrow$ From now on, putting $\mu_0 \rightarrow 1$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0\,, \tag{29}$$

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla\cdot\left[\,\rho\mathbf{v}\mathbf{v} + (p + \tfrac{1}{2}B^2)\,\mathbf{I} - \mathbf{B}\mathbf{B}\,\right] = 0\,, \qquad p = (\gamma - 1)\rho e\,, \tag{30}$$

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2 + \rho e + \tfrac{1}{2}B^2) + \nabla\cdot\left[(\tfrac{1}{2}\rho v^2 + \rho e + p + B^2)\mathbf{v} - \mathbf{v}\cdot\mathbf{B}\mathbf{B}\right] = 0\,, \tag{31}$$

$$\frac{\partial\mathbf{B}}{\partial t} + \nabla\cdot(\mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v}) = 0\,, \qquad \nabla\cdot\mathbf{B} = 0\,. \tag{32}$$

It remains to analyze the meaning of the different terms.

Conservation

- Defining

– *momentum density*: $\boldsymbol{\pi} \equiv \rho \mathbf{v}\,,$ (33)

– *stress tensor*: $\mathbf{T} \equiv \rho \mathbf{v}\mathbf{v} + (p + \frac{1}{2}B^2)\,\mathbf{I} - \mathbf{B}\mathbf{B}\,,$ (34)

– *total energy density*: $\mathcal{H} \equiv \frac{1}{2}\rho v^2 + \frac{1}{\gamma-1}\,p + \frac{1}{2}B^2\,,$ (35)

– *energy flow*: $\mathbf{U} \equiv (\frac{1}{2}\rho v^2 + \frac{\gamma}{\gamma-1}\,p)\mathbf{v} + B^2\mathbf{v} - \mathbf{v}\cdot\mathbf{B}\,\mathbf{B}\,,$ (36)

– *(no name)*: $\mathbf{Y} \equiv \mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v}\,,$ (37)

yields

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\boldsymbol{\pi} = 0 \qquad \textit{(conservation of mass),} \tag{38}$$

$$\frac{\partial \boldsymbol{\pi}}{\partial t} + \nabla\cdot\mathbf{T} = 0 \qquad \textit{(conservation of momentum),} \tag{39}$$

$$\frac{\partial \mathcal{H}}{\partial t} + \nabla\cdot\mathbf{U} = 0 \qquad \textit{(conservation of energy),} \tag{40}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla\cdot\mathbf{Y} = 0 \qquad \textit{(conservation of magnetic flux).} \tag{41}$$

Conservation laws, gravity included

- Including gravity, momentum and energy equation are:

$$\frac{\partial \boldsymbol{\pi}}{\partial t} + \nabla \cdot \mathbf{T} = -\rho \nabla \Phi \qquad \textit{(momentum),} \tag{42}$$

$$\frac{\partial \mathcal{H}}{\partial t} + \nabla \cdot \mathbf{U} = -\rho \mathbf{v} \cdot \nabla \Phi \qquad \textit{(energy).} \tag{43}$$

$$\tag{44}$$

$\Rightarrow$ work done by gravitational force

- include gravitational potential energy: $\mathcal{H}_g \equiv \mathcal{H} + \rho\Phi$ and rewrite to

$$\frac{\partial \mathcal{H}_g}{\partial t} + \nabla \cdot [\mathbf{U} + \rho \mathbf{v} \Phi] = \rho \frac{\partial \Phi}{\partial t} \qquad \textit{(energy)} \tag{45}$$

Global conservation laws

- Defining
  - *total mass*: $$M \equiv \int \rho \, d\tau \,, \tag{46}$$
  - *total momentum*: $$\boldsymbol{\Pi} \equiv \int \boldsymbol{\pi} \, d\tau \,, \tag{47}$$
  - *total energy*: $$H \equiv \int \mathcal{H} \, d\tau \,, \tag{48}$$
  - *total magnetic flux*: $$\Psi \equiv \int \mathbf{B} \cdot \tilde{\mathbf{n}} \, d\tilde{\sigma} \,, \tag{49}$$

  gives, by the application of the right BCs (see later):

$$\dot{M} = \int \dot{\rho} \, d\tau = - \int \nabla \cdot \boldsymbol{\pi} \, d\tau \overset{\text{Gauss}}{=} - \oint \boldsymbol{\pi} \cdot \mathbf{n} \, d\sigma = 0 \,, \tag{50}$$

$$\mathbf{F} = \dot{\boldsymbol{\Pi}} = \int \dot{\boldsymbol{\pi}} \, d\tau = - \int \nabla \cdot \mathbf{T} \, d\tau \overset{\text{Gauss}}{=} - \oint (p + \tfrac{1}{2} B^2) \, \mathbf{n} \, d\sigma \,, \tag{51}$$

$$\dot{H} = \int \dot{\mathcal{H}} \, d\tau = - \int \nabla \cdot \mathbf{U} \, d\tau \overset{\text{Gauss}}{=} - \oint \mathbf{U} \cdot \mathbf{n} \, d\sigma = 0 \,, \tag{52}$$

$$\dot{\Psi} = \int \dot{\mathbf{B}} \cdot \tilde{\mathbf{n}} \, d\tilde{\sigma} = \int \nabla \times (\mathbf{v} \times \mathbf{B}) \cdot \tilde{\mathbf{n}} \, d\tilde{\sigma} \overset{\text{Stokes!}}{=} \oint \mathbf{v} \times \mathbf{B} \cdot \, d\boldsymbol{l} = 0 \,. \tag{53}$$

$\Rightarrow$ *Total mass, momentum, energy, and flux conserved:* **the system is closed!**

Magnetic helicity conservation

- **magnetic helicity** $\mathcal{H} \equiv \int_V \mathbf{A} \cdot \mathbf{B}\, dV$ with $\mathbf{B} = \nabla \times \mathbf{A}$

- *Elsasser (1956) and Woltjer (1958):* helicity is conserved in ideal MHD

  when the volume of integration is bounded by a magnetic surface S*, i.e.* $\mathbf{B} \cdot \mathbf{n}|_S = 0$

- *Berger (1984):* 'relative helicity' $=$ the difference in helicities between a given field and the potential field with the same boundary conditions

$$\mathcal{H}_R \equiv \int_V \mathbf{A} \cdot \mathbf{B}\, dV - \int_V \mathbf{A}_p \cdot \mathbf{B}_p\, dV$$

$\Rightarrow$ is also conserved in ideal MHD:

$$\frac{d\mathcal{H}_R}{dt} = \underbrace{-2 \oint_S (\mathbf{A}_p \cdot \mathbf{v})\mathbf{B} \cdot \mathbf{n}\, dS}_{\textit{shear and twist}} + \underbrace{2 \oint_S (\mathbf{A}_p \cdot \mathbf{B})\mathbf{v} \cdot \mathbf{n}\, dS}_{\textit{flux emergence/disappearence}}$$

Conservative form resistive MHD equations

The resistive MHD equations for ρ, $\mathbf{v}$, e, and $\mathbf{B}$ can be transformed into:

$$
\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \rho\mathbf{v} \\ \frac{\rho v^2}{2} + \rho e + \frac{B^2}{2} \\ \mathbf{B} \end{bmatrix} + \nabla\cdot\begin{bmatrix} \rho\mathbf{v} \\ \rho\mathbf{v}\mathbf{v} + \left(p + \frac{B^2}{2}\right)\mathbf{I} - \mathbf{B}\mathbf{B} \\ \left(\frac{\rho v^2}{2} + \rho e + p\right)\mathbf{v} - (\mathbf{v}\times\mathbf{B} - \eta\mathbf{J})\times\mathbf{B} \\ \mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \eta\,\nabla^2\mathbf{B} + (\mathbf{J})\times\nabla\eta \end{bmatrix}
$$

with $\nabla\cdot\mathbf{B} = 0$

$\Rightarrow$ *conserved quantities* (in ‘closed’ systems):

- *total mass*: $M \equiv \int_V \rho\,\mathrm{d}V$
- *total momentum*: $\mathbf{\Pi} \equiv \int_V \rho\mathbf{v}\,\mathrm{d}V$
- *total energy*: $H \equiv \int_V \left(\frac{\rho v^2}{2} + \rho e + \frac{B^2}{2}\right)\mathrm{d}V$

- magnetic flux, magnetic helicity, and entropy *are not conserved in resistive MHD*
  - $\Rightarrow$ can diffuse, change due to magnetic reconnection,. . .

$\Rightarrow$ the dissipation rate of these quantities is limited by the diffusion time scale

Some relevant parameters are:

- *magnetic Reynolds number*: $R_m \equiv \dfrac{\mu_0 l_0 V_0}{\eta} \gg 1$
  - solar corona: $R_m \sim 10^{13}$, tokamak: $R_m = 10^9$
- *Lundquist number*: $L_u \equiv \dfrac{\mu_0 l_0 v_A}{\eta} \gg 1$
  - used for study of instabilities which also occur in static plasmas

$\Rightarrow$ *Alfvén Mach number*: $\dfrac{R_m}{L_u} = M_A \equiv \dfrac{V_0}{v_A}$

Other dissipative effects

- *viscosity* $\Rightarrow$ momentum equation:

$$\rho(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \rho\,\mathbf{g} + \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} + \mathbf{F}_{\text{visc}}$$

$$\mathbf{F}_{\text{visc}} \approx \rho\nu(\nabla^2\mathbf{v} + \tfrac{1}{3}\nabla\nabla\cdot\mathbf{v})$$

where ν is the kinematic viscosity coefficient

- *thermal conductivity* $\Rightarrow$ internal energy equation:

$$\rho\,[\,\frac{\mathrm{D}e}{\mathrm{D}t} + (\gamma-1)e\nabla\cdot\mathbf{v}\,] = -\nabla\cdot\mathbf{h} + Q$$

  – heat flow: $\mathbf{h} \approx -\kappa\nabla T$ *with κ the coefficient of thermal conductivity*

  – generated heat: $Q \equiv H - L\,,$

$$H = H_{\text{res}} + H_{\text{visc}} + H_{\text{fus}} + \cdots\,, \quad L = L_{\text{rad}} + \cdots$$

Jump conditions

Extending the MHD model

- The BCs for *plasmas surrounded by a solid wall:*

$$\mathbf{n}_w \cdot \mathbf{v} = 0 \qquad \text{(on } W) \quad \Rightarrow \quad \textit{no flow accross the wall,}$$

$$\mathbf{n}_w \cdot \mathbf{B} = 0 \qquad \text{(on } W) \quad \Rightarrow \quad \textit{magnetic field lines do not intersect the wall.}$$

  Under these conditions, conservation laws apply and the system is closed.

- For many applications (both in the laboratory and in astrophysics) this is not enough. One also needs BCs (jump conditions) for *plasmas with an internal boundary* where the magnitudes of the plasma variables 'jump'.

  Example: at the photospheric boundary the density changes $\sim 10^{-9}$.

- Such a boundary is a special case of a *shock,* i.e. an irreversible (entropy-increasing) transition. In gas dynamics, the *Rankine–Hugoniot relations* relate the variables of the subsonic flow downstream the shock with those of the supersonic flow upstream.

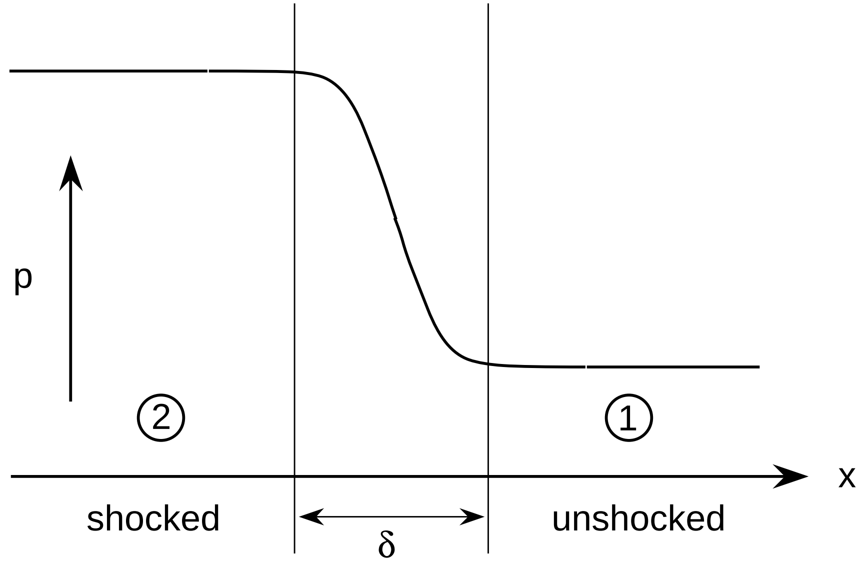
  We will generalize these relations to MHD, but only to get the right form of the jump conditions, not to analyze transonic flows (subject for a much later chapter).

Shock formation

- Excite sound waves in a 1D compressible gas (HD): the local perturbations travel with the sound speed $c \equiv \sqrt{\gamma p / \rho}$.

  $\Rightarrow$ Trajectories in the $x-t$ plane *(characteristics):* $dx/dt = \pm c$.

- Now suddenly increase the pressure, so that p changes in a thin layer of width δ :

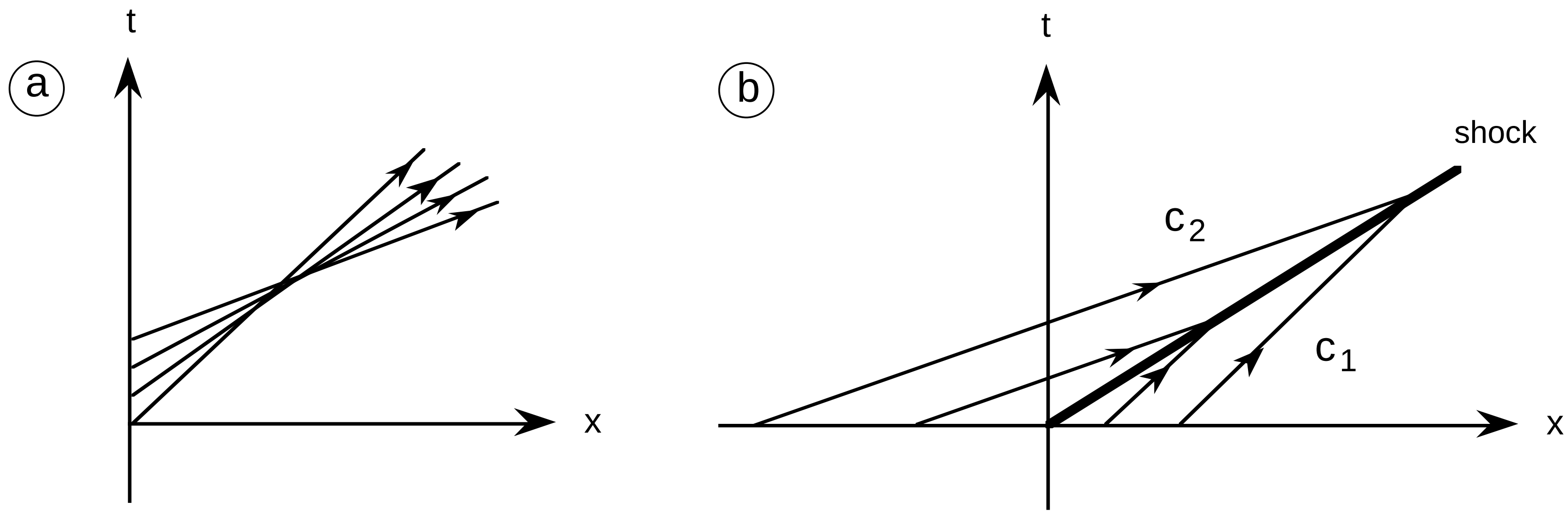


$\Rightarrow$ 'Converging' characteristics in the $x-t$ plane.

$\Rightarrow$ Information from different space-time points accumulates, gradients build up until steady state reached where dissipation and nonlinearities balance $\Rightarrow$ **shock.**

Shock formation (cont'd)

- Without the non-ideal and nonlinear effects, the characteristics would cross (a). With those effects, in the limit $\delta \to 0$, the characteristics meet at the shock front (b).



$\Rightarrow$ *Moving shock front separates two ideal regions.*

- Neglecting the thickness of the shock (not the shock itself of course), all there remains is to derive jump relations across the infinitesimal layer.

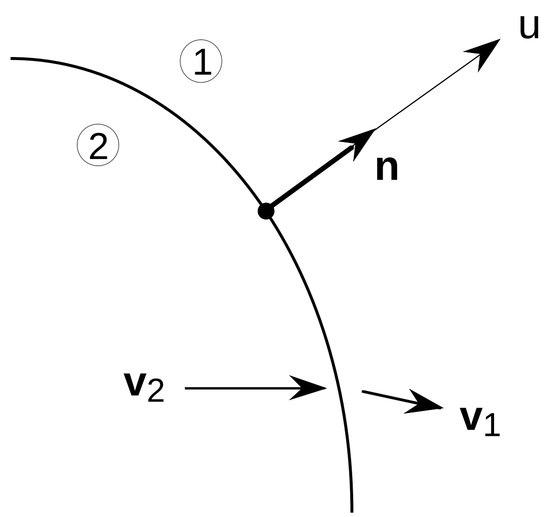
$\Rightarrow$ *Limiting cases of the conservation laws at shock fronts.*

Procedure to derive the jump conditions

Integrate conservation equations across shock from ① (undisturbed) to ② (shocked).

- Only contribution from gradient normal to the front:

$$\lim_{\delta\to 0}\int_1^2 \nabla f\, dl = -\lim_{\delta\to 0}\mathbf{n}\int_1^2 \frac{\partial f}{\partial l}\, dl = \mathbf{n}(f_1 - f_2) \equiv \mathbf{n}\,[\![f]\!]\,. \tag{54}$$



- In frame moving with the shock at normal speed u :

$$\Big(\frac{\mathrm{D}f}{\mathrm{D}t}\Big)_{\rm shock} = \frac{\partial f}{\partial t} - u\,\frac{\partial f}{\partial l}\ \text{finite} \ll \frac{\partial f}{\partial t} \approx u\frac{\partial f}{\partial l} \sim \infty$$

$$\Rightarrow \quad \lim_{\delta\to 0}\int_1^2 \frac{\partial f}{\partial t}\, dl = u\lim_{\delta\to 0}\int_1^2 \frac{\partial f}{\partial l}\, dl = -u\,[\![f]\!]\,. \tag{55}$$

- Hence, *jump conditions follow from the conservation laws by simply substituting*

$$\nabla f \to \mathbf{n}\,[\![f]\!]\,, \qquad \partial f/\partial t \to -u\,[\![f]\!]\,. \tag{56}$$

⇒ MHD jump conditions

- Conservation of mass,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \qquad \Rightarrow \quad -u\,[\![\rho]\!] + \mathbf{n} \cdot [\![\rho \mathbf{v}]\!] = 0\,. \tag{57}$$

- Conservation of momentum,

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot [\, \rho \mathbf{v}\mathbf{v} + (p + \tfrac{1}{2}B^2)\,\mathbf{I} - \mathbf{B}\mathbf{B}\,] = 0$$

$$\Rightarrow \quad -u\,[\![\rho \mathbf{v}]\!] + \mathbf{n} \cdot [\![\rho \mathbf{v}\mathbf{v} + (p + \tfrac{1}{2}B^2)\,\mathbf{I} - \mathbf{B}\mathbf{B}]\!] = 0\,. \tag{58}$$

- Conservation ot total energy,

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2 + \rho e + \tfrac{1}{2}B^2) + \nabla \cdot [(\tfrac{1}{2}\rho v^2 + \rho e + p + B^2)\mathbf{v} - \mathbf{v} \cdot \mathbf{B}\mathbf{B}] = 0$$

$$\Rightarrow \quad -u\,[\![\tfrac{1}{2}\rho v^2 + \tfrac{1}{\gamma - 1}\,p + \tfrac{1}{2}B^2]\!] + \mathbf{n} \cdot [\![(\tfrac{1}{2}\rho v^2 + \tfrac{\gamma}{\gamma - 1}\,p + B^2)\mathbf{v} - \mathbf{v} \cdot \mathbf{B}\mathbf{B}]\!] = 0\,. \tag{59}$$

- Conservation of magnetic flux,

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v}) = 0\,, \qquad \nabla \cdot \mathbf{B} = 0$$

$$\Rightarrow \quad -u\,[\![\mathbf{B}]\!] + \mathbf{n} \cdot [\![\mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v}]\!] = 0\,, \qquad \mathbf{n} \cdot [\![\mathbf{B}]\!] = 0\,. \tag{60}$$

MHD jump conditions in the shock frame

- Simplify jump conditions by *transforming to co-moving shock frame,* where relative plasma velocity is $\mathbf{v}' \equiv \mathbf{v} - u\mathbf{n}$, and split vectors in tangential and normal to shock:

$$[\![\rho v_n']\!] = 0\,, \qquad \textit{(mass)} \qquad (61)$$

$$[\![\rho {v_n'}^2 + p + \tfrac{1}{2}B_t^2]\!] = 0\,, \qquad \textit{(normal momentum)} \qquad (62)$$

$$\rho v_n'\,[\![\mathbf{v}_t']\!] = B_n\,[\![\mathbf{B}_t]\!]\,, \qquad \textit{(tangential momentum)} \qquad (63)$$

$$\rho v_n'\,[\![\tfrac{1}{2}({v_n'}^2 + {v_t'}^2) + (\tfrac{\gamma}{\gamma-1}p + B_t^2)/\rho]\!] = B_n\,[\![\mathbf{v}_t' \cdot \mathbf{B}_t]\!]\,, \qquad \textit{(energy)} \qquad (64)$$

$$[\![B_n]\!] = 0\,, \qquad \textit{(normal flux)} \qquad (65)$$

$$\rho v_n'\,[\![\mathbf{B}_t/\rho]\!] = B_n[\![\mathbf{v}_t']\!]\,. \qquad \textit{(tangential flux)} \qquad (66)$$

$\Rightarrow$ 6 relations for the 6 jumps $[\![\rho]\!]$, $[\![v_n]\!]$, $[\![\mathbf{v}_t]\!]$, $[\![p]\!]$, $[\![B_n]\!]$, $[\![\mathbf{B}_t]\!]$.

- *Do not use entropy conservation law* since shock is entropy-increasing transition:

$$\text{not}\quad \frac{\partial}{\partial t}(\rho S) + \nabla\cdot(\rho S\mathbf{v}) = 0 \;\;\Rightarrow\;\; \rho v_n'\,[\![S]\!] = 0\,, \quad \text{but}\quad [\![S]\!] \equiv [\![\rho^{-\gamma}p]\!] \le 0\,. \qquad (67)$$

$\Rightarrow$ This is the only remnant of the dissipative processes in the thin layer.

⇒ Two classes of discontinuities:

(1) *Boundary conditions for moving plasma-plasma interfaces,* where there is no flow accross the discontinuity ($v'_n = 0$) $\Rightarrow$ will continue with this here.

(2) *Jump conditions for shocks* ($v'_n \neq 0$) $\Rightarrow$ leave for advanced MHD lectures.

BCs at co-moving interfaces

- When $v'_n = 0\,,$ jump conditions (61)–(66) reduce to:

$$[\![p + \tfrac{1}{2}B_t^2]\!] = 0\,, \qquad \textit{(normal momentum)} \qquad (68)$$

$$B_n\,[\![\mathbf{B}_t]\!] = 0\,, \qquad \textit{(tangential momentum)} \qquad (69)$$

$$B_n\,[\![\mathbf{v}'_t \cdot \mathbf{B}_t]\!] = 0\,, \qquad \textit{(energy)} \qquad (70)$$

$$[\![B_n]\!] = 0\,, \qquad \textit{(normal flux)} \qquad (71)$$

$$B_n\,[\![\mathbf{v}'_t]\!] = 0\,. \qquad \textit{(tangential flux)} \qquad (72)$$

- Two possibilities, depending on whether $\mathbf{B}$ intersects the interface or not:

  (a) *Contact discontinuities* when $B_n \neq 0\,,$

  (b) *Tangential discontinuities* if $B_n = 0\,.$

(a) Contact discontinuities

- For co-moving interfaces with an intersecting magnetic field, $B_n \neq 0\,,$ the jump conditions (68)–(72) only admit a jump of the density (or temperature, or entropy) whereas all other quantities should be continuous:

$$
\begin{aligned}
&\text{– jumping:} && [\![\rho]\!] \neq 0\,, \\
&\text{– continuous:} && v_n' = 0\,, \quad [\![\mathbf{v}_t']\!] = 0\,, \quad [\![p]\!] = 0\,, \quad [\![B_n]\!] = 0\,, \quad [\![\mathbf{B}_t]\!] = 0\,.
\end{aligned}
\tag{73}
$$

  Examples: photospheric footpoints of coronal loops where density jumps,
  ‘divertor’ tokamak plasmas with $\mathbf{B}$ intersecting boundary.

- These BCs are most typical for astrophysical plasmas, *modelling plasmas with very different properties of the different spatial regions involved* (e.g. close to a star and far away): difficult! Computing waves in such systems usually requires extreme resolutions to follow the disparate time scales in the problem.

(b) Tangential discontinuities

- For co-moving interfaces with purely tangential magnetic field, $B_n = 0\,,$ the jump conditions (68)–(72) are much less restrictive:

$$\begin{aligned} &\text{– jumping:} && [\![\rho]\!] \neq 0\,, \quad [\![\mathbf{v}'_t]\!] \neq 0\,, \quad [\![p]\!] \neq 0\,, \quad [\![\mathbf{B}_t]\!] \neq 0\,, \\ &\text{– continuous:} && v'_n = 0\,, \quad B_n = 0\,, \quad [\![p + \tfrac{1}{2}B_t^2]\!] = 0\,. \end{aligned} \tag{74}$$

  Examples: tokamak plasma separated from wall by tenuous plasma (or ‘vacuum’), dayside magnetosphere where IMF meets Earth’s dipole.

- *Plasma–plasma interface BCs* by transforming back to lab frame, $v_n - u \equiv v'_n = 0$:

$$\boxed{\mathbf{n}\cdot\mathbf{B} = 0} \qquad (\mathbf{B} \parallel \text{interface})\,, \tag{75}$$

$$\boxed{\mathbf{n}\cdot[\![\mathbf{v}]\!] = 0} \qquad (\text{normal velocity continuous})\,, \tag{76}$$

$$\boxed{[\![p + \tfrac{1}{2}B^2]\!] = 0} \qquad (\text{total pressure continuous})\,. \tag{77}$$

- Jumps tangential components, $[\![\mathbf{B}_t]\!]$ & $[\![\mathbf{v}_t]\!]$, due to *surface current & surface vorticity:*

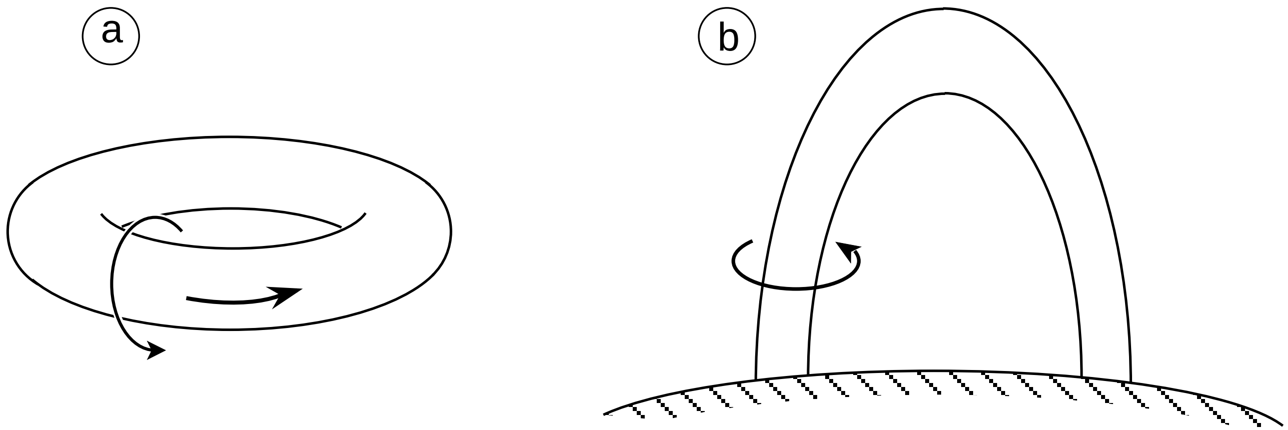
$$\mathbf{j} = \nabla\times\mathbf{B} \quad \Rightarrow \quad \mathbf{j}^\star \equiv \lim_{\delta\to 0,\, |\mathbf{j}|\to\infty}(\delta\,\mathbf{j}) = \mathbf{n}\times[\![\mathbf{B}]\!]\,, \tag{78}$$

$$\boldsymbol{\omega} \equiv \nabla\times\mathbf{v} \quad \Rightarrow \quad \boldsymbol{\omega}^\star \equiv \lim_{\delta\to 0,\, |\boldsymbol{\omega}|\to\infty}(\delta\,\boldsymbol{\omega}) = \mathbf{n}\times[\![\mathbf{v}]\!]\,. \tag{79}$$

Model problems

- We are now prepared to formulate **complete models for plasma dynamics** $\equiv$ **MHD equations** $+$ **specification of magnetic geometries** $\Rightarrow$ **appropriate BCs.**

- For example, recall two generic magnetic structures: (a) tokamak; (b) coronal loop.



- Generalize this to **six model problems,** separated in two classes:

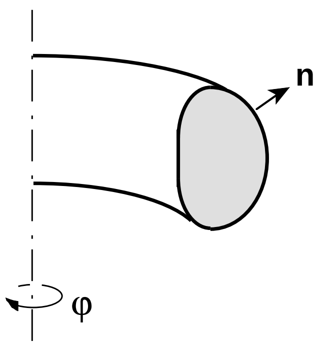
  $\Rightarrow$ **Models I–III (laboratory plasmas) with tangential discontinuities;**

  $\Rightarrow$ **Models IV–VI (astrophysical plasmas) with contact discontinuities.**

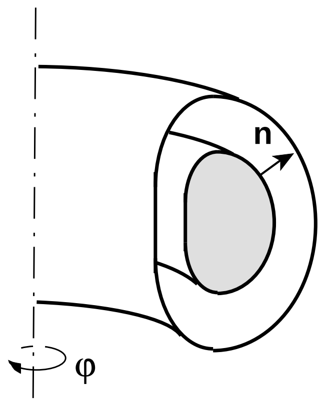
Laboratory plasmas (models I–III)

toroidal

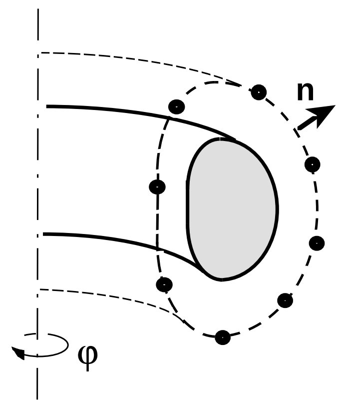
a



b



c



model I

a′ plasma–wall

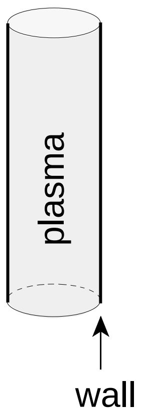
model II (∗)

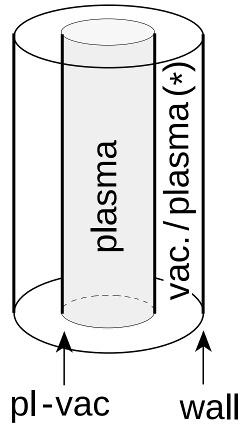
b′ plasma–vac–wall

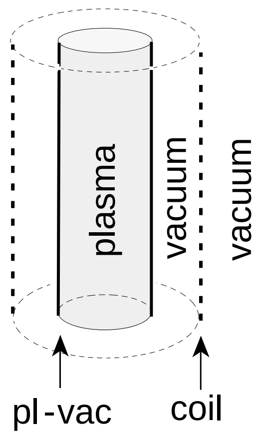
model III

c′ plasma–vac–coil–vac

cylindrical







Model I: plasma confined inside rigid wall

- Model I: axisymmetric (2D) plasma contained in a ‘donut’-shaped vessel (tokamak) which *confines the magnetic structure to a finite volume.* Vessel $+$ external coils need to be firmly fixed to the laboratory floor since magnetic forces are huge.

  $\Rightarrow$ Plasma–wall, impenetrable wall needs not be conducting (remember why?).

  $\Rightarrow$ Boundary conditions are

$$\mathbf{n} \cdot \mathbf{B} = 0 \qquad \textit{(at the wall)}\,, \tag{80}$$

$$\mathbf{n} \cdot \mathbf{v} = 0 \qquad \textit{(at the wall)}\,. \tag{81}$$

  $\Rightarrow$ just two BCs for 8 variables!

- These BCs guarantee conservation of mass, momentum, energy and magnetic flux: *the system is closed off from the outside world.*

- Most widely used simplification: cylindrical version (1D) with symmetry in θ and z.

  $\Rightarrow$ Non-trivial problem only in the radial direction, therefore: one-dimensional.

Model II: plasma-vacuum system inside rigid wall

- Model II: as I, but plasma separated from wall by vacuum (tokamak with a 'limiter').

  $\Rightarrow$ Plasma–vacuum–wall, wall now perfectly conducting (since vacuum in front).

- Vacuum has no density, velocity, current, only $\hat{\mathbf{B}}$ $\Rightarrow$ pre-Maxwell dynamics:

$$\nabla \times \hat{\mathbf{B}} = 0 \,, \qquad \nabla \cdot \hat{\mathbf{B}} = 0 \,, \tag{82}$$

$$\nabla \times \hat{\mathbf{E}} = -\frac{\partial \hat{\mathbf{B}}}{\partial t} \,, \qquad \nabla \cdot \hat{\mathbf{E}} = 0 \,. \tag{83}$$

  BC at exterior interface (only on $\hat{\mathbf{B}}$, consistent with $\hat{\mathbf{E}}_t = 0$):

$$\mathbf{n} \cdot \hat{\mathbf{B}} = 0 \qquad \textit{(at conducting wall)} \,. \tag{84}$$

- BCs at interior interface ($\mathbf{B}$ not pointing into vacuum and total pressure balance):

$$\mathbf{n} \cdot \mathbf{B} = \mathbf{n} \cdot \hat{\mathbf{B}} = 0 \qquad \textit{(at plasma–vacuum interface)} \,, \tag{85}$$

$$[\![p + \tfrac{1}{2} B^2]\!] = 0 \qquad \textit{(at plasma–vacuum interface)} \,. \tag{86}$$

  $\Rightarrow$ Consequence (not a BC) is jump in $\mathbf{B}_t$, i.e. skin current:

$$\mathbf{j}^\star = \mathbf{n} \times [\![\mathbf{B}]\!] \qquad \textit{(at plasma–vacuum interface)} \,. \tag{87}$$

Model II*: plasma-plasma system inside rigid wall

- Variant of Model II with vacuum replaced by tenuous plasma (negligible density, with or without current), where again the impenetrable wall needs not be conducting.

  $\Rightarrow$ Applicable to tokamaks to incorporate effects of outer plasma.

  $\Rightarrow$ Also for astrophysical plasmas (coronal loops) where 'wall' is assumed far away.

- BCs at exterior interface for outer plasma:

$$\mathbf{n}\cdot\hat{\mathbf{B}} = 0 \qquad \textit{(at the wall)}\,,$$

$$\mathbf{n}\cdot\hat{\mathbf{v}} = 0 \qquad \textit{(at the wall)}\,.$$

- BCs at interior interface for tangential plasma-plasma discontinuity:

$$\mathbf{n}\cdot\mathbf{B} = \mathbf{n}\cdot\hat{\mathbf{B}} = 0 \qquad \textit{(at plasma–plasma interface)}\,,$$

$$\mathbf{n}\cdot[\![\mathbf{v}]\!] = 0 \qquad \textit{(at plasma–plasma interface)}\,,$$

$$[\![p + \tfrac{1}{2}B^2]\!] = 0 \qquad \textit{(at plasma–plasma interface)}\,.$$

Note: Model II obtained by just dropping conditions on $\mathbf{v}$ and $\hat{\mathbf{v}}$.

Model III: plasma-vacuum system with external currents

- Model III is an *open* plasma–vacuum configuration excited by magnetic fields $\hat{\mathbf{B}}(t)$ that are externally created by a coil (antenna) with skin current.

  $\Rightarrow$ Open system: forced oscillations *pump energy into the plasma.*

  $\Rightarrow$ Applications in laboratory and astrophysical plasmas: original creation of the confining magnetic fields and excitation of MHD waves.

- BCs at coil surface:

$$\mathbf{n}\cdot[\![\hat{\mathbf{B}}]\!] = 0 \qquad \textit{(at coil surface)}\,, \tag{88}$$

$$\mathbf{n}\times[\![\hat{\mathbf{B}}]\!] = \mathbf{j}_c^\star(\mathbf{r},t) \qquad \textit{(at coil surface)}\,. \tag{89}$$

  where $\mathbf{j}_c^\star(\mathbf{r},t)$ is the prescribed skin current in the coil.

- Magnetic field outside coil subject to exterior BC (84) at wall (possibly moved to ∞), combined with plasma-vacuum interface conditions (85) and (86):

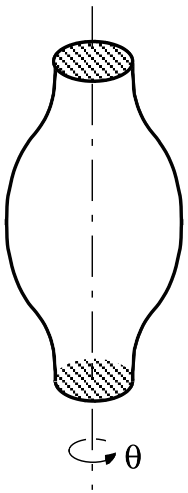
$$\mathbf{n}\cdot\mathbf{B} = \mathbf{n}\cdot\hat{\mathbf{B}} = 0 \qquad \textit{(at plasma–vacuum interface)}\,,$$

$$[\![p + \tfrac{1}{2}B^2]\!] = 0 \qquad \textit{(at plasma–vacuum interface)}\,.$$

Astrophysical plasmas (models IV–VI)

2D

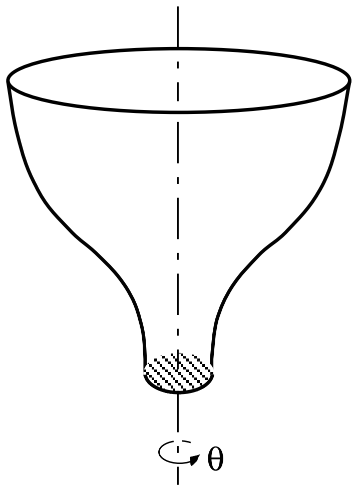
a



model IV

closed loop

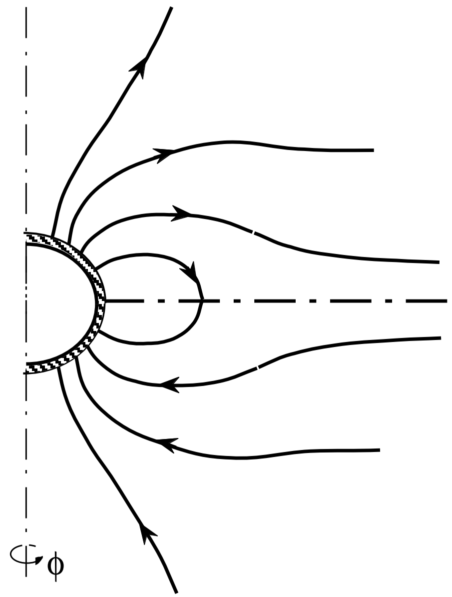
b



model V

open loop

c

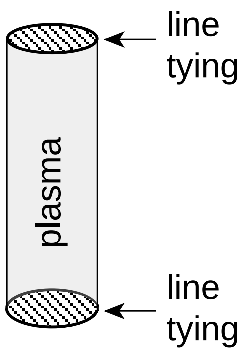


model VI

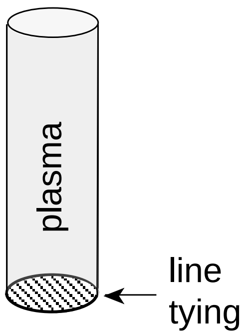
stellar wind

“1D”

a′



b′



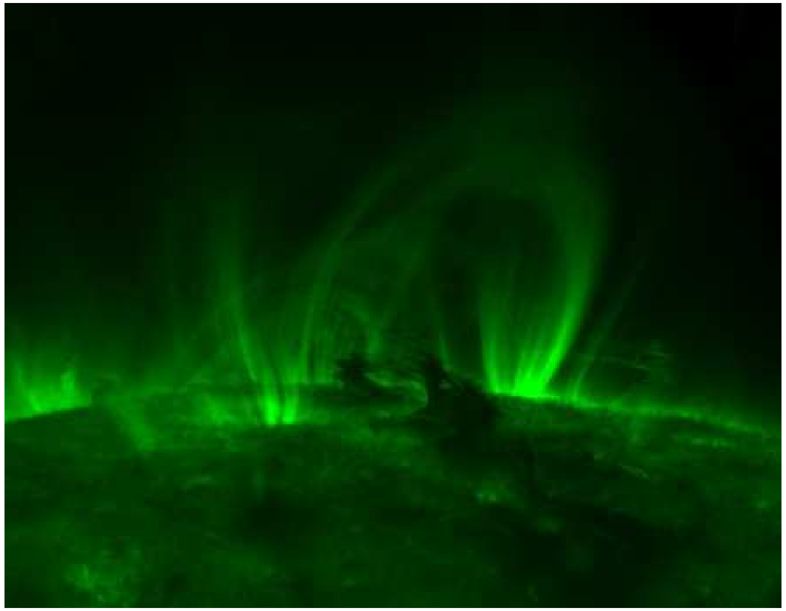
Model IV: 'closed' coronal magnetic loop

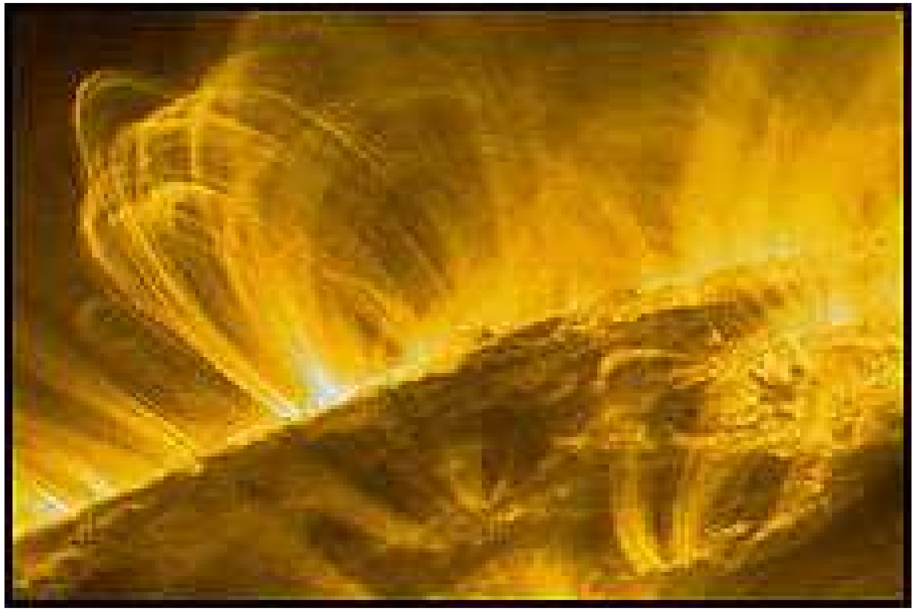
- In model IV, the field lines of finite plasma column (coronal loop) are line-tied on both sides to plasma of such high density (photosphere) that it is effectively immobile.

  $\Rightarrow$ Line-tying boundary conditions:

$$\mathbf{v} = 0 \qquad \textit{(at photospheric end planes)}\,. \tag{90}$$

  $\Rightarrow$ Applies to waves in solar coronal flux tubes, no back-reaction on photosphere:





- In this model, loops are straightened out to 2D configuration (depending on r and z). Also neglecting fanning out of field lines $\Rightarrow$ quasi-1D (finite length cylinder).

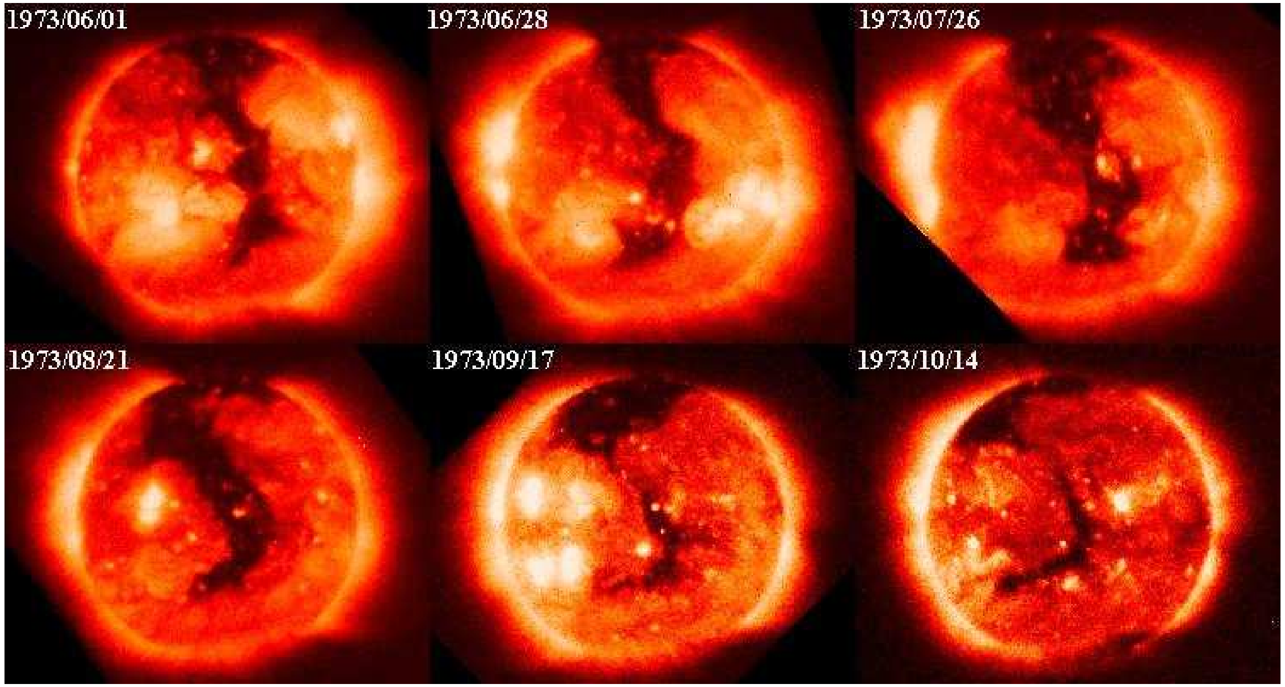
Model V: open coronal magnetic loop

- In model V, the magnetic field lines of a semi-infinite plasma column are line-tied on one side to a massive plasma.

  $\Rightarrow$ Line-tying boundary condition:

  $$\mathbf{v} = 0 \qquad \textit{(at photospheric end plane)} \, .$$

  $\Rightarrow$ Applies to dynamics in coronal holes, where (fast) solar wind escapes freely:

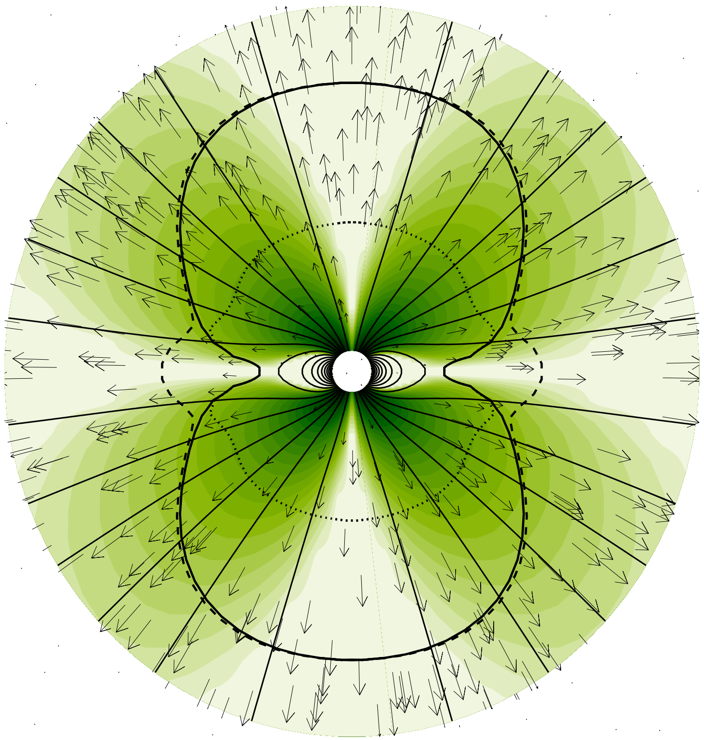


- *Truly open* variants of models IV & V: photospheric excitation ($\mathbf{v}(t) \neq 0$ prescribed).

Model VI: Stellar wind

- In model VI, a plasma is ejected from photosphere of a star and accelerated along the open magnetic field lines into outer space.

  $\Rightarrow$ Combines closed & open loops (models IV & V), line-tied at dense photosphere, but stress on outflow rather than waves (requires more advanced discussion).



- Output from an actual simulation with the Versatile Advection code: 2D (axisymm.) magnetized wind with 'wind' and 'dead' zone. Sun at the center, field lines drawn, velocity vectors, density coloring. Dotted, drawn, dashed: slow, Alfvén, fast critical surfaces.

  [Keppens & Goedbloed, Ap. J. **530**, 1036 (2000)]

Linear MHD waves and characteristics

Overview

- **MagnetohydraSTATICS**
- **Physics and accounting:** use example of sound waves to illustrate method of linearization and counting of variables and solutions; [book: Sec. 5.1]
- **MHD waves:** different representations and reductions of the linearized MHD equations, obtaining the three main waves, dispersion diagrams; [book: Sec. 5.2]
- **Phase and group diagrams:** propagation of plane waves and wave packets, asymptotic properties of the three MHD waves; [book: Sec. 5.3]
- **Characteristics:** numerical method, classification of PDEs, application to MHD. [book: Sec. 5.4]

Magnetohydro**statics**

- static equilibrium configurations can be found by assuming no time dependence and by putting the velocity equal to zero in the MHD equations:

$$\rho = \rho_0(\mathbf{r}), \quad e = e_0(\mathbf{r}), \quad \mathbf{B} = \mathbf{B}_0(\mathbf{r}), \quad \mathbf{v}_0 \equiv 0$$

$\Rightarrow$ the MHD equations than reduce to:

$$\nabla p_0 = \rho_0 \mathbf{g} + (\nabla \times \mathbf{B}_0) \times \mathbf{B}_0$$

the equilibrium force balance

$$\nabla \cdot \mathbf{B}_0 = 0$$

which now fully counts! (not just an initial condition)

$\Rightarrow$ four equations for the determination of $\rho_0(\mathbf{r}),\ p_0(\mathbf{r}),\ \mathbf{B}_0(\mathbf{r})$

$\Rightarrow$ **a lot of freedom is left!**

Sound waves

- Perturb the gas dynamic equations ($\mathbf{B} = 0$),

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0\,, \tag{1}$$

$$\rho\,\Big(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\Big) + \nabla p = 0\,, \tag{2}$$

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0\,, \tag{3}$$

about infinite, homogeneous gas at rest,

$$\begin{aligned} \rho(\mathbf{r}, t) &= \rho_0 + \rho_1(\mathbf{r}, t) \qquad &&(\text{where } |\rho_1| \ll \rho_0 = \text{const})\,, \\ p(\mathbf{r}, t) &= p_0 + p_1(\mathbf{r}, t) \qquad &&(\text{where } |p_1| \ll p_0 = \text{const})\,, \\ \mathbf{v}(\mathbf{r}, t) &= \mathbf{v}_1(\mathbf{r}, t) \qquad &&(\text{since } \mathbf{v}_0 = 0)\,. \end{aligned} \tag{4}$$

$\Rightarrow$ *Linearised equations of gas dynamics:*

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1 = 0\,, \tag{5}$$

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} + \nabla p_1 = 0\,, \tag{6}$$

$$\frac{\partial p_1}{\partial t} + \gamma p_0 \nabla \cdot \mathbf{v}_1 = 0\,. \tag{7}$$

Wave equation

- Equation for ρ_1 does not couple to the other equations: drop. Remaining equations give *wave equation for sound waves*:

$$\frac{\partial^2 \mathbf{v}_1}{\partial t^2} - c^2 \, \nabla\nabla \cdot \mathbf{v}_1 = 0 \,, \tag{8}$$

where

$$c \equiv \sqrt{\gamma p_0 / \rho_0} \tag{9}$$

is *the velocity of sound* of the background medium.

- Plane wave solutions

$$\mathbf{v}_1(\mathbf{r}, t) = \sum_{\mathbf{k}} \hat{\mathbf{v}}_{\mathbf{k}} \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \tag{10}$$

turn the wave equation (8) into an algebraic equation:

$$(\, \omega^2 \mathsf{I} - c^2 \, \mathbf{k}\mathbf{k} \,) \cdot \hat{\mathbf{v}} = 0 \,. \tag{11}$$

- For $\mathbf{k} = k \, \mathbf{e}_z$, the solution is:

$$\omega = \pm k \, c \,, \qquad \hat{v}_x = \hat{v}_y = 0 \,, \quad \hat{v}_z \text{ arbitrary} \,, \tag{12}$$

$\Rightarrow$ *Sound waves* propagating to the right ($+$) and to the left ($-$): *compressible* ($\nabla \cdot \mathbf{v} \neq 0$) and *longitudinal* ($\mathbf{v} \parallel \mathbf{k}$) waves.

Counting

- There are also other solutions:

$$\omega^2 = 0\,, \qquad \hat{v}_x\,, \hat{v}_y \text{ arbitrary}\,, \qquad \hat{v}_z = 0\,, \tag{13}$$

$\Rightarrow$ *incompressible transverse ($\mathbf{v}_1 \perp \mathbf{k}$) translations.* They do not represent interesting physics, but simply establish completeness of the velocity representation.

- Problem: 1st order system (5)–(7) for ρ_1, $\mathbf{v}_1$, p_1 has 5 degrees of freedom, whereas 2nd order system (8) for $\mathbf{v}_1$ appears to have 6 degrees of freedom ($\partial^2/\partial t^2 \rightarrow -\omega^2$). However, the 2nd order system actually only has 4 degrees of freedom, since ω^2 does not double the number of translations (13). Spurious doubling of the eigenvalue $\omega = 0$ happened when we applied the operator $\partial/\partial t$ to Eq. (6) to eliminate p_1.

- Hence, we *lost one degree of freedom* in the reduction to the wave equation in terms of $\mathbf{v}_1$ alone. This happened when we dropped Eq. (5) for ρ_1. Inserting $\mathbf{v}_1 = 0$ in the original system gives the signature of this lost mode:

$$\omega\hat{\rho} = 0 \quad \Rightarrow \quad \omega = 0\,, \quad \hat{\rho} \text{ arbitrary}\,, \quad \text{but } \hat{\mathbf{v}} = 0 \quad \text{and} \quad \hat{p} = 0\,. \tag{14}$$

$\Rightarrow$ *entropy wave*: perturbation of the density and, hence, of the entropy $S \equiv p\rho^{-\gamma}$. Like the translations (13), this mode does not represent important physics but is needed to account for the degrees of freedom of the different representations.

MHD waves

- Similar analysis for MHD in terms of ρ , $\mathbf{v}$, e $\left(\equiv \frac{1}{\gamma-1}\, p/\rho \right)$, and $\mathbf{B}$:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \,, \tag{15}$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} + (\gamma - 1) \nabla (\rho e) + (\nabla \mathbf{B}) \cdot \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{B} = 0 \,, \tag{16}$$

$$\frac{\partial e}{\partial t} + \mathbf{v} \cdot \nabla e + (\gamma - 1) e \nabla \cdot \mathbf{v} = 0 \,, \tag{17}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{B} + \mathbf{B} \nabla \cdot \mathbf{v} - \mathbf{B} \cdot \nabla \mathbf{v} = 0 \,, \qquad \nabla \cdot \mathbf{B} = 0 \,, \tag{18}$$

- *Linearise about plasma at rest,* $\mathbf{v}_0 = 0$, ρ_0 , e_0 , $\mathbf{B}_0 = \text{const}$:

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1 = 0 \,, \tag{19}$$

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} + (\gamma - 1)(e_0 \nabla \rho_1 + \rho_0 \nabla e_1) + (\nabla \mathbf{B}_1) \cdot \mathbf{B}_0 - \mathbf{B}_0 \cdot \nabla \mathbf{B}_1 = 0 \,, \tag{20}$$

$$\frac{\partial e_1}{\partial t} + (\gamma - 1) e_0 \nabla \cdot \mathbf{v}_1 = 0 \,, \tag{21}$$

$$\frac{\partial \mathbf{B}_1}{\partial t} + \mathbf{B}_0 \nabla \cdot \mathbf{v}_1 - \mathbf{B}_0 \cdot \nabla \mathbf{v}_1 = 0 \,, \qquad \nabla \cdot \mathbf{B}_1 = 0 \,. \tag{22}$$

Transformation

- *Sound* and vectorial *Alfvén speed*,

$$c \equiv \sqrt{\frac{\gamma p_0}{\rho_0}}\,, \qquad \mathbf{b} \equiv \frac{\mathbf{B}_0}{\sqrt{\rho_0}}\,, \tag{23}$$

and dimensionless variables,

$$\tilde{\rho} \equiv \frac{\rho_1}{\gamma\,\rho_0}\,, \qquad \tilde{\mathbf{v}} \equiv \frac{\mathbf{v}_1}{c}\,, \qquad \tilde{e} \equiv \frac{e_1}{\gamma\,e_0}\,, \qquad \tilde{\mathbf{B}} \equiv \frac{\mathbf{B}_1}{c\sqrt{\rho_0}}\,, \tag{24}$$

⇒ *linearised MHD equations* with coefficients c and $\mathbf{b}$:

$$\gamma\,\frac{\partial\tilde{\rho}}{\partial t} + c\,\nabla\cdot\tilde{\mathbf{v}} = 0\,, \tag{25}$$

$$\frac{\partial\tilde{\mathbf{v}}}{\partial t} + c\,\nabla\tilde{\rho} + c\,\nabla\tilde{e} + (\nabla\tilde{\mathbf{B}})\cdot\mathbf{b} - \mathbf{b}\cdot\nabla\tilde{\mathbf{B}} = 0\,, \tag{26}$$

$$\frac{\gamma}{\gamma-1}\,\frac{\partial\tilde{e}}{\partial t} + c\,\nabla\cdot\tilde{\mathbf{v}} = 0\,, \tag{27}$$

$$\frac{\partial\tilde{\mathbf{B}}}{\partial t} + \mathbf{b}\,\nabla\cdot\tilde{\mathbf{v}} - \mathbf{b}\cdot\nabla\tilde{\mathbf{v}} = 0\,, \qquad \nabla\cdot\tilde{\mathbf{B}} = 0\,. \tag{28}$$

Symmetry

- *Plane wave solutions*, with $\mathbf{b}$ and $\mathbf{k}$ arbitrary now:

$$\tilde{\rho} = \tilde{\rho}(\mathbf{r}, t) = \hat{\rho}\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\,, \quad \text{etc.} \tag{29}$$

yields an algebraic system of eigenvalue equations:

$$\begin{aligned}
c\,\mathbf{k}\cdot\hat{\mathbf{v}} &= \gamma\,\omega\,\hat{\rho}\,,\\
\mathbf{k}\,c\,\hat{\rho} + \mathbf{k}\,c\,\hat{e} + (\mathbf{k}\mathbf{b}\cdot - \mathbf{k}\cdot\mathbf{b})\,\hat{\mathbf{B}} &= \omega\,\hat{\mathbf{v}}\,,\\
c\,\mathbf{k}\cdot\hat{\mathbf{v}} &= \tfrac{\gamma}{\gamma-1}\,\omega\,\hat{e}\,,\\
(\mathbf{b}\mathbf{k}\cdot - \mathbf{b}\cdot\mathbf{k})\,\hat{\mathbf{v}} &= \omega\,\hat{\mathbf{B}}\,, \qquad \mathbf{k}\cdot\hat{\mathbf{B}} = 0\,.
\end{aligned} \tag{30}$$

$\Rightarrow$ **Symmetric eigenvalue problem!** (The equations for $\hat{\rho}$, $\hat{\mathbf{v}}$, $\hat{e}$, and $\hat{\mathbf{B}}$ appear to know about each other.) .

- The symmetry of the linearized system is closely related to an analogous property of the original nonlinear equations: *the nonlinear ideal MHD equations are symmetric hyperbolic partial differential equations*.

Matrix eigenvalue problem

- Choose $\mathbf{b} = (0, 0, b)\,, \quad \mathbf{k} = (k_\perp, 0, k_\|)$:

$$\begin{pmatrix} 0 & k_\perp c & 0 & k_\| c & 0 & 0 & 0 & 0 \\ k_\perp c & 0 & 0 & 0 & k_\perp c & -k_\| b & 0 & k_\perp b \\ 0 & 0 & 0 & 0 & 0 & 0 & -k_\| b & 0 \\ k_\| c & 0 & 0 & 0 & k_\| c & 0 & 0 & 0 \\ 0 & k_\perp c & 0 & k_\| c & 0 & 0 & 0 & 0 \\ 0 & -k_\| b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k_\| b & 0 & 0 & 0 & 0 & 0 \\ 0 & k_\perp b & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat\rho \\ \hat v_x \\ \hat v_y \\ \hat v_z \\ \hat e \\ \hat B_x \\ \hat B_y \\ \hat B_z \end{pmatrix} = \omega \begin{pmatrix} \gamma\,\hat\rho \\ \hat v_x \\ \hat v_y \\ \hat v_z \\ \frac{\gamma}{\gamma-1}\,\hat e \\ \hat B_x \\ \hat B_y \\ \hat B_z \end{pmatrix} . \tag{31}$$

$\Rightarrow$ **Another representation of the symmetry of linearized MHD equations.**

- New features of MHD waves compared to sound: occurrence of Alfvén speed b and anisotropy expressed by the two components $k_\|$ and $k_\perp$ of the wave vector. We could compute the dispersion equation from the determinant and study the associated waves, but we prefer again to exploit the much simpler velocity representation.

MHD wave equation

- Ignoring the magnetic field constraint $\mathbf{k}\cdot\hat{\mathbf{B}} = 0$ in the 8×8 eigenvalue problem (31) would yield *one spurious eigenvalue $\omega = 0$*. This may be seen by operating with the projector $\mathbf{k}\cdot$ onto Eq. (30)(d), which gives $\omega\,\mathbf{k}\cdot\hat{\mathbf{B}} = 0$.

- Like in the gas dynamics problem, a *genuine but unimportant marginal entropy mode* is obtained for $\omega = 0$ with $\hat{\mathbf{v}} = 0$, $\hat{p} = 0$, and $\hat{\mathbf{B}} = 0$:

$$\omega = 0\,, \qquad \hat{p} = \hat{e} + \hat{\rho} = 0\,, \qquad \hat{S} = \gamma\hat{e} = -\gamma\hat{\rho} \neq 0\,. \tag{32}$$

- Both of these marginal modes are eliminated by exploiting *the velocity representation.* The perturbations ρ_1, e_1, $\mathbf{B}_1$ are expressed in terms of $\mathbf{v}_1$ by means of Eqs. (19), (21), and (22), and substituted into the momentum equation (20). This yields the **MHD wave equation for a homogeneous medium**:

$$\frac{\partial^2\mathbf{v}_1}{\partial t^2} - \big[\,(\mathbf{b}\cdot\nabla)^2\,\mathbf{I} + (b^2 + c^2)\,\nabla\,\nabla - \mathbf{b}\cdot\nabla\,(\nabla\,\mathbf{b} + \mathbf{b}\,\nabla)\,\big]\cdot\mathbf{v}_1 = 0\,. \tag{33}$$

The sound wave equation (8) is obtained for the special case $\mathbf{b} = 0$.

MHD wave equation (cont'd)

- Inserting plane wave solutions gives the required eigenvalue equation:

$$\Big\{\big[\,\omega^2 - (\mathbf{k}\cdot\mathbf{b})^2\,\big]\,\mathbf{I} - (b^2 + c^2)\,\mathbf{k}\,\mathbf{k} + \mathbf{k}\cdot\mathbf{b}\,(\mathbf{k}\,\mathbf{b} + \mathbf{b}\,\mathbf{k})\Big\}\cdot\hat{\mathbf{v}} = 0\,, \qquad (34)$$

or, in components:

$$\begin{pmatrix} -\,k_\perp^2(b^2+c^2) - k_\parallel^2 b^2 & 0 & -k_\perp k_\parallel c^2 \\ 0 & -k_\parallel^2 b^2 & 0 \\ -\,k_\perp k_\parallel c^2 & 0 & -k_\parallel^2 c^2 \end{pmatrix} \begin{pmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{v}_z \end{pmatrix} = -\omega^2 \begin{pmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{v}_z \end{pmatrix}. \qquad (35)$$

Hence, a 3×3 symmetric matrix equation is obtained in terms of the variable $\hat{\mathbf{v}}$, with *quadratic eigenvalue* ω^2, corresponding to the original 6×6 representation with eigenvalue ω (resulting from elimination of the two marginal modes).

- Determinant yields the **dispersion equation:**

$$\det = \omega\,(\omega^2 - k_\parallel^2 b^2)\,\Big[\,\omega^4 - k^2(b^2+c^2)\,\omega^2 + k_\parallel^2 k^2 b^2 c^2\,\Big] = 0 \qquad (36)$$

(where we have artificially included a factor ω for the marginal entropy wave).

Roots

1) Entropy waves:

$$\boxed{\omega = \omega_E \equiv 0\,,} \tag{37}$$

$$\hat{\mathbf{v}} = \hat{\mathbf{B}} = 0\,, \quad \hat{p} = 0\,, \quad \text{but} \quad \hat{s} \neq 0\,. \tag{38}$$

$\Rightarrow$ just perturbation of thermodynamic variables.

2) Alfvén waves:

$$\boxed{\omega^2 = \omega_A^2 \equiv k_\parallel^2 b^2 \quad \rightarrow \quad \omega = \pm\omega_A\,,} \tag{39}$$

$$\hat{v}_x = \hat{v}_z = \hat{B}_x = \hat{B}_z = \hat{s} = \hat{p} = 0\,, \quad \hat{B}_y = -\hat{v}_y \neq 0\,. \tag{40}$$

$\Rightarrow$ transverse $\hat{\mathbf{v}}$ and $\hat{\mathbf{B}}$ so that field lines follow the flow.

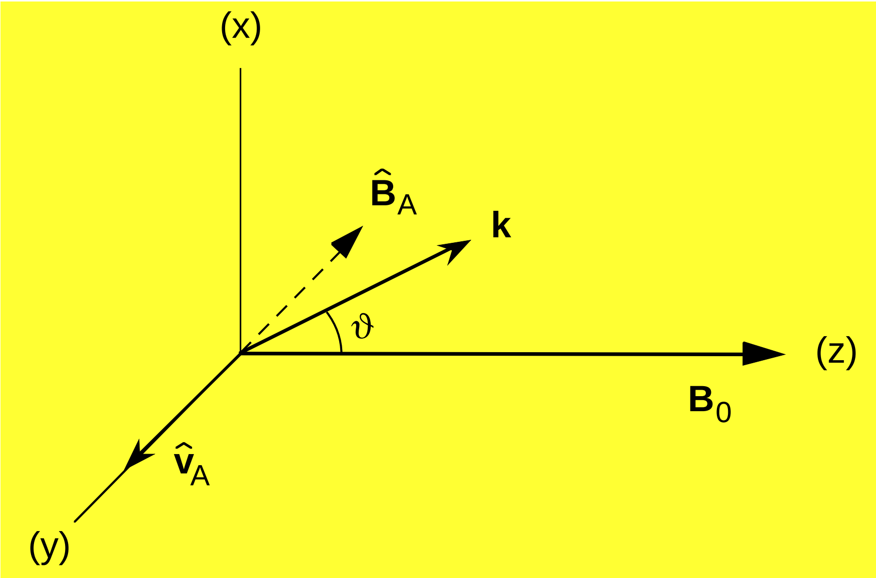
3) Fast (+) and Slow (−) magnetoacoustic waves:

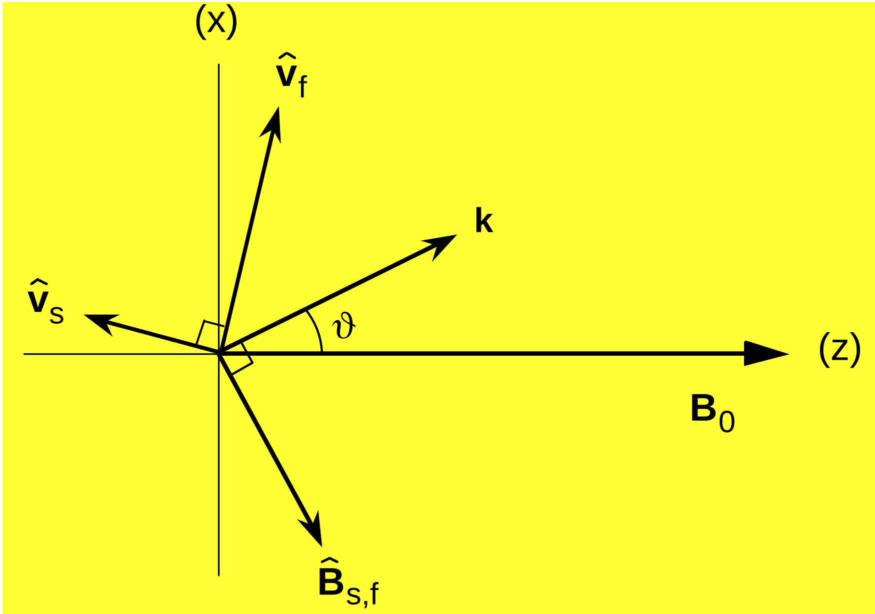
$$\boxed{\omega^2 = \omega_{s,f}^2 \equiv \tfrac{1}{2}k^2(b^2+c^2)\left[\,1 \pm \sqrt{1 - \frac{4k_\parallel^2 b^2 c^2}{k^2(b^2+c^2)^2}}\,\right] \quad \rightarrow \quad \omega = \begin{cases} \pm\omega_s \\ \pm\omega_f \end{cases}} \tag{41}$$

$$\hat{v}_y = \hat{B}_y = \hat{s} = 0\,, \quad \text{but} \quad \hat{v}_x\,, \hat{v}_z\,, \hat{p}\,, \hat{B}_x\,, \hat{B}_z \neq 0\,, \tag{42}$$

$\Rightarrow$ perturbations $\hat{\mathbf{v}}$ and $\hat{\mathbf{B}}$ in the plane through $\mathbf{k}$ and $\mathbf{B}_0$.

Eigenfunctions





Alfvén waves **Magnetosonic waves**

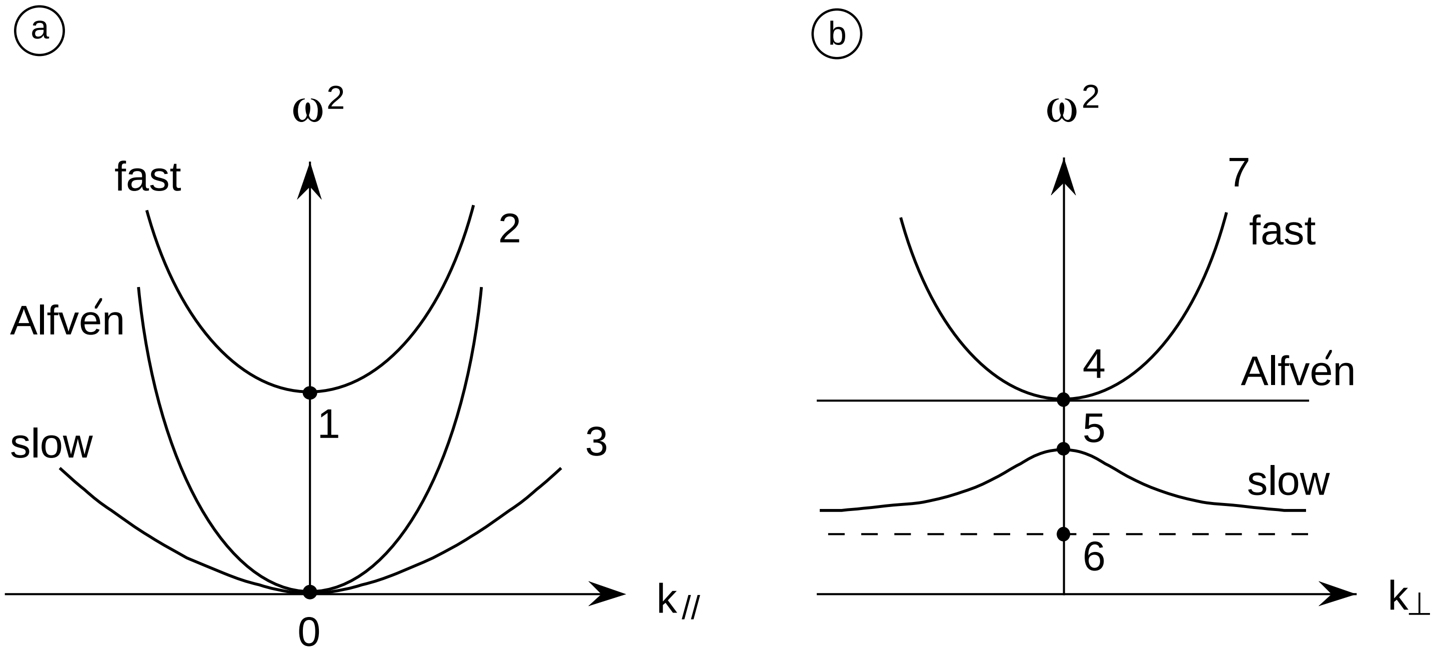
- Note: **the eigenfunctions are mutually orthogonal:**

$$\hat{\mathbf{v}}_s \perp \hat{\mathbf{v}}_A \perp \hat{\mathbf{v}}_f \,. \tag{43}$$

$\Rightarrow$ Arbitrary velocity field may be decomposed at all times (e.g. at $t = 0$) in the three MHD waves: **the initial value problem is a well-posed problem**.

Dispersion diagrams (schematic)

[exact diagrams in book: Fig. 5.3, scaling $\bar{\omega} \equiv (l/b)\,\omega\,,\ \bar{k} \equiv k\,l$]



- Note: $\omega^2(k_\parallel = 0) = 0$ for Alfvén and slow waves $\Rightarrow$ potential onset of *instability.*
- Asymptotics of $\omega^2(k_\perp \to \infty)$ characterizes *local* behavior of the three waves:

$$\left\{ \begin{array}{lll} \partial\omega/\partial k_\perp > 0\,, & \omega_f^2 \to \infty & \text{for fast waves,} \\ \partial\omega/\partial k_\perp = 0\,, & \omega_A^2 \to k_\parallel^2 b^2 & \text{for Alfvén waves,} \\ \partial\omega/\partial k_\perp < 0\,, & \omega_s^2 \to k_\parallel^2 \dfrac{b^2 c^2}{b^2 + c^2} & \text{for slow waves.} \end{array} \right. \tag{44}$$

Phase and group velocity

Dispersion equation $\omega = \omega(\mathbf{k})$ $\Rightarrow$ two fundamental concepts:

1. A *single plane wave* propagates in the direction of $\mathbf{k}$ with the **phase velocity**

$$\mathbf{v}_{\rm ph} \equiv \frac{\omega}{k}\,\mathbf{n}\,, \qquad \mathbf{n} \equiv \mathbf{k}/k = (\sin\vartheta, 0, \cos\vartheta)\,; \tag{45}$$

$\Rightarrow$ MHD waves are non-dispersive (only depend on angle ϑ, not on $|\mathbf{k}|$):

$$(\mathbf{v}_{\rm ph})_A \equiv b\cos\vartheta\,\mathbf{n}\,, \tag{46}$$

$$(\mathbf{v}_{\rm ph})_{s,f} \equiv \sqrt{\tfrac{1}{2}(b^2+c^2)}\sqrt{1 \pm \sqrt{1-\sigma\cos^2\vartheta}}\,\mathbf{n}\,, \quad \sigma \equiv \frac{4b^2c^2}{(b^2+c^2)^2}\,. \tag{47}$$

2. A *wave packet* propagates with the **group velocity**

$$\mathbf{v}_{\rm gr} \equiv \frac{\partial\omega}{\partial\mathbf{k}} \quad \Big[\equiv \frac{\partial\omega}{\partial k_x}\mathbf{e}_x + \frac{\partial\omega}{\partial k_y}\mathbf{e}_y + \frac{\partial\omega}{\partial k_z}\mathbf{e}_z\Big]\,; \tag{48}$$

$\Rightarrow$ MHD caustics in directions $\mathbf{b}$, and mix of $\mathbf{n}$ and $\mathbf{t}$ ($\perp$ $\mathbf{n}$):

$$(\mathbf{v}_{\rm gr})_A = \mathbf{b}\,, \tag{49}$$

$$(\mathbf{v}_{\rm gr})_{s,f} = (v_{\rm ph})_{s,f}\left[\mathbf{n} \pm \frac{\sigma\sin\vartheta\cos\vartheta}{2\sqrt{1-\sigma\cos^2\vartheta}\,\left[1 \pm \sqrt{1-\sigma\cos^2\vartheta}\,\right]}\,\mathbf{t}\right]. \tag{50}$$

Wave packet

Wave packet of plane waves satisfying dispersion equation $\omega = \omega(\mathbf{k})$:

$$\Psi_i(\mathbf{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} A_i(\mathbf{k})\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega(\mathbf{k})t)}\, d^3k\,. \tag{51}$$

Evolves from initial shape given by *Fourier synthesis,*

$$\Psi_i(\mathbf{r}, 0) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} A_i(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{r}}\, d^3k\,, \tag{52}$$

where amplitudes $A_i(\mathbf{k})$ are related to initial values $\Psi_i(\mathbf{r}, 0)$ by *Fourier analysis,*

$$A_i(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \Psi_i(\mathbf{r}, 0)\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, d^3r\,. \tag{53}$$

MHD: Ψ_i – perturbations ($\tilde{\rho}_1$,) $\tilde{\mathbf{v}}_1$ (, $\tilde{e}_1$, $\tilde{\mathbf{B}}_1$); A_i – Fourier amplitudes ($\hat{\rho}_1$,) $\hat{\mathbf{v}}_1$ (, $\hat{e}_1$, $\hat{\mathbf{B}}_1$).

Example: Gaussian wave packet of harmonics centered at some wave vector $\mathbf{k}_0$,

$$A_i(\mathbf{k}) = \hat{A}_i\, e^{-\frac{1}{2}|(\mathbf{k}-\mathbf{k}_0)\cdot\mathbf{a}|^2}\,, \tag{54}$$

corresponds to initial packet with main harmonic $\mathbf{k}_0$ and modulated amplitude centered at $\mathbf{r} = 0$:

$$\Psi_i(\mathbf{r}, 0) = e^{i\mathbf{k}_0\cdot\mathbf{r}} \times \frac{\hat{A}_i}{a_x a_y a_z}\, e^{-\frac{1}{2}[(x/a_x)^2 + (y/a_y)^2 + (z/a_z)^2]}\,. \tag{55}$$

Wave packet (cont'd)

For arbitrary wave packet with localized range of wave vectors, we may expand the dispersion equation about the central value $\mathbf{k}_0$:

$$\omega(\mathbf{k}) \approx \omega_0 + (\mathbf{k} - \mathbf{k}_0) \cdot \left(\frac{\partial \omega}{\partial \mathbf{k}} \right)_{\mathbf{k}_0} , \qquad \omega_0 \equiv \omega(\mathbf{k}_0) \,. \tag{56}$$

Inserting this approximation in the expression (51) for the wave packet gives

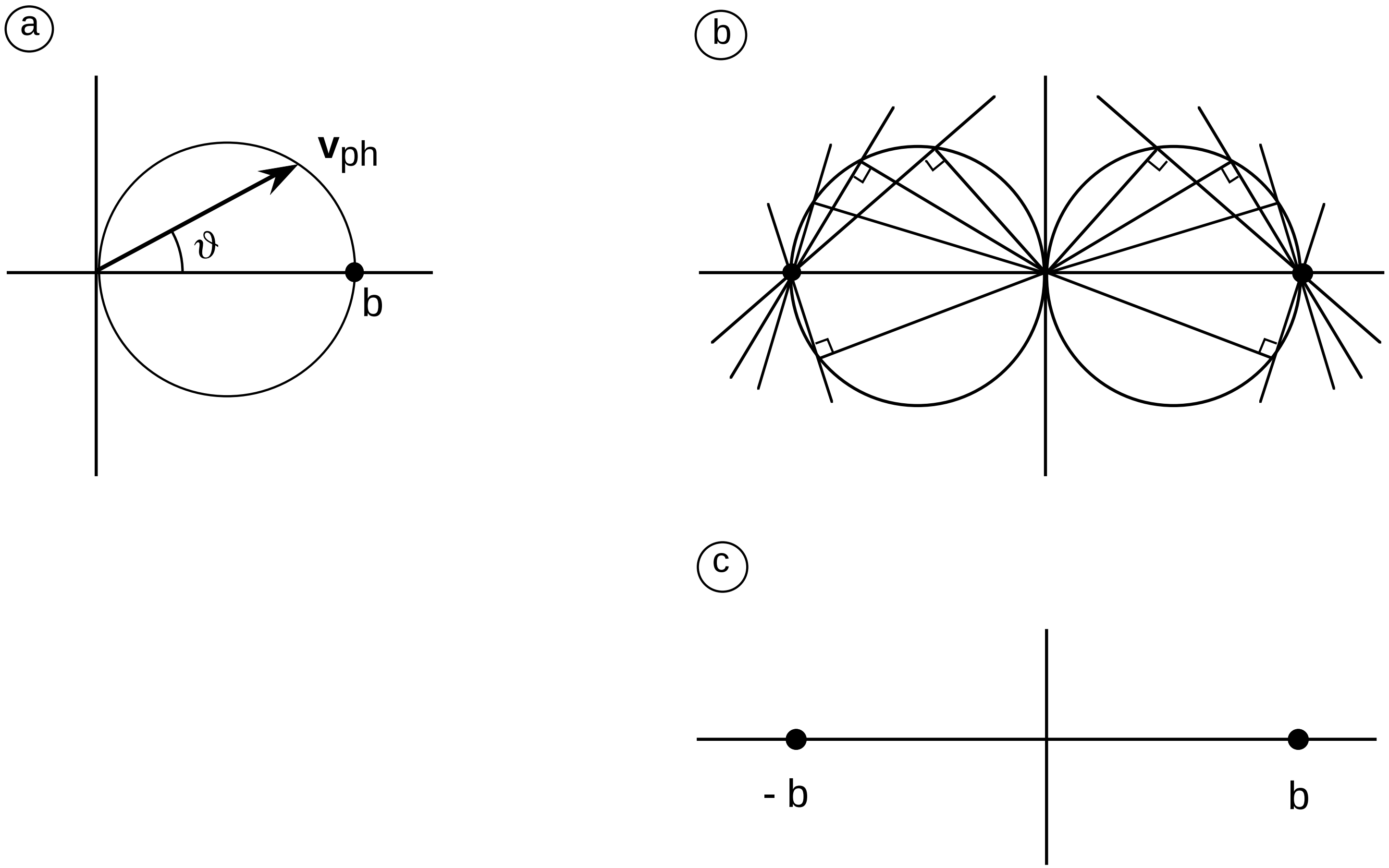
$$\Psi_i(\mathbf{r}, t) \approx e^{i(\mathbf{k}_0 \cdot \mathbf{r} - \omega_0 t)} \times \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} A_i(\mathbf{k})\, e^{i(\mathbf{k}-\mathbf{k}_0)\cdot(\mathbf{r} - (\partial\omega/\partial\mathbf{k})_{\mathbf{k}_0} t)}\, d^3 k \,, \tag{57}$$

representing a carrier wave $\exp i(\mathbf{k}_0 \cdot \mathbf{r} - \omega_0 t)$ with an amplitude-modulated envelope. Through constructive interference of the plane waves, *the envelope maintains its shape* during an extended interval of time, whereas *the surfaces of constant phase of the envelope move with the group velocity,*

$$\mathbf{v}_{\rm gr} = \left(\frac{d\mathbf{r}}{dt} \right)_{\rm const.\ phase} = \left(\frac{\partial \omega}{\partial \mathbf{k}} \right)_{\mathbf{k}_0} , \tag{58}$$

in agreement with the definition (48).

Example: Alfvén waves

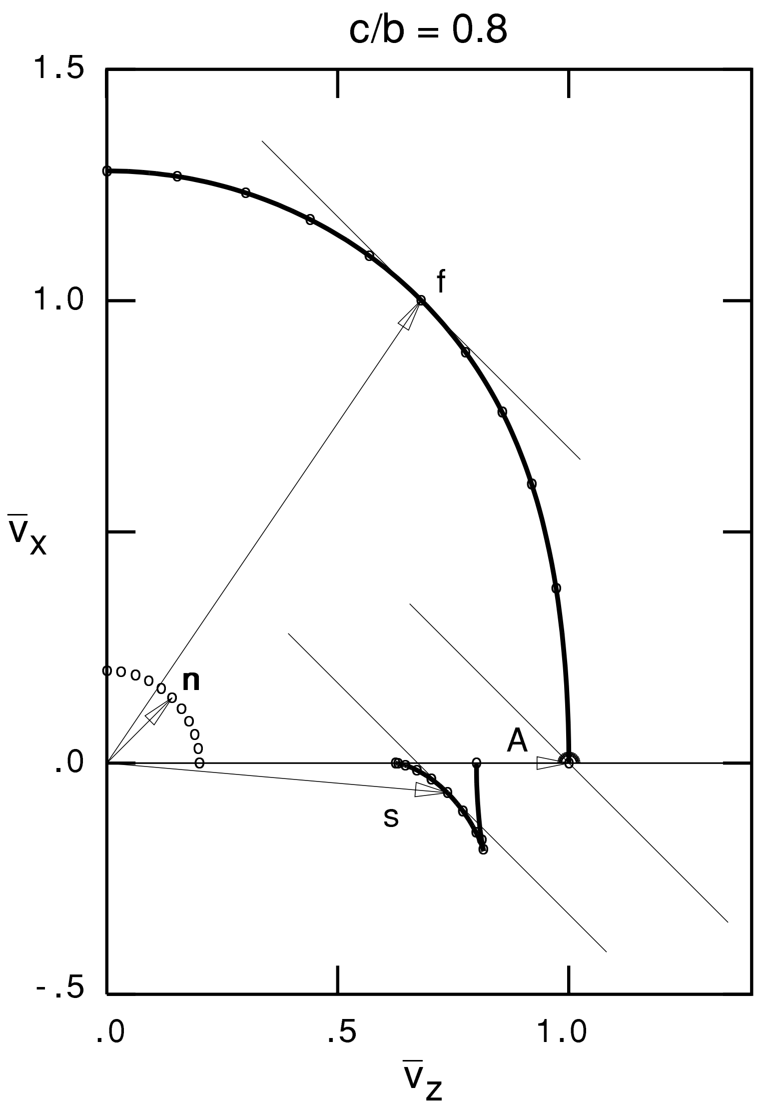


(a) Phase diagram for Alfvén waves is circle $\Rightarrow$ (b) wavefronts pass through points $\pm b$
$\Rightarrow$(c) those points are the group diagram.

Group diagram: queer behavior

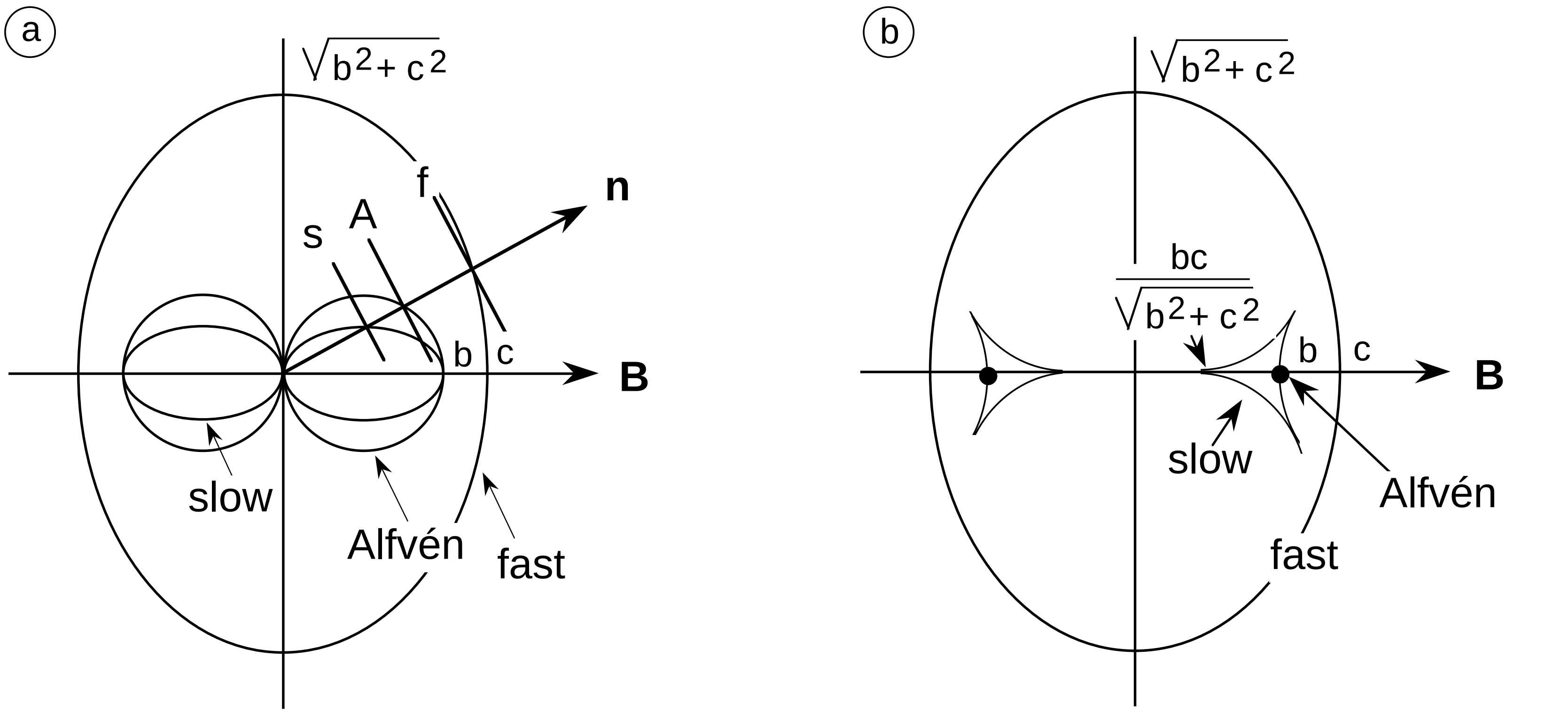
- *Group diagrams with $\mathbf{v}_{\rm gr}$ relative to $\mathbf{n}$ for the three MHD waves in the first quadrant.*

  Group velocities exhibit *mutually exclusive directions of propagation:* When $\mathbf{n}$ goes from $\vartheta = 0$ ($\parallel$ $\mathbf{B}$) to $\vartheta = \pi/2$ ($\perp$ $\mathbf{B}$), the fast group velocity changes from parallel to perpendicular (though it does not remain parallel to $\mathbf{n}$), the Alfvén group velocity remains purely parallel, but the slow group velocity initially changes *clockwise* from parallel to some negative angle and then back again to purely parallel. In the perpendicular direction, *slow wave packages propagate opposite to direction of $\mathbf{n}$!*



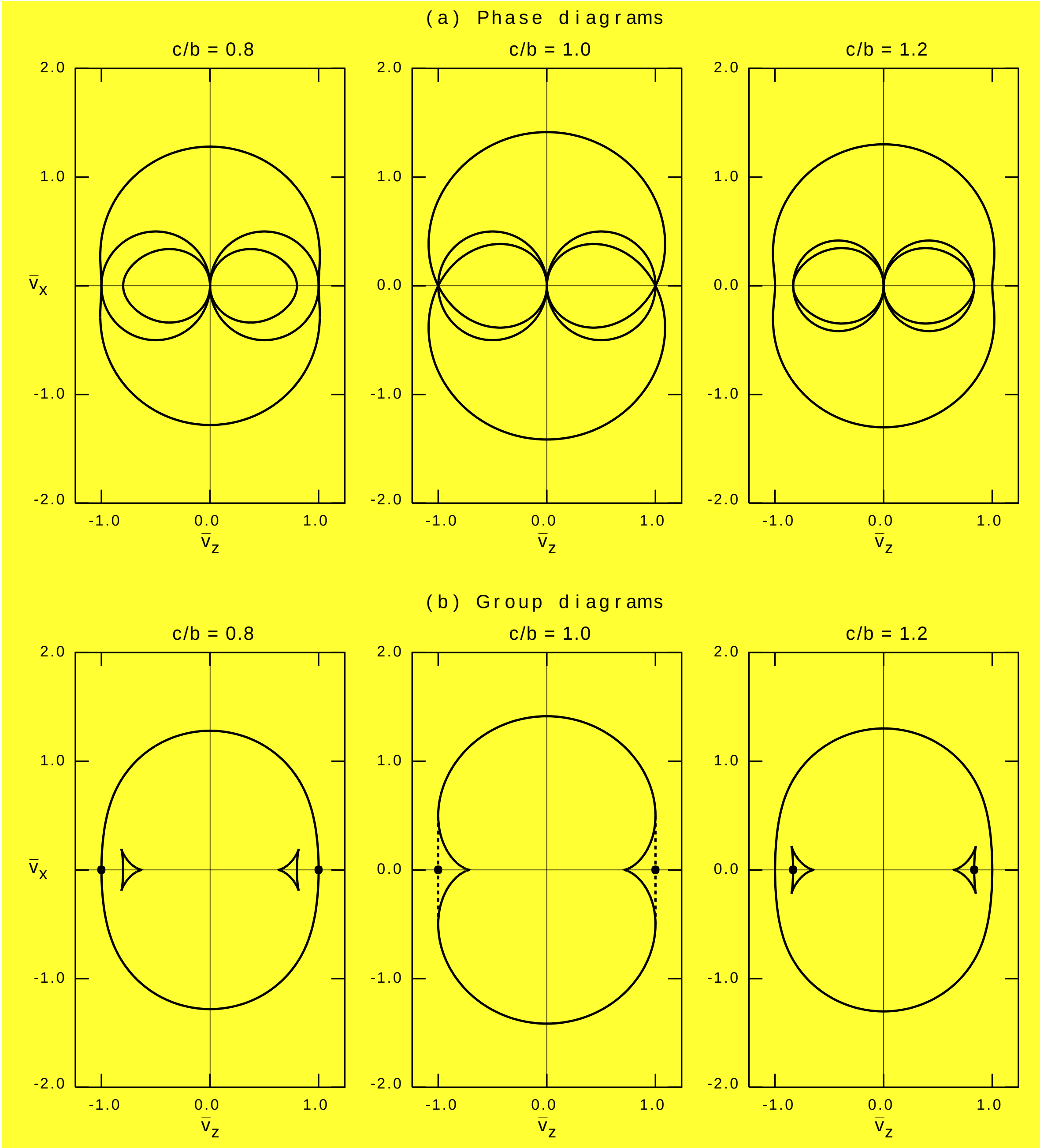
Friedrichs diagrams (schematic)

[exact diagrams in book: Fig. 5.5, parameter $c/b = \frac{1}{2}\gamma\beta\,,\ \beta \equiv 2p/B^2$]



Phase diagram
(plane waves)

Group diagram
(point disturbances)



(a) Phase diagrams and (b) group diagrams of the MHD waves for three values of the ratio c/b of the sound speed to the Alfvén speed. The phase and group velocities are normalised as $V \equiv v/\max(b,c)$.

Method

- *Linear advection equation* in one spatial dimension with unknown $\Psi(x,t)$,

$$\frac{\partial \Psi}{\partial t} + u \frac{\partial \Psi}{\partial x} = 0\,, \tag{59}$$

  and given advection velocity u. For $u = \mathrm{const}$, the solution is trivial:

$$\Psi = f(x - ut)\,, \quad \text{where} \quad f = \Psi_0 \equiv \Psi(x, t=0)\,. \tag{60}$$

  $\Rightarrow$ Initial data Ψ_0 propagate along *characteristics:* parallel straight lines $dx/dt = u$.

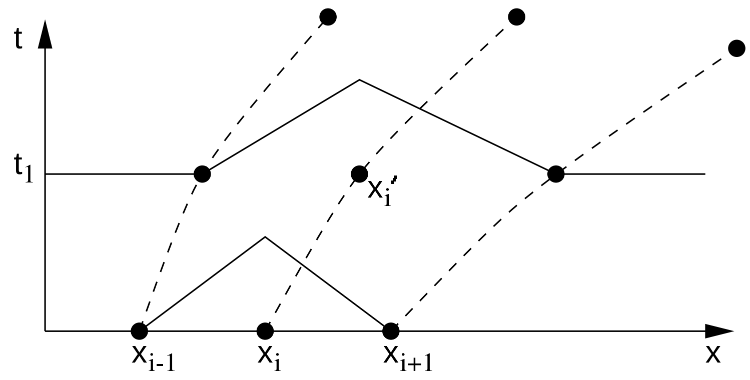
- For u not constant, characteristics become solutions of the ODEs

$$\frac{dx}{dt} = u(x,t)\,. \tag{61}$$

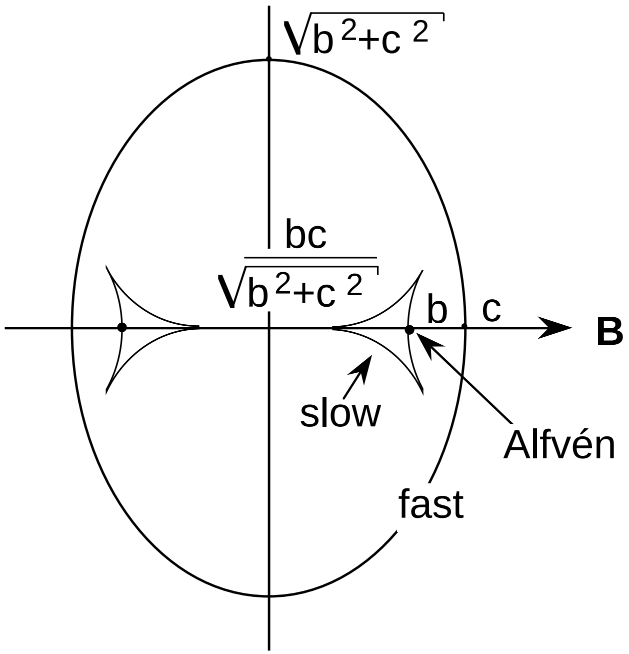
  Along these curves, solution $\Psi(x,t)$ of (59) is const:

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} \equiv \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi}{\partial x}\frac{dx}{dt} = 0\,. \tag{62}$$

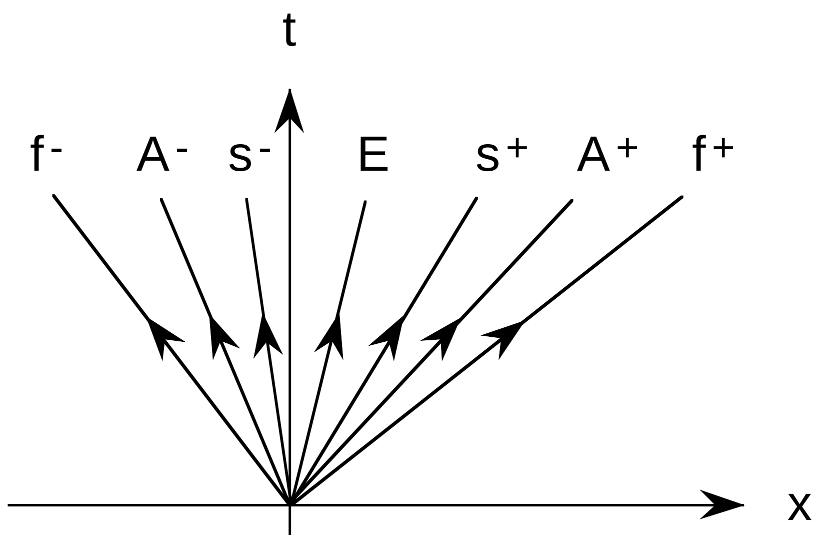
  $\Rightarrow$ For given initial data, the solution can be determined at any time $t_1 > 0$ by constructing *characteristics through suitable set of points.* E.g., $\Psi(x_i', t_1) = \Psi_0(x_i)$ for 'tent' function.



Application (cont'd)



Group diagram is the *ray surface*, i.e. the spatial part of characteristic manifold at certain time t_0.



x-t cross-sections of 7 characteristics (x-axis oblique with respect to $\mathbf{B}$; inclination of entropy mode E indicates plasma background flow).

- *Locality* of group diagrams and characteristics neglects *global plasma inhomogeneity.*
  $\Rightarrow$ Next topic is *waves and instabilities in inhomogeneous plasmas*.

Waves/instab. in inhomogeneous plasmas

Overview

- **Hydrodynamics of the solar interior:** radiative equilibrium model of the Sun, convection zone; [book: Sec. 7.1]

  $\Rightarrow$ *please read at home!*

- **Hydrodynamic waves & instabilities of a gravitating slab:** HD wave equation, convective instabilities, gravito-acoustic waves, helioseismology; [book: Sec. 7.2]

  $\Rightarrow$ *not treated this year*

- **MHD wave equation for a gravitating magnetized plasma slab:** derivation MHD wave equation for gravitating slab, gravito-MHD waves; [book: Sec. 7.3]

  $\Rightarrow$ *not treated this year*

- **Continuous spectrum and spectral structure:** singular differential equations, Alfvén and slow continua, oscillation theorems; [book: Sec. 7.4]

Motivation

- plasma **WAVES and INSTABILITIES** play an important role...
  - in the *dynamics* of plasma perturbations
  - in *energy conversion and transport*
  - in the *heating & acceleration* of plasma

- characteristics (ν, λ, amplitude...) are determined by the ambient plasma

$\Rightarrow$ can be exploited as a *diagnostic tool* for plasma parameters, *e.g.*

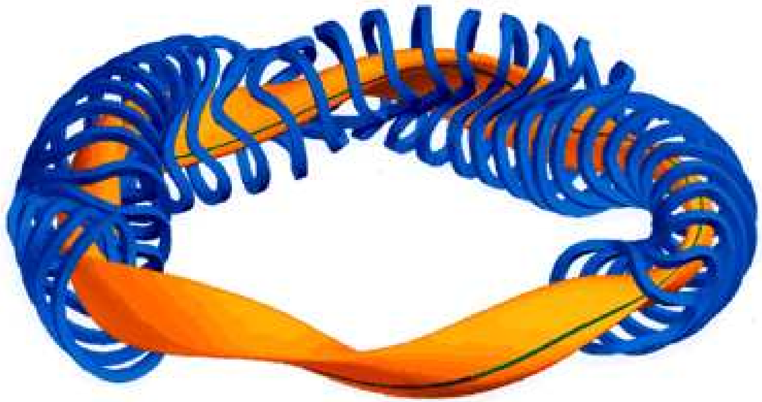
- wave generation, propagation, and dissipation in a confined plasma
  - $\Rightarrow$ *helioseismology* *(e.g. Gough '83)*
  - $\Rightarrow$ *MHD spectroscopy* *(e.g. Goedbloed et al. '93)*
- interaction of external waves with (magnetic) plasma structures
  - $\Rightarrow$ *sunspot seismology* *(e.g. Thomas et al. '82, Bogdan '91)*
  - $\Rightarrow$ *AR / coronal seismology* *(e.g. Nakariakov et al. 2000)*

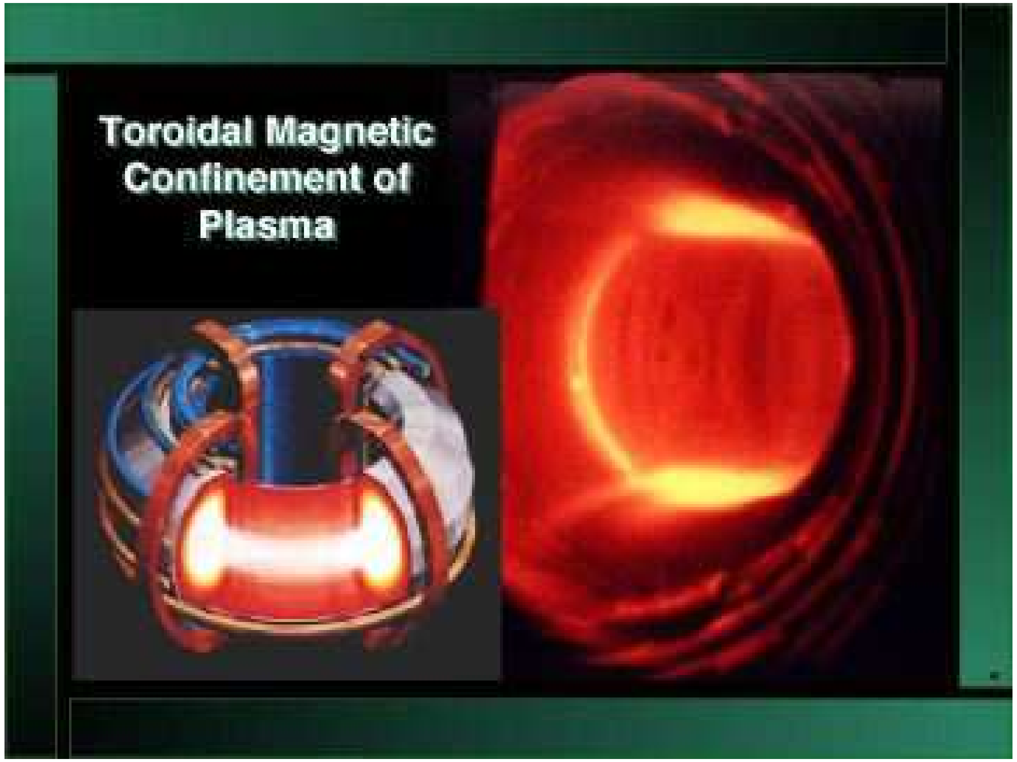
Fusion plasmas

controlled thermo-nuclear fusion:

- tokamaks / stellerators / inertial plasmas, . . .
- MHD spectroscopy

⇒ complicated by **inhomogeneity**



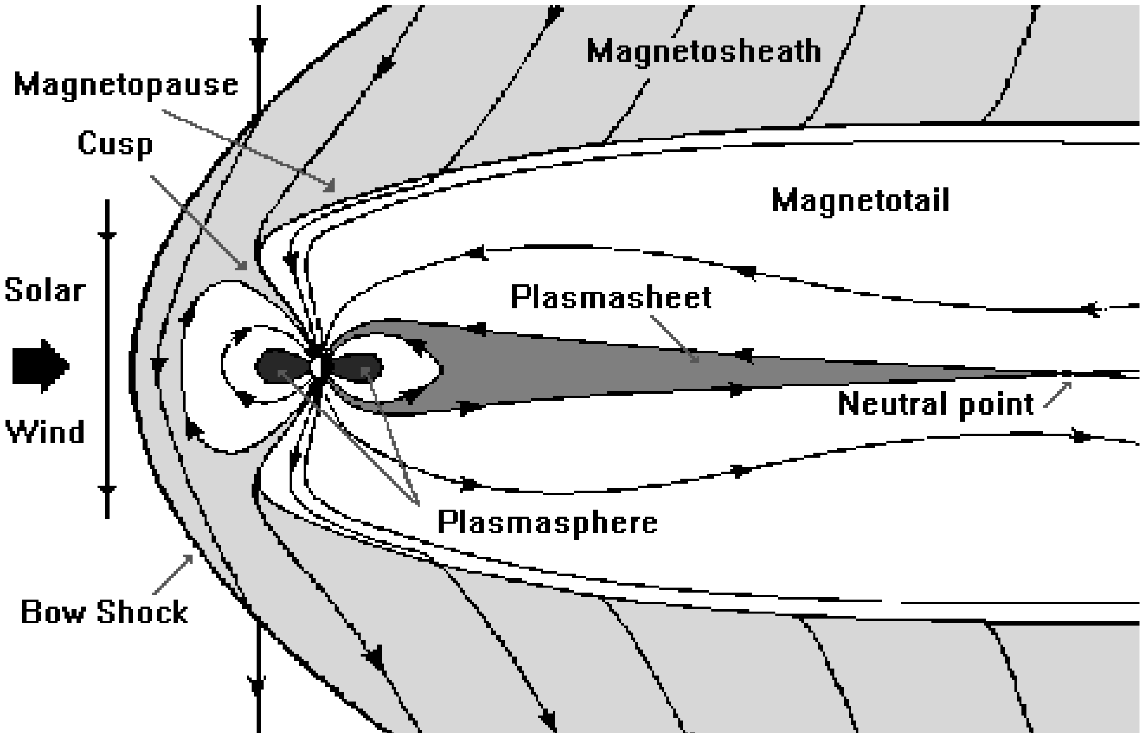


Solar wind – magnetosphere coupling

- interaction of time-varying solar wind with the geomagnetic field near the magnetopause results in *wave mode conversion*

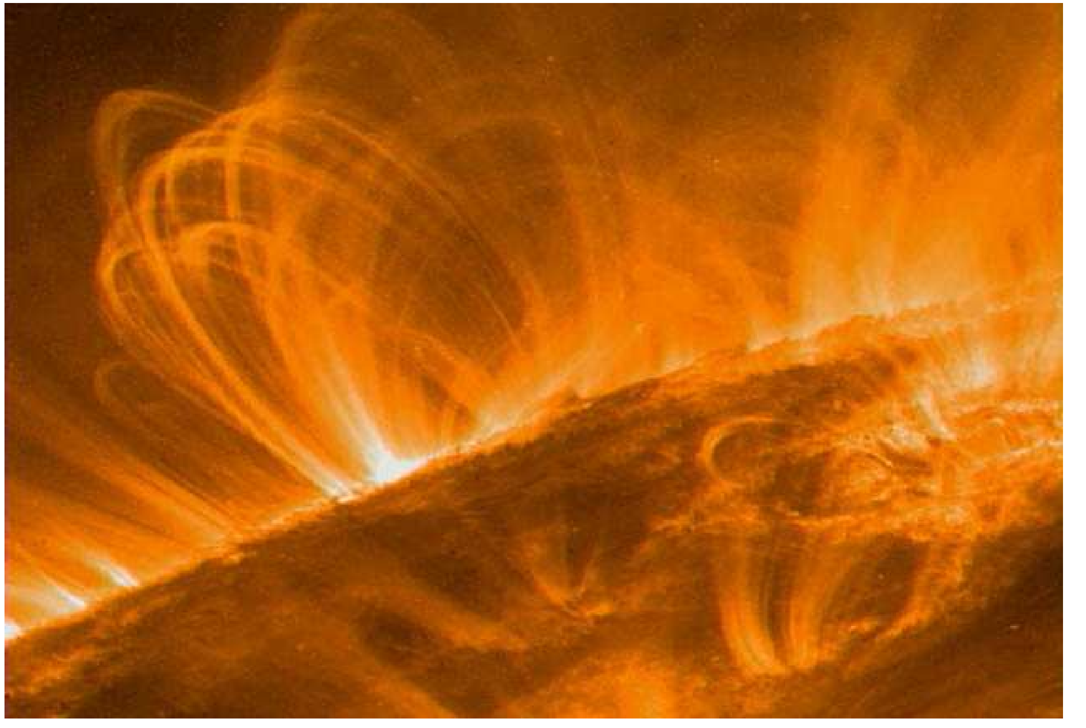
- ultra-low frequency (ULF) waves (periods of seconds to minutes)

⇒ standing AWs with fixed ends in the ionosphere



⇒ interaction of time-varying solar wind with the geomagnetic field near the magnetopause results in *resonant wave mode conversion*

- corona: highly inhomogeneous
  in both space and time
  (Skylab, Yohkoh, Soho)
- structure dominated by magnetic field
- average temperature $2 - 3 \times 10^6\,\mathrm{K}$
- hot material concentrated in loops *(Rossner et al. '78)*
- TRACE: $\frac{n_{loop}}{n_{backgr}} \sim 10$ *(Aschwanden '00)*
- outline magnetic field *(Orrall '81)*



Hot coronal loops (TRACE)

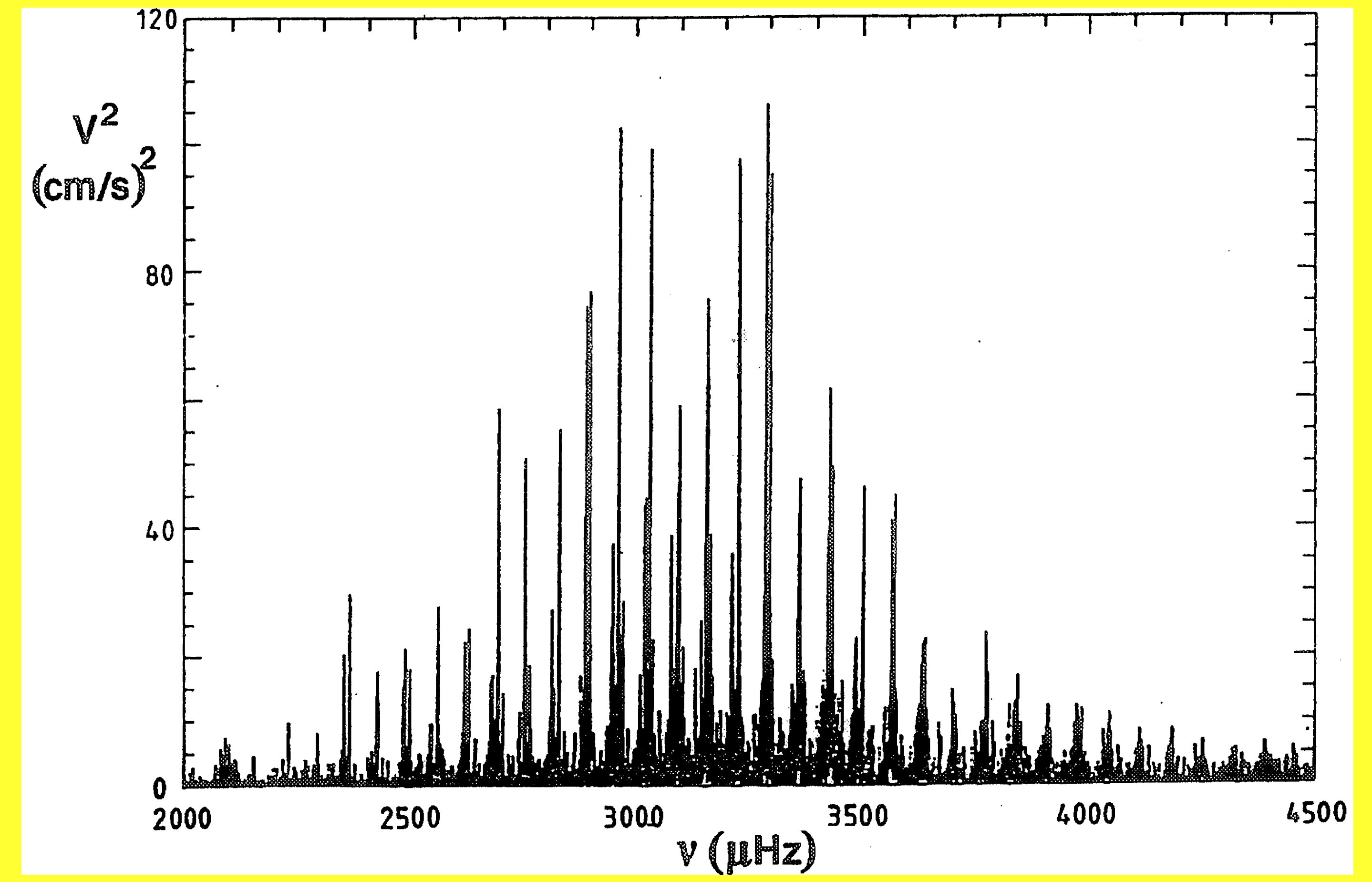
⇒ *waves?* *(generation, propagation, dissipation?)*

⇒ *heating mechanism(s)?* *(what is the role of waves?)*

- **magnetohydrodynamics (MHD)**: simplest ⇒ most popular

Helioseismology

- Power spectrum of solar oscillations, from Doppler velocity measurements in light integrated over solar disk (Christensen-Dalsgaard, *Stellar Oscillations,* 1989):



⇒ *Powerful tool for probing the interior of the sun!*

- Done by comparison with theoretically calculated spectrum for standard solar model (of course, spherical geometry) (Christensen-Dalsgaard, 1989).

- Orders of magnitude :

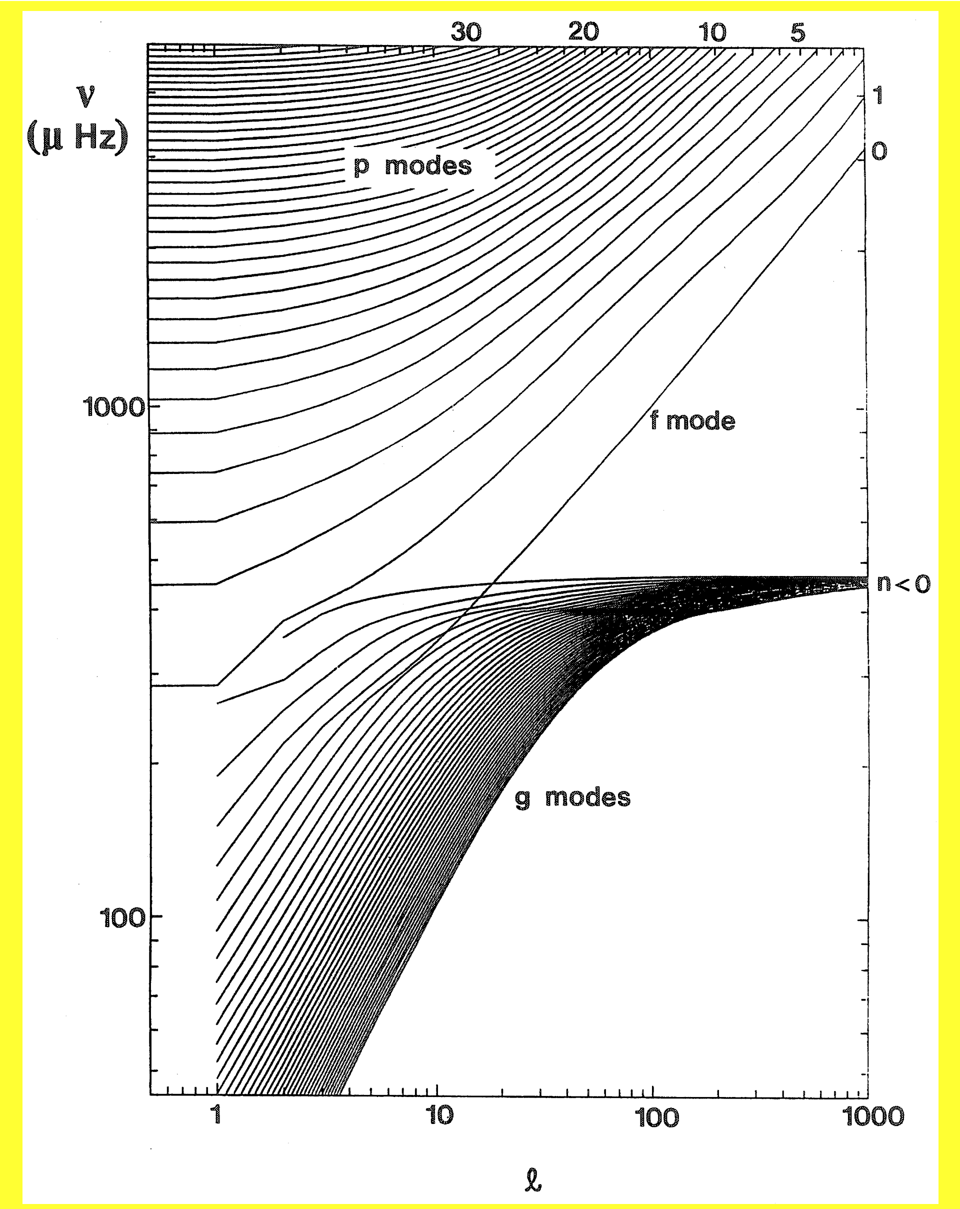
  $\tau \sim 5\,\mathrm{min} \quad \Rightarrow \quad \nu \sim 3\,\mathrm{mHz}$

  $\tilde{v}_r < 1\mathrm{km/s} \approx 5 \times 10^{-4} R_\odot / 5\,\mathrm{min}$

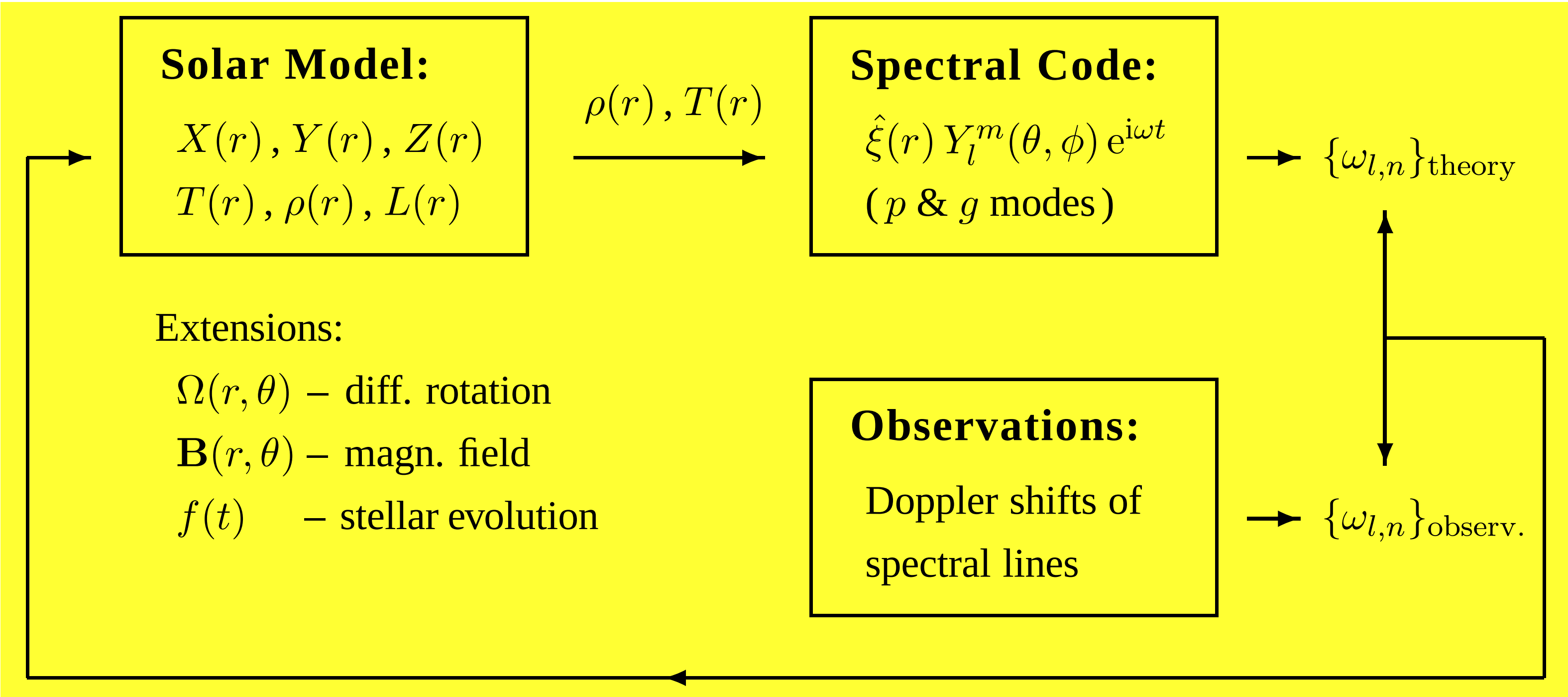
  $\Rightarrow$ linear theory OK!

- p-modes of low order l penetrate deep in the Sun, high l modes are localized on outside. g-modes are *cavity modes* trapped deeper than convection zone and, hence, quite difficult to observe.

- Frequencies deduced from the Doppler shifts of spectral lines agree with calculated ones for p-modes to within 0.1%!


ν
(μ Hz)
1000
100
30
20
10
5
1
0
p modes
f mode
g modes
n < 0
1
10
100
1000
ℓ

Systematics of helioseismology



- Similar activities:

  – *MHD spectroscopy* for laboratory fusion plasmas (Goedbloed et al., 1993),

  – *Sunspot seismology* (Bogdan and Braun, 1995),

  – *Magneto-seismology of accretion disks* (Keppens et al., 2002).

Different approaches

- the system of linear PDEs $\boxed{\mathbf{L}\cdot\frac{\partial \mathbf{u}}{\partial t} = \mathbf{R}\cdot\mathbf{u}}$ can be approached in three different ways (after spatial discretization of $\mathbf{L}$ and $\mathbf{R}$):

  1) steady state approach: t-dependence is *prescribed*, e.g. $\sim e^{i\omega_d t}$

     $\Rightarrow$ linear algebraic system: $(\mathbf{A} - i\omega_d\mathbf{B})\cdot\mathbf{x} = \mathbf{f}$

     with force $\mathbf{f}$: from BCs (driver)

  2) time evolution approach: t-dependence is *calculated*

     $\Rightarrow$ initial value problem: $\mathbf{A}\cdot\mathbf{x} = \mathbf{B}\cdot\frac{\partial \mathbf{x}}{\partial t}$ with $\mathbf{x}(\mathbf{r}, t = 0)$ given

     $\Rightarrow$ 'driven' problem: $\mathbf{A}\cdot\mathbf{x} = \mathbf{B}\cdot\frac{\partial \mathbf{x}}{\partial t} + \mathbf{f}$

  3) eigenvalue approach: t-dependence $\sim e^{\lambda t}$, with λ calculated

     $\Rightarrow$ eigenvalue problem: $(\mathbf{A} - \lambda\mathbf{B})\cdot\mathbf{x} = 0$

- Starting point is the general *MHD spectral equation:*

$$\mathbf{F}(\boldsymbol{\xi}) \equiv -\nabla\pi - \mathbf{B}\times(\nabla\times\mathbf{Q}) + (\nabla\times\mathbf{B})\times\mathbf{Q} + \nabla\Phi\,\nabla\cdot(\rho\boldsymbol{\xi}) = \rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\rho\omega^2\boldsymbol{\xi}\,, \qquad (1)$$

where $\quad \pi = -\gamma p\nabla\cdot\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla p\,, \qquad \mathbf{Q} = \nabla\times(\boldsymbol{\xi}\times\mathbf{B})\,. \qquad (2)$

Aside:

- Recall *homogeneous* plasmas (Chap. 5) with plane wave solutions $\hat{\boldsymbol{\xi}}(\mathbf{k})\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})$:

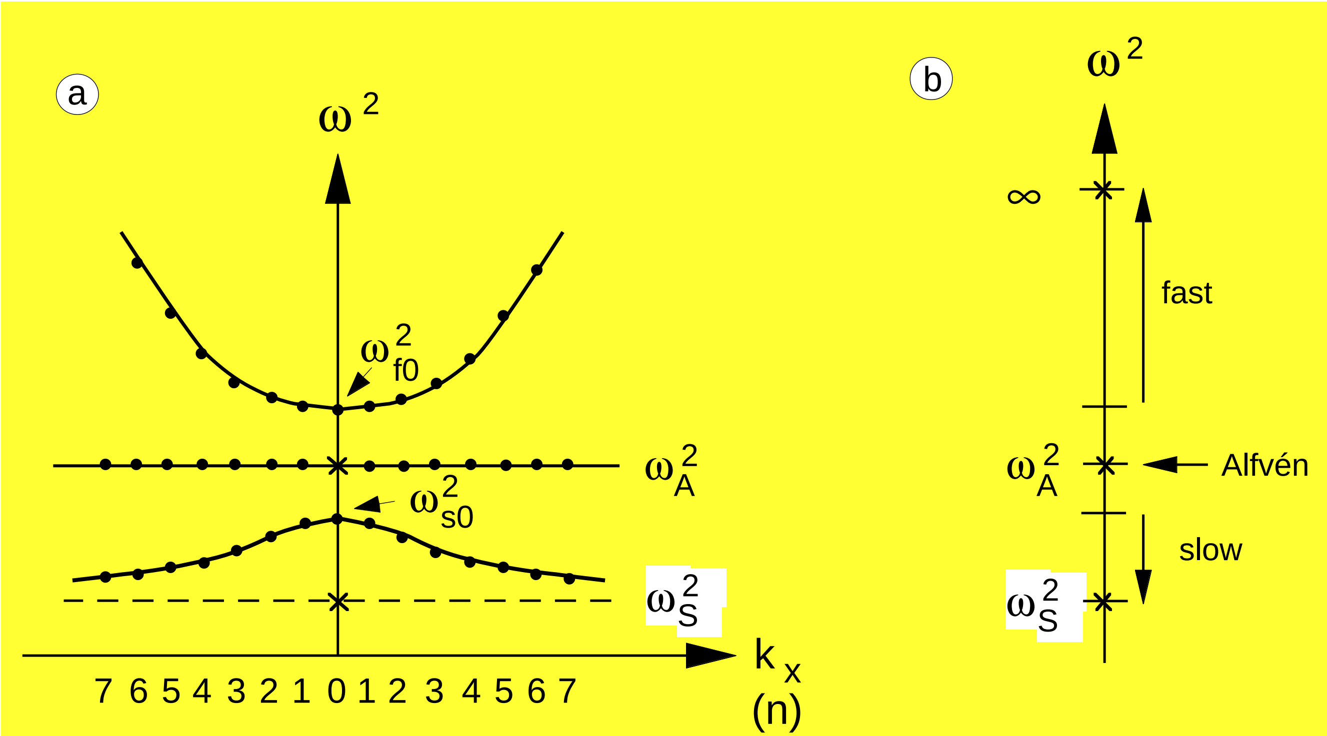
$$\rho^{-1}\mathbf{F}(\hat{\boldsymbol{\xi}}) = \big[-(\mathbf{k}\cdot\mathbf{b})^2\,\mathbf{I} - (b^2+c^2)\,\mathbf{k}\mathbf{k} + \mathbf{k}\cdot\mathbf{b}\,(\mathbf{k}\mathbf{b}+\mathbf{b}\mathbf{k})\big]\cdot\hat{\boldsymbol{\xi}} = -\omega^2\hat{\boldsymbol{\xi}}\,. \qquad (3)$$

In components:

$$\begin{pmatrix} -k_x^2(b^2+c^2) - k_z^2 b^2 & -k_x k_y(b^2+c^2) & -k_x k_z c^2 \\ -k_x k_y(b^2+c^2) & -k_y^2(b^2+c^2) - k_z^2 b^2 & -k_y k_z c^2 \\ -k_x k_z c^2 & -k_y k_z c^2 & -k_z^2 c^2 \end{pmatrix} \begin{pmatrix} \xi_x \\ \xi_y \\ \xi_z \end{pmatrix} = -\omega^2 \begin{pmatrix} \xi_x \\ \xi_y \\ \xi_z \end{pmatrix}. \qquad (4)$$

Corresponds to Eq. (5.35) [book (5.52)] with $k_y \neq 0$: Coordinate system rotated to distinguish between k_x (becomes differential operator in inhomogeneous systems) and k_y (remains number).

- Dispersion diagram $\omega^2(k_x)$ exhibits relevant asymptotics for $k_x \to \infty$:



Yields the essential spectrum:

$$\omega_F^2 \equiv \lim_{k_x\to\infty} \omega_f^2 \approx \lim_{k_x\to\infty} k_x^2(b^2+c^2) = \infty\,, \quad \text{(fast cluster point)} \tag{5}$$

$$\omega_A^2 \equiv \lim_{k_x\to\infty} \omega_a^2 = \omega_a^2 = k_\parallel^2 b^2\,, \quad \text{(Alfvén infinitely degenerate)} \tag{6}$$

$$\omega_S^2 \equiv \lim_{k_x\to\infty} \omega_s^2 = k_\parallel^2 \frac{b^2c^2}{b^2+c^2}\,. \quad \text{(slow cluster point)} \tag{7}$$

End aside

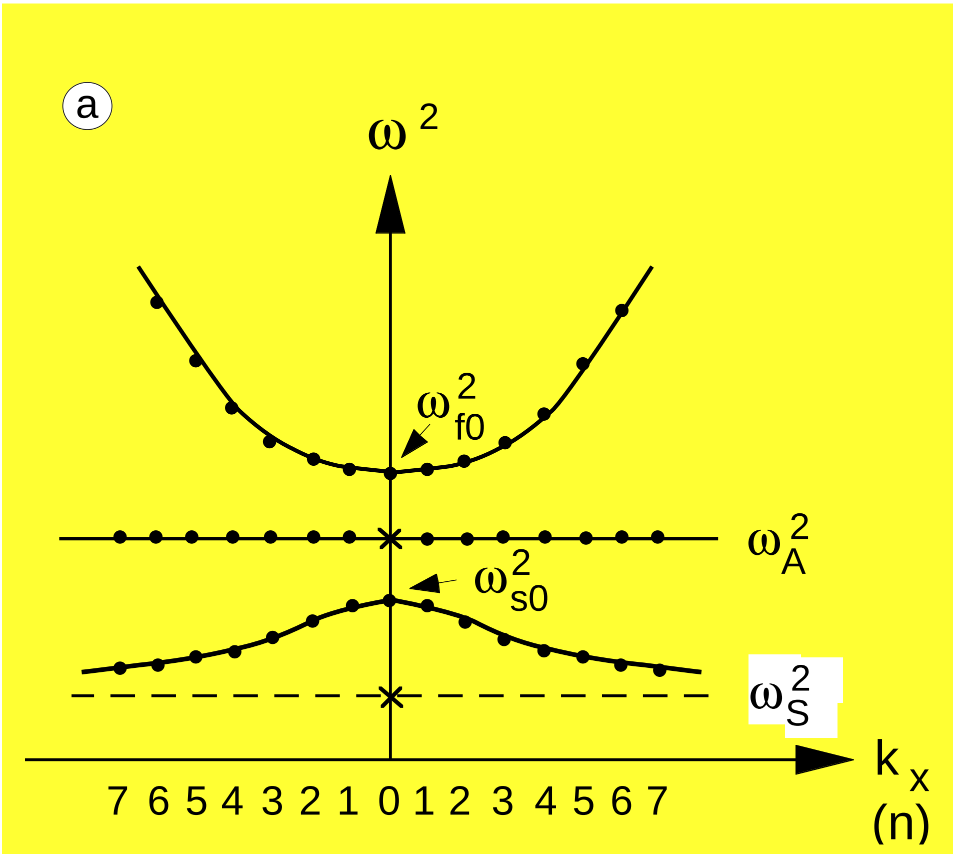
Finite homogeneous plasma slab

- equilibrium: $\mathbf{B}_0 = B_0 \mathbf{e}_z$
  - with $\rho_0 \, , p_0 \, , B_0 = \mathrm{const}$
  - enclosed by plates at $x = \pm a$
- normal modes: $\sim \exp(-i\omega\, t)$

  $\Rightarrow$ *eigenvalueproblem*
- plane wave solutions $\sim \exp(\vec{k} \cdot \vec{x})$

  $\Rightarrow$ $k_x = \frac{\pi}{a} n$ is quantized

$\Rightarrow$ **three MHD waves: FMW, AW, SMW**



Dispersion diagram $\omega^2 = \omega^2(k_x)$ for k_y and k_z fixed

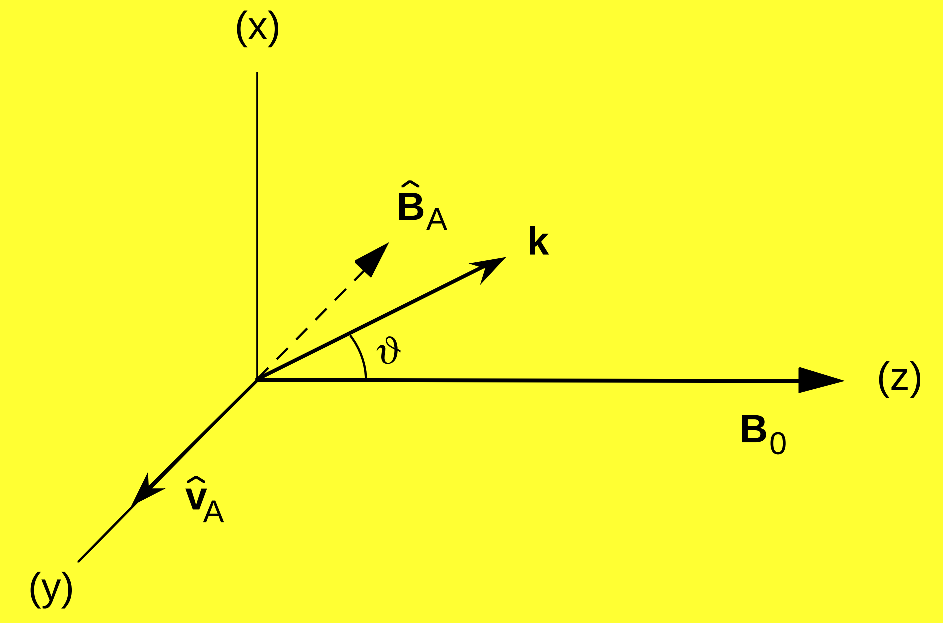
Alfvén waves

- *eigenfrequency*: $\omega = \pm\omega_A$

$$\omega_A \equiv k_\parallel b = k\, b\, \cos\vartheta$$

with $\quad b = \dfrac{B_0}{\sqrt{\rho_0}}$

- *eigenfunctions*:

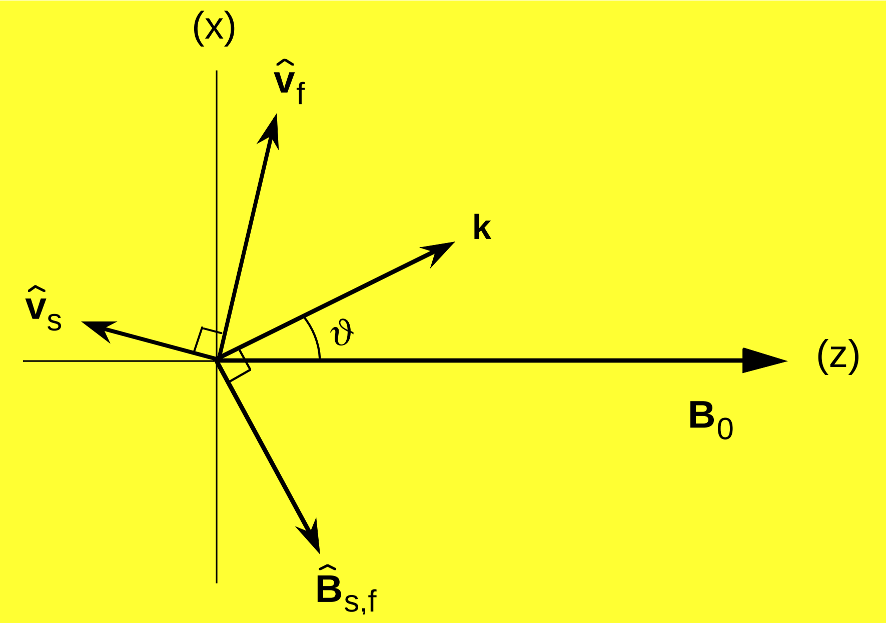


Fast & slow magnetoacoustic waves

- *eigenfrequency*: $\omega = \pm\omega_{s,f}$

$$\omega_{s,f} \equiv k\sqrt{\tfrac{1}{2}(b^2+c^2) \pm \tfrac{1}{2}\sqrt{(b^2+c^2)^2 - 4(k_\parallel^2/k^2)\, b^2c^2}}$$

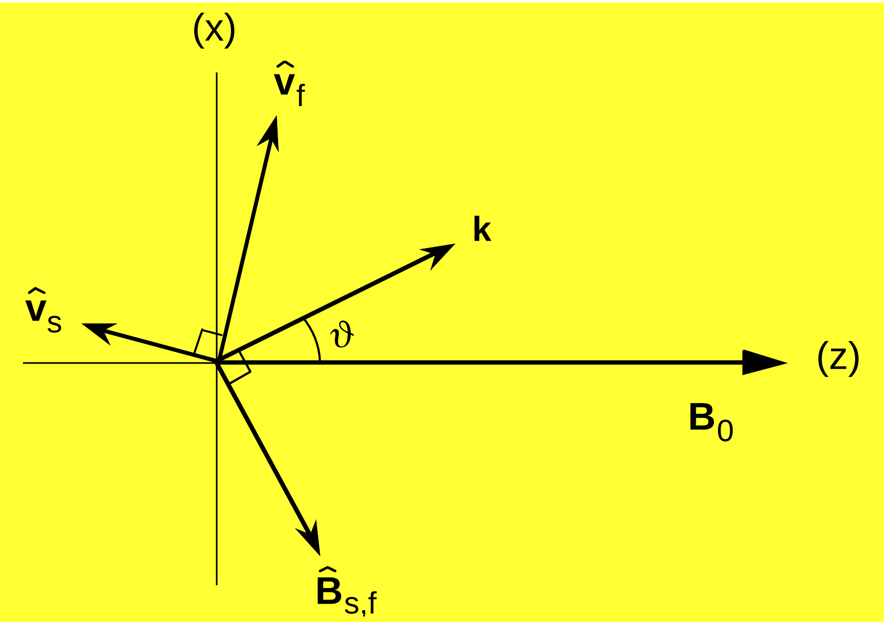
- *eigenfunctions*:



- *the eigenfunctions are mutually orthogonal*:

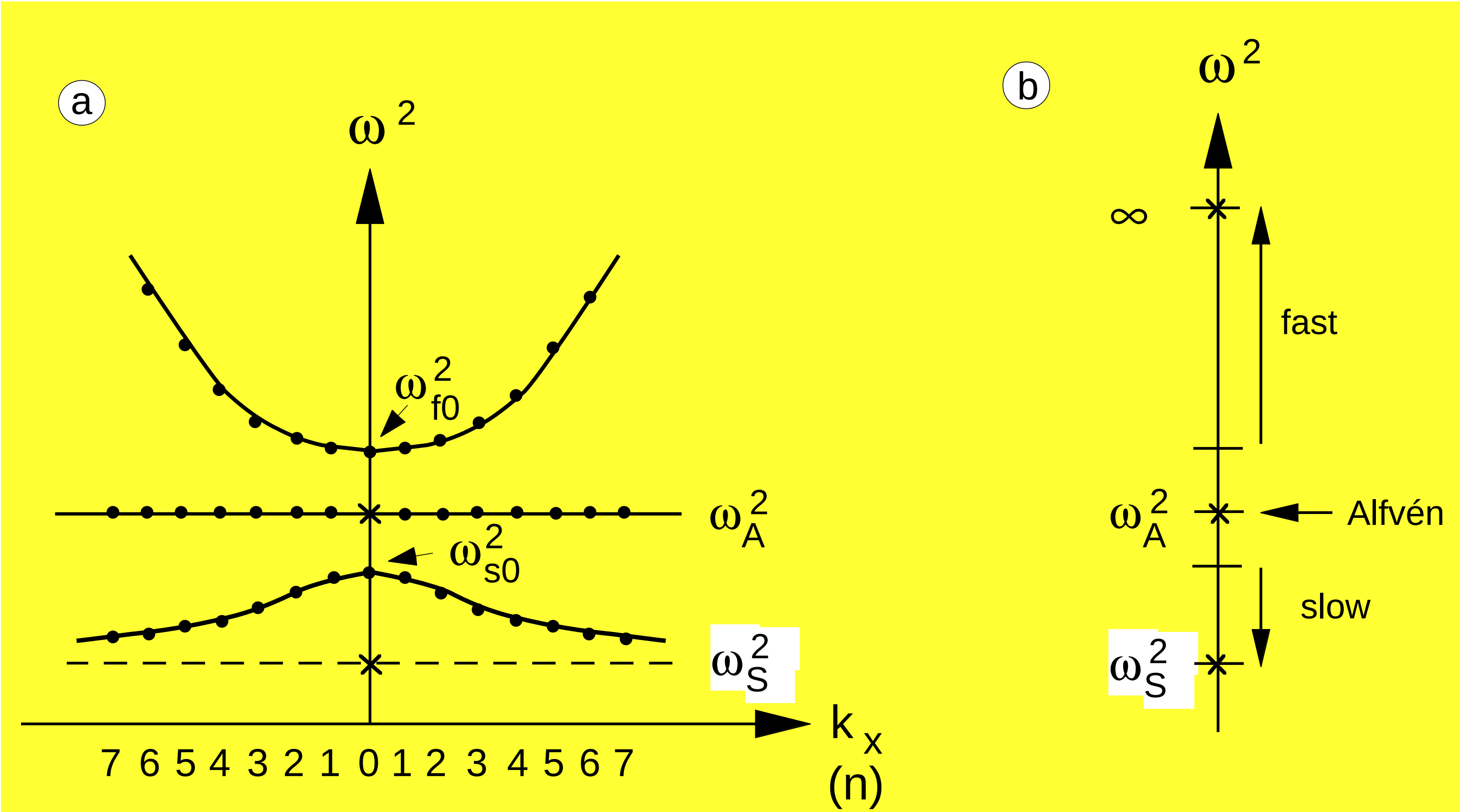
$$\hat{\mathbf{v}}_s \perp \hat{\mathbf{v}}_A \perp \hat{\mathbf{v}}_f$$

$\Rightarrow$ arbitrary velocity field may be decomposed in the three waves!



- **Remark**: for $\theta = 0$ the FMW is polarized almost perpendicular to $\vec{B}_0$ but *in* the $(\vec{k}, \vec{B}_0)$-plane

$\Rightarrow$ corresponds to the direction *normal to the magnetic flux surfaces* in the inhomogeneous plasmas discussed below



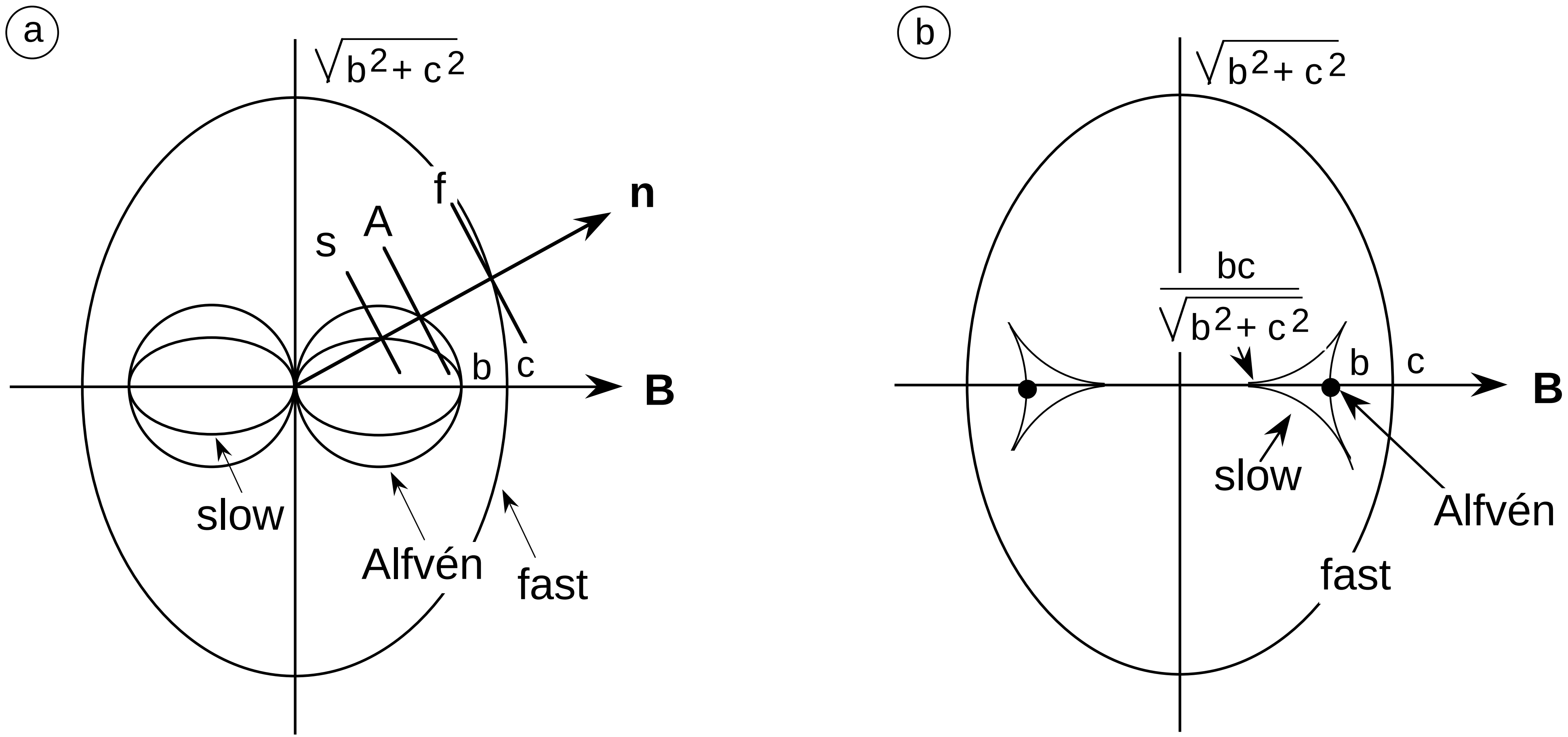
(a) Dispersion diagram $\omega^2 = \omega^2(k_x)$ for k_y and k_z fixed; (b) Corresponding structure of the spectrum.

- *the eigenfrequencies are well-ordered*:

$$0 \leq \omega_s^2 \leq \omega_{s0}^2 \leq \omega_A^2 \leq \omega_{f0}^2 \leq \omega_f^2 < \infty$$

$\Rightarrow$ crucial for spectral theory of MHD waves!

- three MHD waves exhibit a strong anisotropy depending on the direction of the wave vector $\mathbf{k}$ with respect to the magnetic field $\mathbf{B}_0$



Friedrichs diagrams: Schematic representation of (a) reciprocal normal surface (or phase diagram) and (b) ray surface (or group diagram) of the MHD waves ($b < c$).

Finite inhomogeneous plasma slab

- $\mathbf{B}_0 = B_{0y}(x)\,\mathbf{e}_y + B_{0z}(x)\,\mathbf{e}_z\,, \quad \rho_0 = \rho_0(x)\,, \quad p_0 = p_0(x)$

- influence of inhomogeneity on the spectrum of MHD waves?

  $\Rightarrow$ different $\mathbf{k}$'s couple $\quad\Rightarrow\quad$ *wave transformations can occur*

  (e.g. fast wave character in one place, Alfvén character in another)

  $\Rightarrow$ two new phenomena, viz. *instabilities* and *continuous spectra*

- wave or spectral equation can be written in terms of

$$\xi \equiv \mathbf{e}_x \cdot \boldsymbol{\xi} = \xi_x, \quad \eta \equiv i\mathbf{e}_\perp \cdot \boldsymbol{\xi}, \quad \zeta \equiv i\mathbf{e}_\parallel \cdot \boldsymbol{\xi}$$

$\Rightarrow$ eliminate η and ζ with 2nd and 3rd component (algebraic in η and ζ):

$$\boxed{\frac{d}{dx}\frac{N}{D}\frac{d\xi}{dx} + \left[\,\rho(\omega^2 - f^2 b^2)\,\right]\xi = 0}$$

(Hain, Lust, Goedbloed equation)

$\Rightarrow$ resulting wave or spectral equation for a plane slab:

$$\begin{pmatrix} \frac{d}{dx}(\gamma p + B^2)\frac{d}{dx} - f^2B^2 & \frac{d}{dx}g(\gamma p + B^2) & \frac{d}{dx}f\gamma p \\ -g(\gamma p + B^2)\frac{d}{dx} & -g^2(\gamma p + B^2) - f^2B^2 & -gf\gamma p \\ -f\gamma p\frac{d}{dx} & -f\,g\,\gamma p & -f^2\gamma p \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = -\rho\omega^2 \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}$$

where $\quad \xi \equiv \mathbf{e}_x \cdot \boldsymbol{\xi} = \xi_x, \quad \eta \equiv i\mathbf{e}_\perp \cdot \boldsymbol{\xi}, \quad \zeta \equiv i\mathbf{e}_\parallel \cdot \boldsymbol{\xi}$

$\Rightarrow$ eliminate η and ζ with 2nd and 3rd component (algebraic in η and ζ):

$$\boxed{\frac{d}{dx}\frac{N}{D}\frac{d\xi}{dx} + [\,\rho(\omega^2 - f^2b^2)\,]\,\xi = 0}$$

where $\quad N = N(x;\omega^2) \equiv \rho(\omega^2 - f^2b^2)\,\left[\,(b^2 + c^2)\,\omega^2 - f^2b^2c^2\,\right]$

$$D = D(x;\omega^2) \equiv \omega^4 - k_0^2(b^2 + c^2)\,\omega^2 + k_0^2 f^2 b^2 c^2$$

- the coefficient factor N/D of the ODE plays an important role in the analysis

$\Rightarrow$ may be written in terms of the four ω^2's introduced for homogeneous plasmas:

$$\boxed{\frac{N}{D} = \rho(b^2+c^2)\,\frac{[\,\omega^2-\omega_A^2(x)\,]\,[\,\omega^2-\omega_S^2(x)\,]}{[\,\omega^2-\omega_{s0}^2(x)\,]\,[\,\omega^2-\omega_{f0}^2(x)\,]}}$$

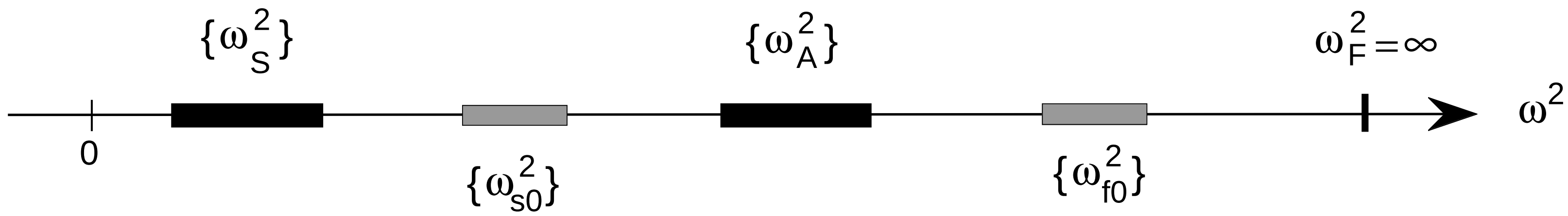
where

$$\omega_A^2(x) \equiv f^2 b^2 \equiv F^2/\rho\,, \qquad \omega_S^2(x) \equiv f^2\frac{b^2c^2}{b^2+c^2} \equiv \frac{\gamma p}{\gamma p + B^2}F^2/\rho\,,$$

$$\omega_{s0,f0}^2(x) \equiv \tfrac{1}{2}k_0^2(b^2+c^2)\left[\,1 \pm \sqrt{1-\frac{4f^2b^2c^2}{k_0^2(b^2+c^2)^2}}\,\right]$$

$\Rightarrow$ only **two continuous spectra** (2 apparent singularities)

$\Rightarrow$ the four finite 'limiting frequencies' now *spread out to a continuous range* :

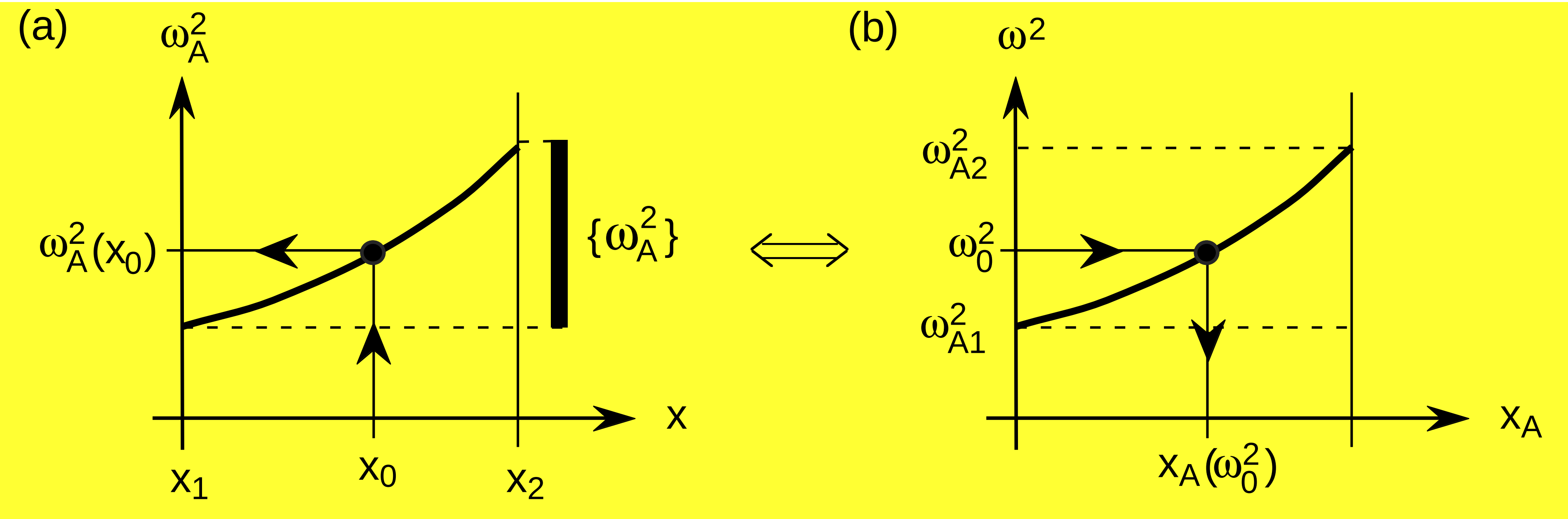


The continuous spectrum

- assume slow and Alfvén continuum do NOT overlap

- assume Alfvén frequency is monotone

$\Rightarrow$ inversion of the Alfvén frequency function: (a) $\omega_A^2 = \omega_A^2(x)$; (b) $x_A = x_A(\omega^2)$



- expansion about the singularity:

$$\omega^2 - \omega_A^2(x) \approx -\omega_A^{2\,\prime}(x_s)\,(x - x_s) = -\omega_A^{2\,\prime}(x_s)\,[\,x - x_A(\omega^2)\,]$$

$\Rightarrow$ close to the singularity $s \equiv x - x_A(\omega^2) = 0$, the ODE then reduces to

$$\frac{d}{ds}[\,s\,(1+\cdots)\,\frac{d\xi}{ds}\,] - \alpha\,(1+\cdots)\,\xi = 0$$

$\Rightarrow$ indicial equation $\nu^2 = 0$, so that the indices are equal: $\nu_1 = \nu_2 = 0$

$$\Rightarrow \begin{cases} \xi_1 = u(s;\omega^2) & (\textit{small}\ \text{solution}) \\ \xi_2 = u(s;\omega^2)\ln|s| + v(s;\omega^2) & (\textit{large}\ \text{solution}) . \end{cases}$$

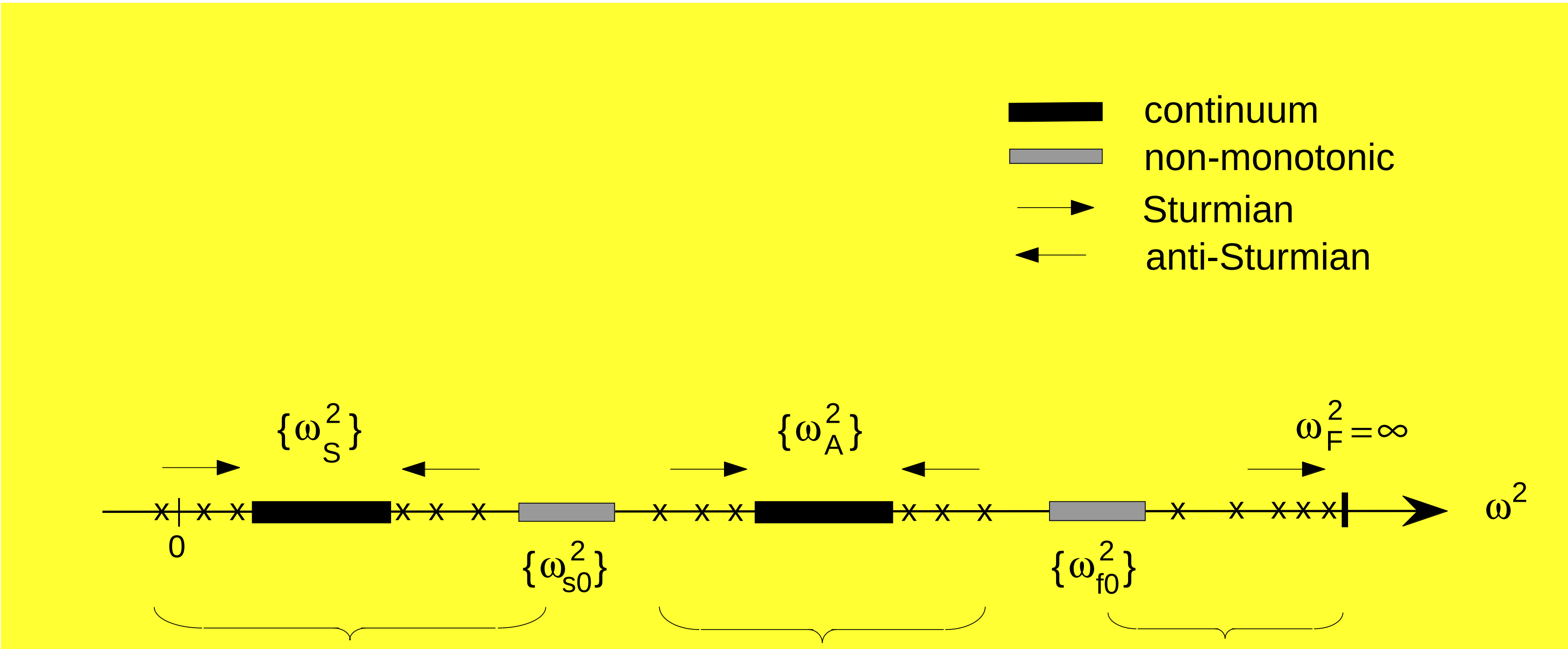
- since

$$\eta \sim (\omega^2 - \omega_S^2)\,\xi'\,, \qquad \zeta \sim (\omega^2 - \omega_A^2)\,\xi'$$

- logarithmic contribution in ξ-component

$\Rightarrow$ but the dominant *(non-square integrable)* part of the eigenfunctions:

$$\xi_A \approx 0\,, \qquad \eta_A \approx \mathcal{P}\,\frac{1}{x - x_A(\omega^2)} + \lambda(\omega^2)\,\delta(x - x_A(\omega^2))\,, \qquad \zeta_A \approx 0\,,$$
$$\xi_S \approx 0\,, \qquad \eta_S \approx 0\,, \qquad \zeta_S \approx \mathcal{P}\,\frac{1}{x - x_S(\omega^2)} + \lambda(\omega^2)\,\delta(x - x_S(\omega^2))\,,$$



Schematic structure of the spectrum of an inhomogeneous plasma with gravity.

- MHD example: resistive MHD spectrum of a cylindrical plasma column *(from Poedts et al. '89)*

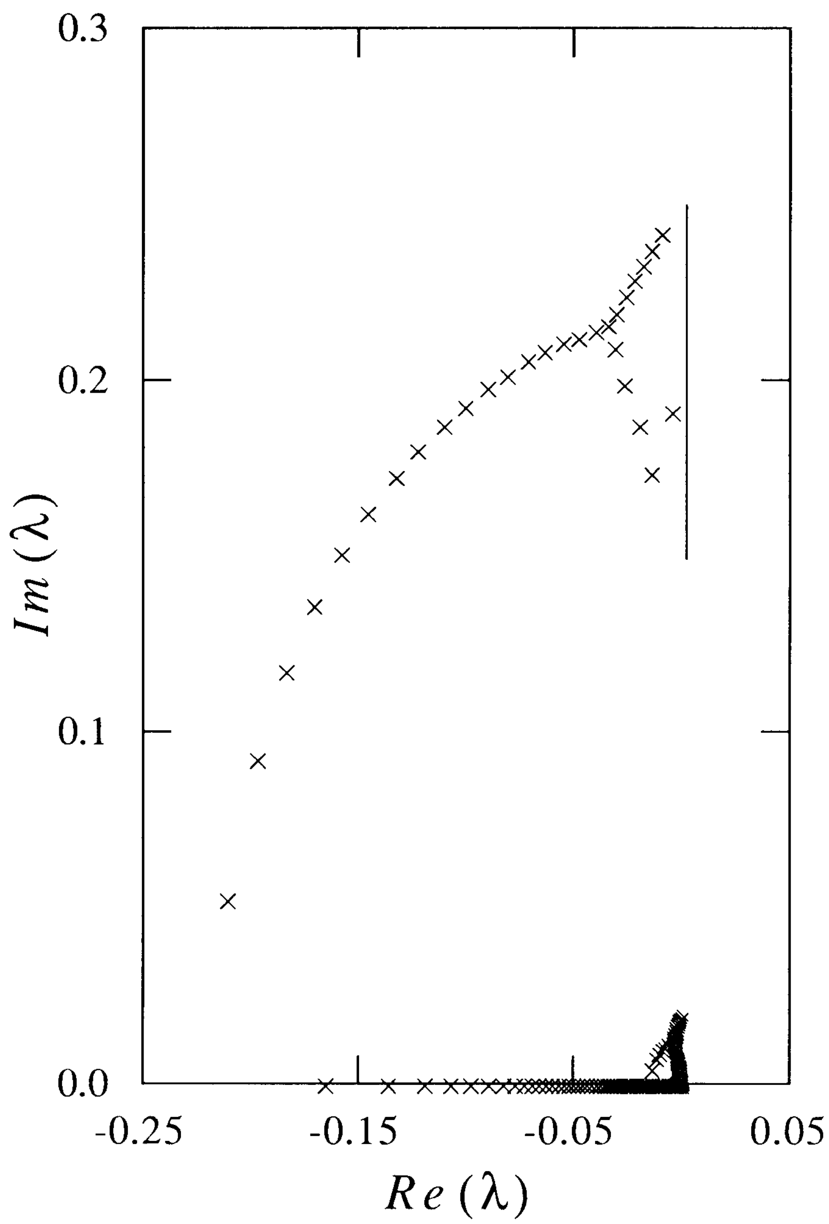
  (only Alfvén and slow magnetosonic subspectrum are shown)

$\Rightarrow$ resistive modes lie on fixed curves in complex frequency plane (independent of resistivity!)

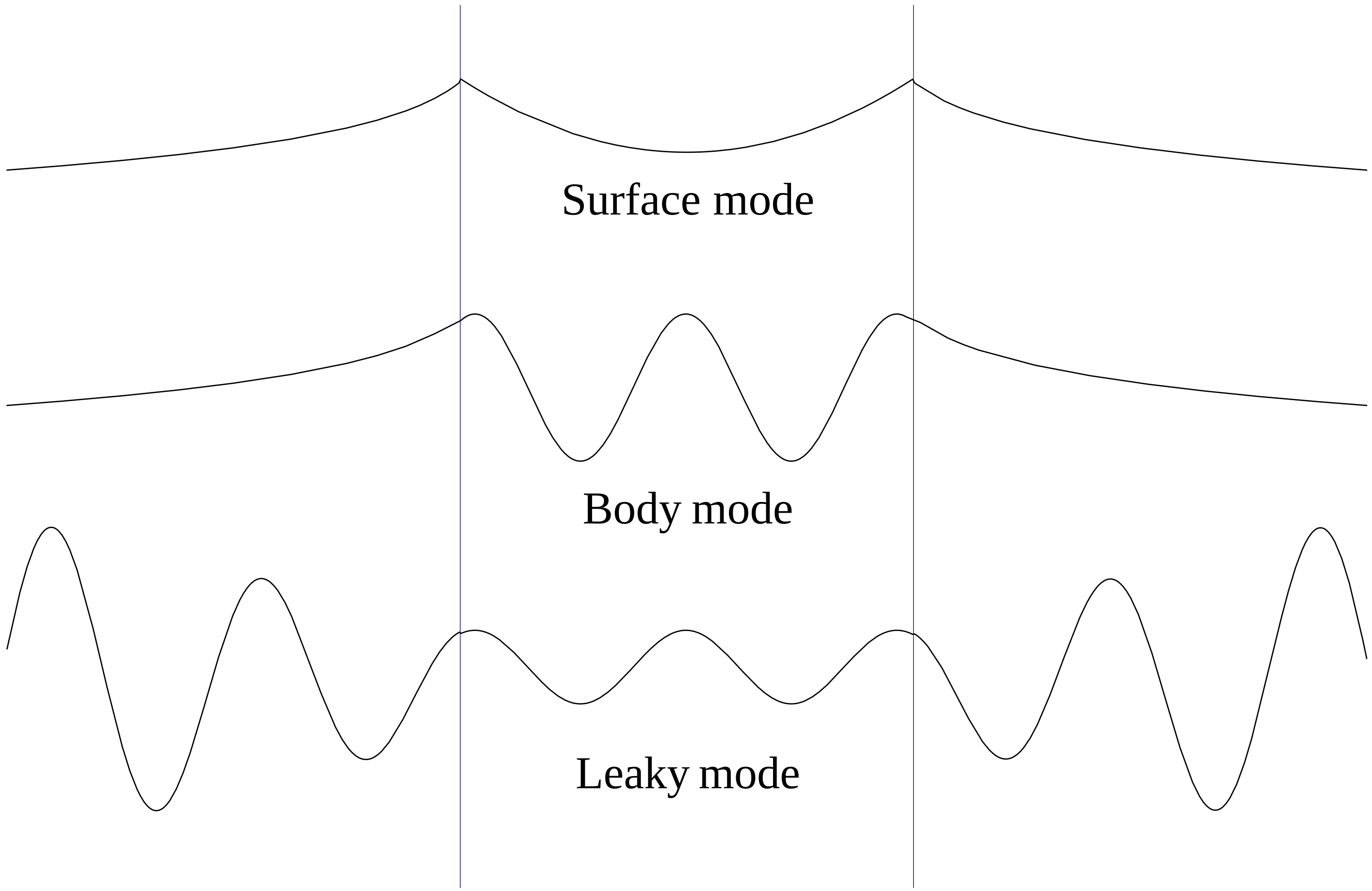
$\Rightarrow$ ideal continuous spectrum only approximated at end points

$\Rightarrow$ ideal quasi-mode clearly visible!

- collective mode
- weakly damped



Body modes, surface modes, and leaky waves



Different wave types in flux tubes.

Magnetic structures and dynamics

Overview

- **Origin of solar magnetism:** solar model, helioseismology, solar cycle, dynamo, convection; [book: Sec. 8.2.1]
- **Solar magnetic structures:** solar atmosphere and magnetic structuring and dynamics; [book: Sec. 8.2.2]
- **Planetary magnetic fields:** geodynamo, journey through the solar system; [book: Sec. 8.3]
- **Solar wind and space weather:** solar wind, interaction with magnetospheres; [book: Sec. 8.4]
- **Astro-plasma physics:** launching collimated astrophysical jets.

Questions

Eventually, all theory has to be confronted with empirical reality. This should lead to attempt to answer the following questions:

- Is the MHD model developed so far an adequate starting point for the *description of observed plasma dynamics?*
- Are important *theoretical pieces still missing?*
- What should be the *main goals of present research* to resolve these questions?

$\Rightarrow$ Inspiration for answers to these questions to be obtained from **phenomenology of magnetic structures and associated dynamics.**

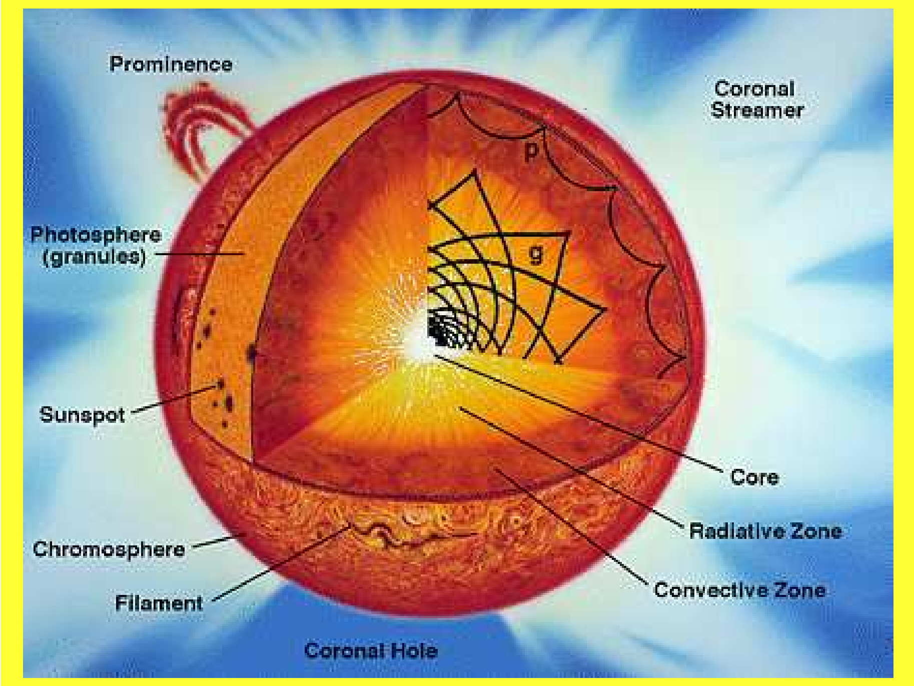
Solar magnetism

- Central question: **Where does the solar magnetism come from?**

Recall structure of the Sun:

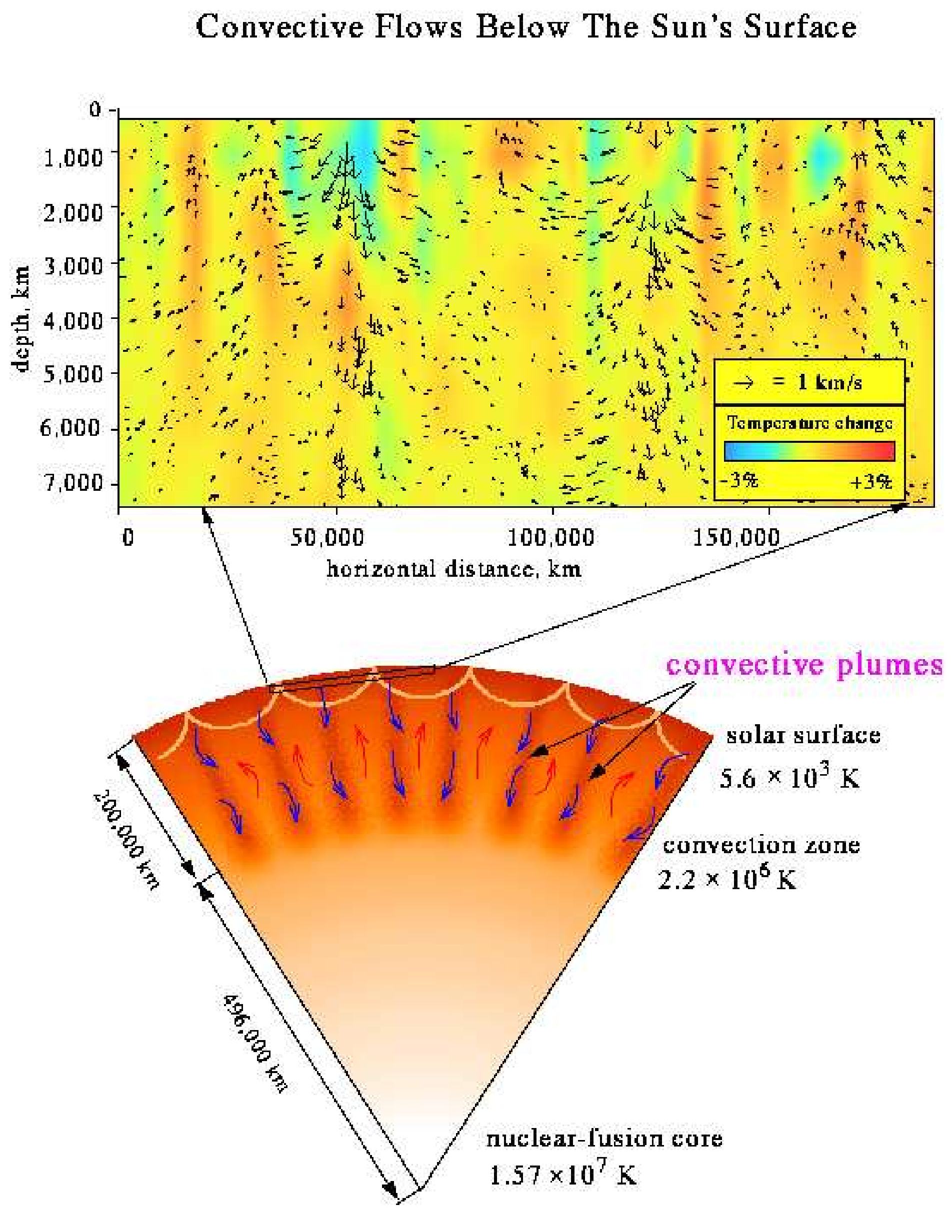
- core, $r \leq 0.25 R_\odot$:
  thermonuclear conversion of hydrogen into helium;
- radiative zone, $0.25 R_\odot \leq r \leq 0.71 R_\odot$:
  outward radiative transport of produced energy;
- **convection zone,** $0.71 R_\odot \leq r \leq R_\odot$:
  temperature gradient so steep that the plasma is convectively unstable

  $\Rightarrow$ **seat of the solar dynamo!**



(from SOHO web site)

Convective flows



(from SOHO web site)

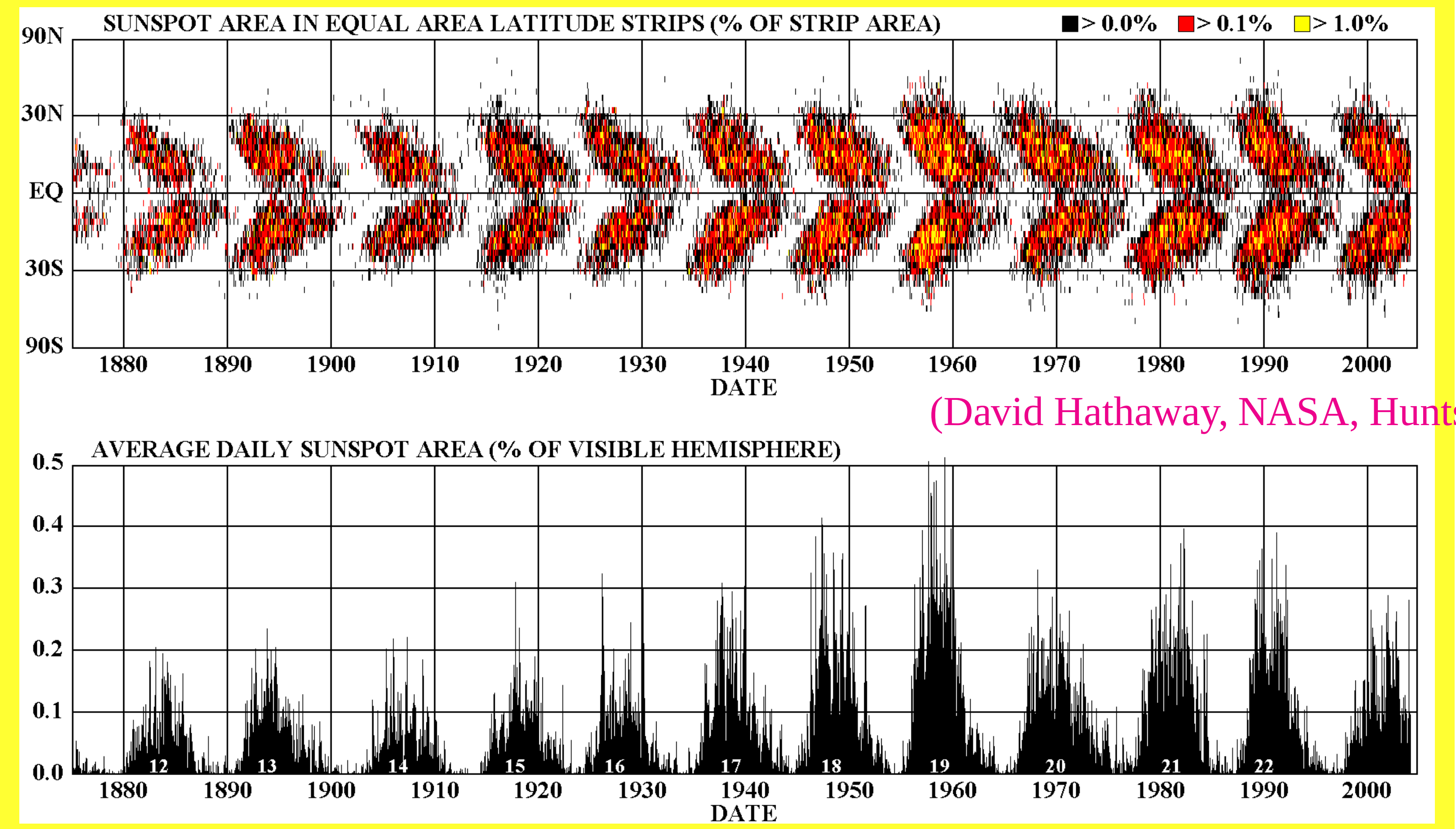
Sunspots

- Dark spots in the (visible) photosphere that are cooler (darker) than surroundings.
- Can live days–month and rotate west–east across the disk in bands up to $\pm 35^\circ$ about equator.
- Reveal existence of *several 1000 Gauss magnetic field!*

(from SOHO web site)

Solar cycle

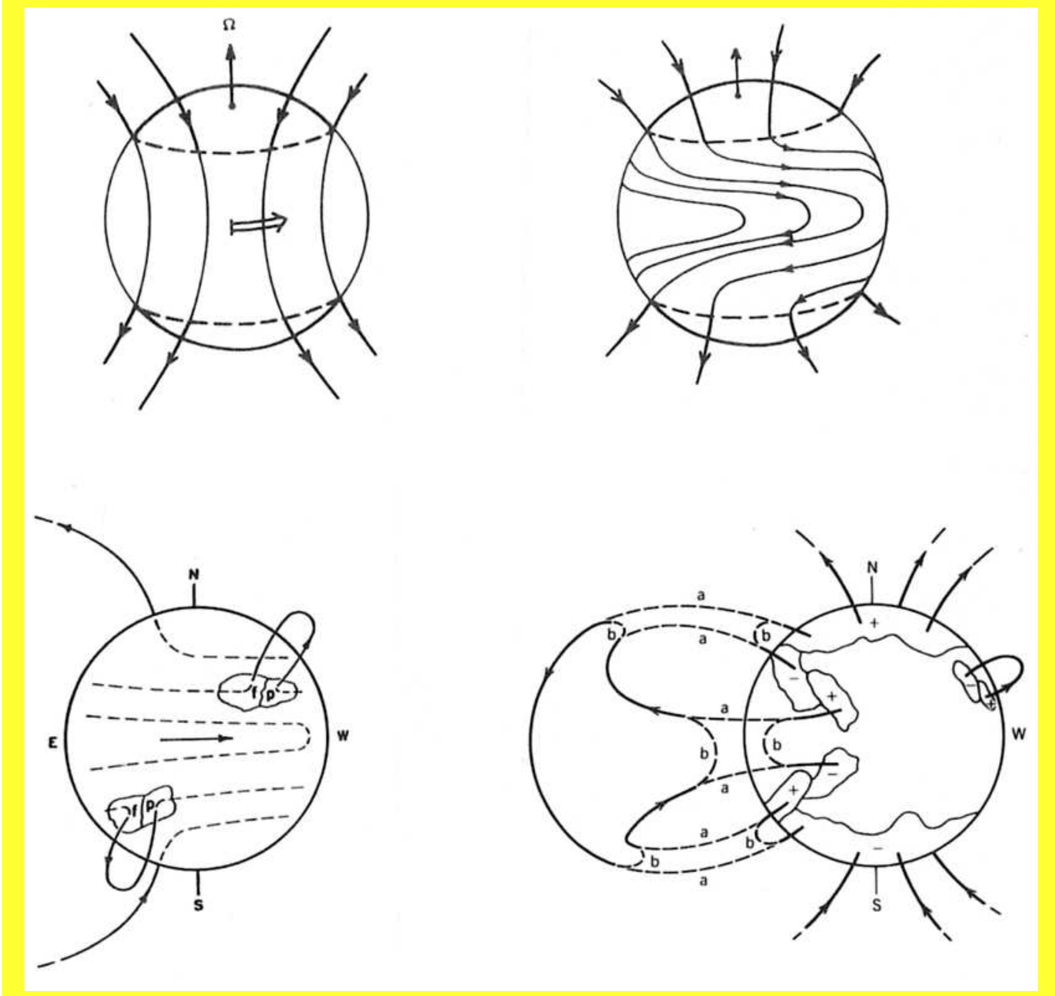
- *Butterfly diagram of the solar cycle* shows variation sunspot number with years:
  - drifting in latitude with roughly 11 year periodicity.



(David Hathaway, NASA, Huntsville)

Solar dynamo

- The solar cycle (reversal of magnetic polarity every 11 years) is *a magnetic oscillation* driven by the periodic *solar dynamo:* conversion of mechanical into magnetic energy.
- Solar dynamo ingredients:
  – differential rotation,
  – convection,
  – (small) magnetic diffusivity.
- Illustrated by Babcock cartoon model (1961) of the solar cycle: [H. Babcock, Ap. J. **133**, 572 (1961)]



Crude approach:

- **Mean field dynamo**, which parameterizes the unknown diffusivity enhancement:

  – *Inhomogeneity of the magnetic field will decay* in time τ_D determined by resistivity η and length scale $l_0 \sim \nabla^{-1}$ of the inhomogeneity:

$$\tau_D = \mu_0 l_0^2/\eta = l_0^2/\tilde{\eta}\,.$$

  – Classical values for l_0 and η yield τ_D many orders of magnitude too long: reduce by factors parameterizing *turbulent vortex interactions.*

- Procedure: Split $\mathbf{v}$ and $\mathbf{B}$ in mean $+$ fluctuations: $\mathbf{v} = \langle \mathbf{v} \rangle + \mathbf{v}'$, $\mathbf{B} = \langle \mathbf{B} \rangle + \mathbf{B}'$, which yields averaged form of induction equation:

$$\frac{\partial \langle \mathbf{B} \rangle}{\partial t} = \nabla \times \big(\langle \mathbf{v} \rangle \times \langle \mathbf{B} \rangle \big) + \nabla \times \langle \mathbf{v}' \times \mathbf{B}' \rangle - \nabla \times \big(\tilde{\eta} \nabla \times \langle \mathbf{B} \rangle \big)\ .$$

- **IF we assume**

$$\langle \mathbf{v}' \times \mathbf{B}' \rangle \approx \alpha \, \langle \mathbf{B} \rangle - \beta \, \nabla \times \langle \mathbf{B} \rangle + \ldots\ ,$$

  THEN there is field amplification through α and decay through $\tilde{\eta} + \beta$ from

$$\frac{\partial \langle \mathbf{B} \rangle}{\partial t} = \nabla \times \big(\langle \mathbf{v} \rangle \times \langle \mathbf{B} \rangle \big) + \nabla \times \left(\alpha \, \langle \mathbf{B} \rangle \right) - \nabla \times \big[\left(\tilde{\eta} + \beta \right) \nabla \times \langle \mathbf{B} \rangle \big]\ ,$$

  since $\tilde{\eta} + \beta \approx \beta \approx v' l \gg \tilde{\eta}$, *using length/time scale for convective granulation.*

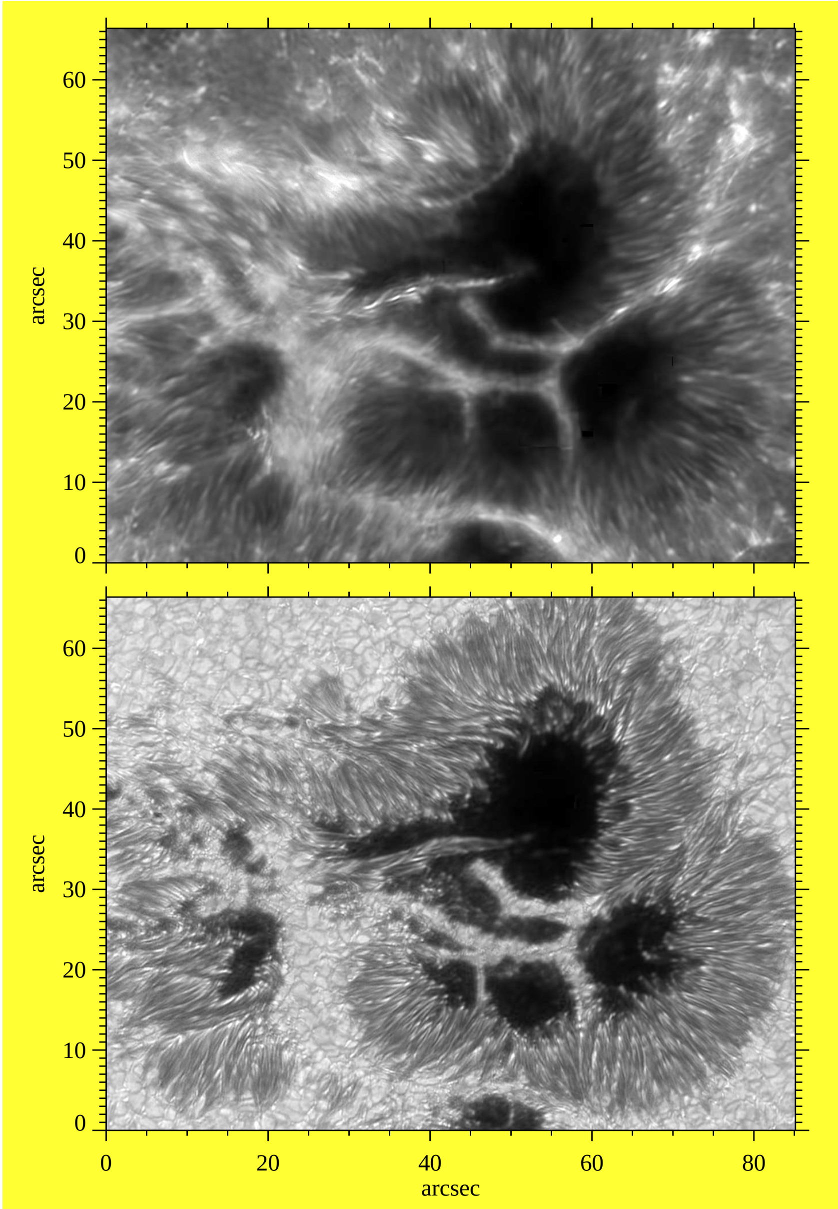
Improving by computational MHD:

- Since *kinematic dynamo problem* ignores back reaction on the flow,
  – need for full 3D magnetoconvection models in rotating boxes,
  – would allow for quantification of correlation coefficients α.
- Hence, ingredients **flux tube dynamo simulations**:
  – can store and amplify magnetic fields stably in overshoot region below the convection zone (tachocline),
  – will reach field strengths of order 10^5 G, forming toroidal flux tube,
  – becomes unstable to long-wavelength undular deformation (Parker instability),
  – rises without strong deformation through convection zone,
  – consistent with Joy's laws (tilt dependence on latitude of active regions due to Coriolis effects), observed asymmetries $p - f$ spots, bipole orientation w.r.t. equator,
  – expands and gets shredded just prior to photospheric emergence.

Sunspots

- Dark umbra, at typical 3700 K, with near vertical field.
- Filamentary penumbra, intercombed dark/bright, with inclined fields.
- Subsurface structure: 'spaghetti' (umbral dots).
- Sampling sunspot with height:

  Dutch Open Telescope movie
  (dotmovie.mpeg)



⇒ Need sunspot (local) seismology

Active region seismology

- Interaction of p-modes with sunspots:

  – decompose in in- & outgoing waves,

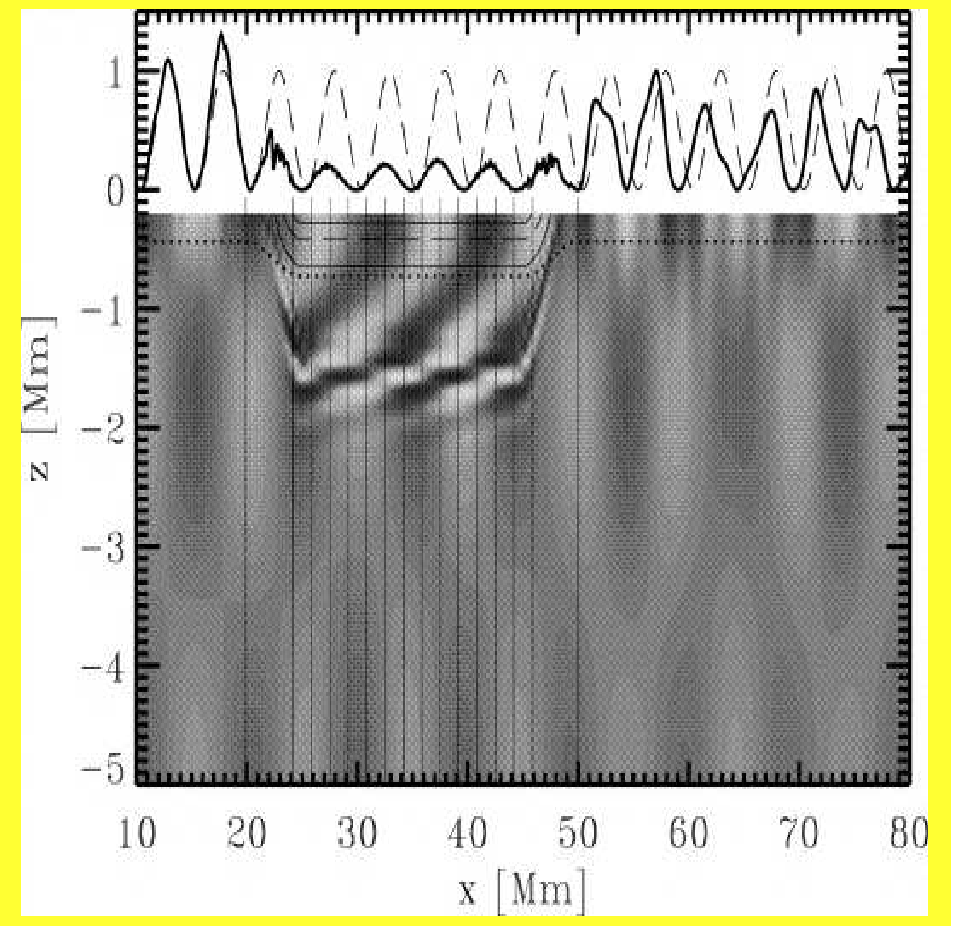
  – sunspots absorb up to 50 % of the impingent acoustic power!

- Candidate linear MHD processes:

  – driving frequencies in Alfvén continuum range causing local resonant absorption (dissipation),

- True 2D stratification:
  mode conversion (magflux.mpeg)
  to downward propagating s-modes at $\beta \approx 1$ layers.

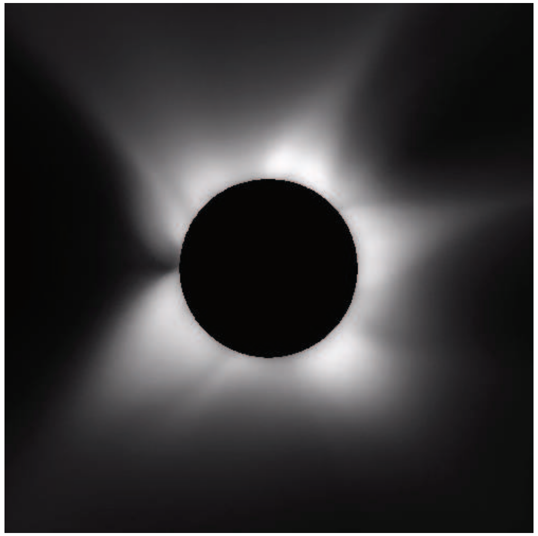
Cally and Bogdan, ApJL **486**, L67 (1997).

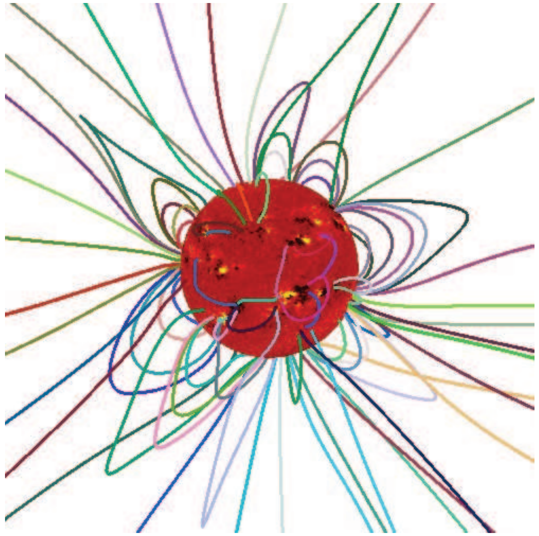


http://www.hao.ucar.edu/
(modelling interaction of p-modes with sunspots)

Corona: eclipse images

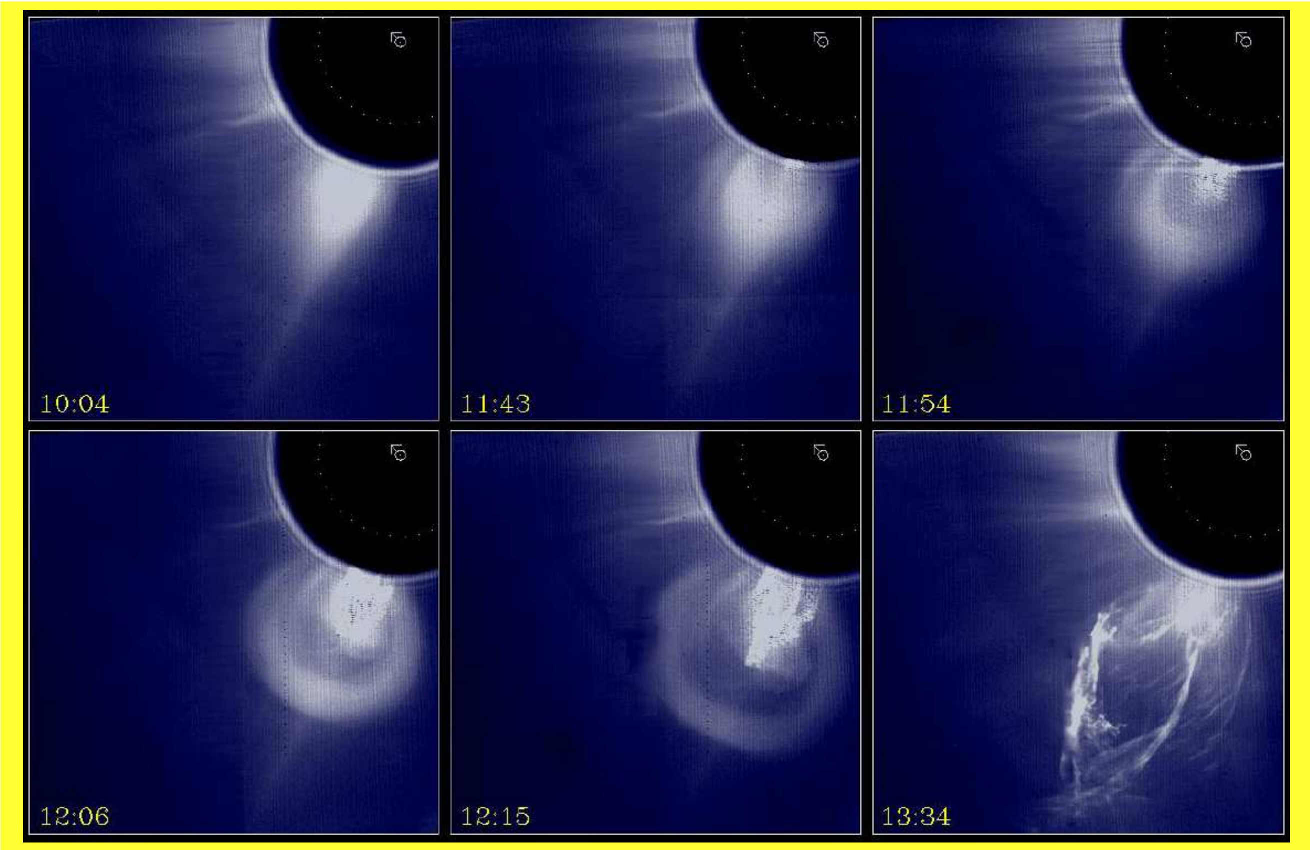
- At solar max: coronal helmet streamers.
- 3D structure can be ‘predicted’ from MHD models.





Corona: coronagraph

- Monitoring Coronal Mass Ejections (CMEs):
  - 10^{12} kg ejected, few 100–1000 km/s,
  - Solar Maximum Mission (1980–1989).

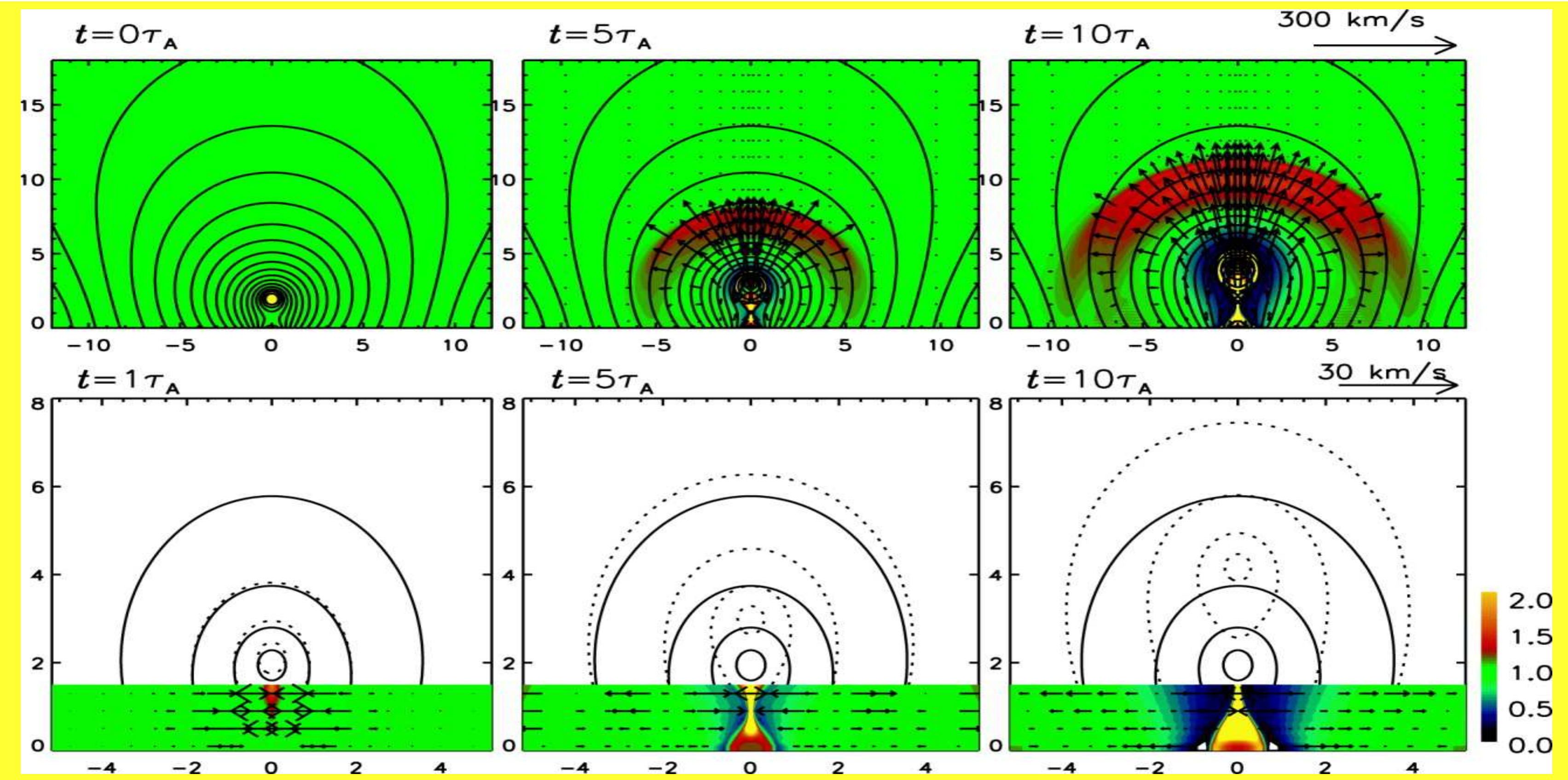


Coronal dynamics

- **SOHO** Extreme UV imaging Telescope (EIT).
  - visualizes solar cycle variations of coronal structure:

- Solar flares (10^{24} J 'explosions'):
  - reconnection, particle accelerations, associated CME.
- EIT identified **flare-associated waves** (EITrndif.mov)
  - circularly propagate away from flare site, enhanced transient coronal emission.

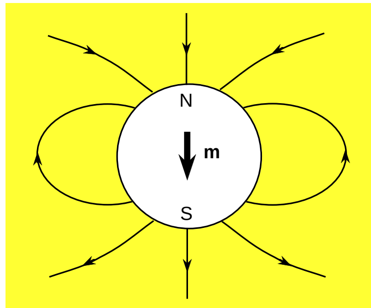
- MHD model for EIT waves and related chromospheric waves:
  - CME-induced due to rising flux ropes, Chen *et al.* **572**, L99 (2002);
  - Overarching shock front: 'legs' produce chromospheric Moreton waves;
  - EIT waves mark site of successive opening of covering field lines.



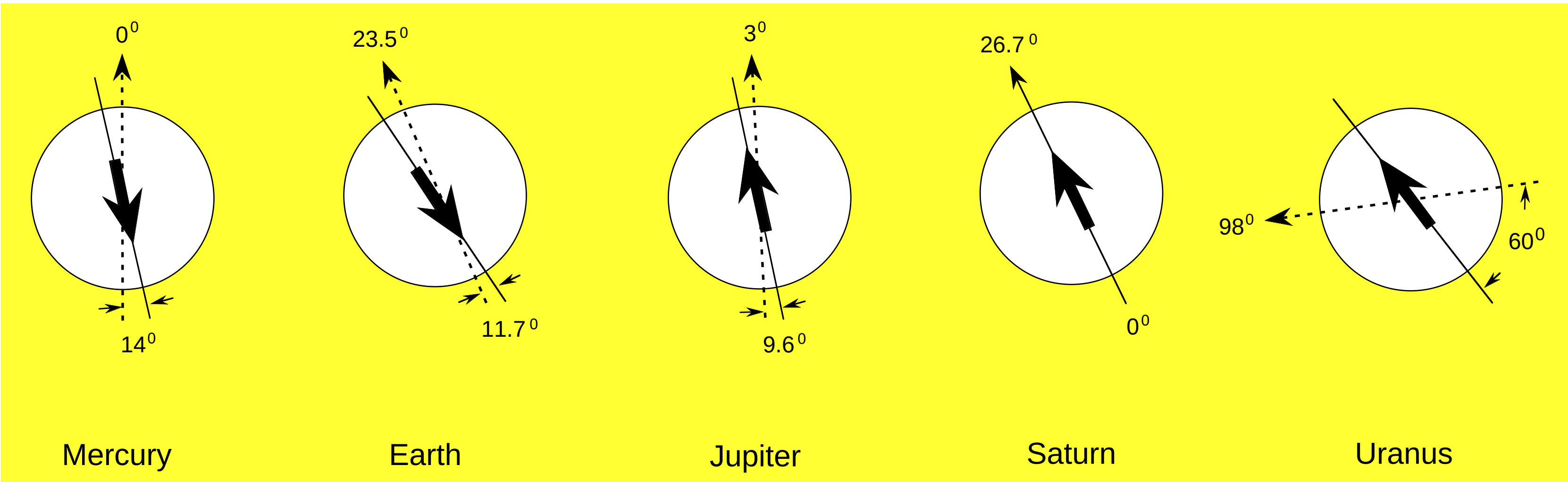
Planetary magnetic fields by themselves:

Magnetic dipole

- Dipole magnetic field: $\mathbf{B}(\mathbf{r}) = (\mu_0/(4\pi r^3))\,(3\mathbf{m}\cdot\mathbf{e}_r\,\mathbf{e}_r - \mathbf{m})$.
  - Earth: $m_{\rm E} = 8.1 \times 10^{22}\,{\rm A\,m^2}$.



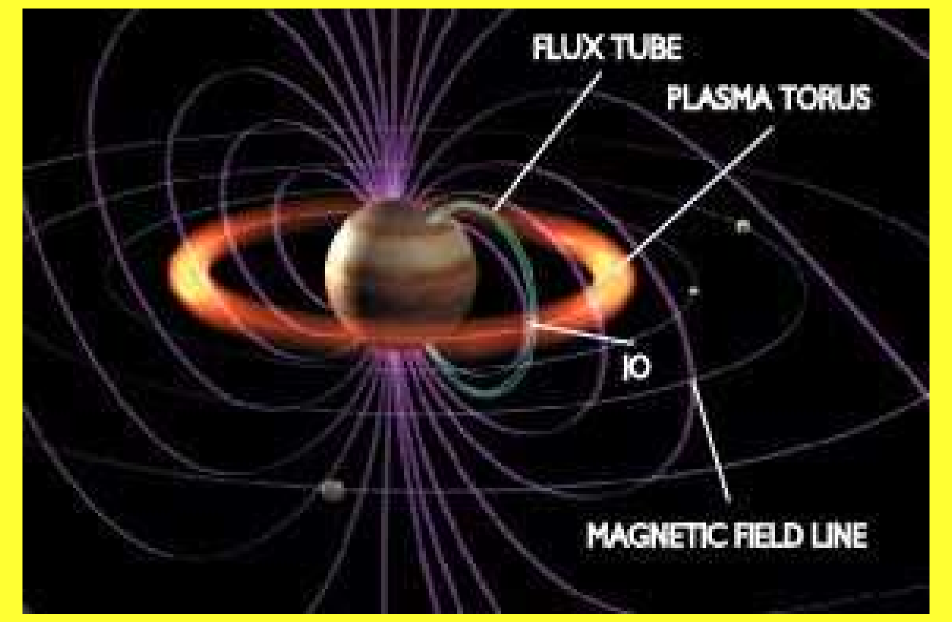
- Table B.8 for values of solar system planets:
  - sizeable fields for Earth and giant planets (Jupiter, Saturn, Uranus, Neptune),
  - interesting orientation w.r.t. planetary rotation axis (preferred alignment?).

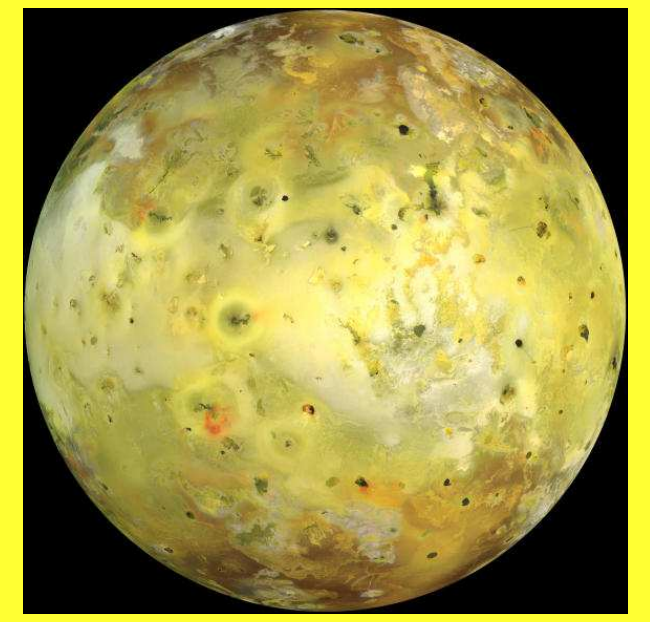


Jovian system

- **Jupiter** has largest magnetosphere in solar system:
  - magnetosphere extends 150 to 200 Jupiter radii,
  - magnetopause (boundary static Jovian plasma/solar wind) at $\sim 65 R_{\rm J}$,
  - equatorial $\mathbf{B}$ tenfold of Earth's,
  - liquid metallic hydrogen in inner mantle drives dynamo,
  - 16 moons (Io, Ganymede, Europa, Callisto, . . .).
- NASA Galileo mission (1989–1999) arrived at Jupiter in 1995.

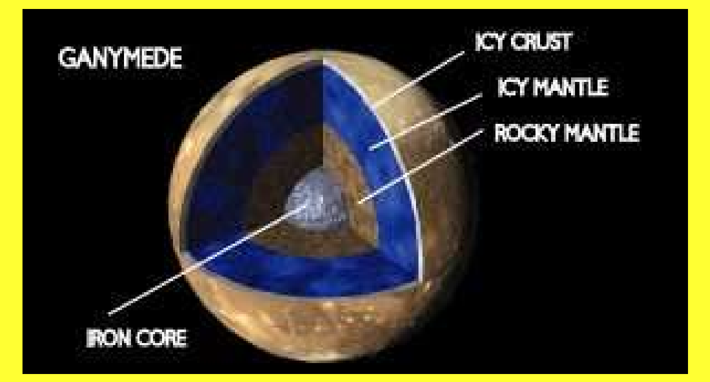
  http://home.freeuk.com/catherine-uk/
  (Galileo mission to Jupiter, pictures of Jovian system
  with 'Io plasma torus' and Ganymede on next page)





http://antwrp.gsfc.nasa.gov/apod/ap990920.html (Io)

- **Io:** strong tidal forces (eccentric orbit) induce volcanic activity,
  - injects plasma into torus about Jupiter,
  - drives auroral displays on Jupiter poles.
- **Ganymede:** dynamo in iron core (?),
  - B of 10 % Earth,
  - magnetosphere in Jovian magnetosphere.



Solar wind by itself:

Solar wind: Parker model

- Coronal plasma at 10^6 K, density drops for increasing r.
  - Pressure gradient drives continuous outflow.
  - Predicted by Parker in 1958, later observed by satellites.
- Model with hydrodynamic equations, spherical symmetry:
  - Look for stationary solutions $\partial/\partial t = 0$;
  - Assume isothermal corona (fixed temperature T), include gravity:

$$\frac{d}{dr}(r^2 \rho v) = 0 \qquad \Rightarrow \qquad r^2 \rho v = \mathrm{const}\,,$$

$$\rho v \frac{dv}{dr} + v_{\mathrm{th}}^2 \frac{d\rho}{dr} + G M_\odot \frac{\rho}{r^2} = 0\,.$$

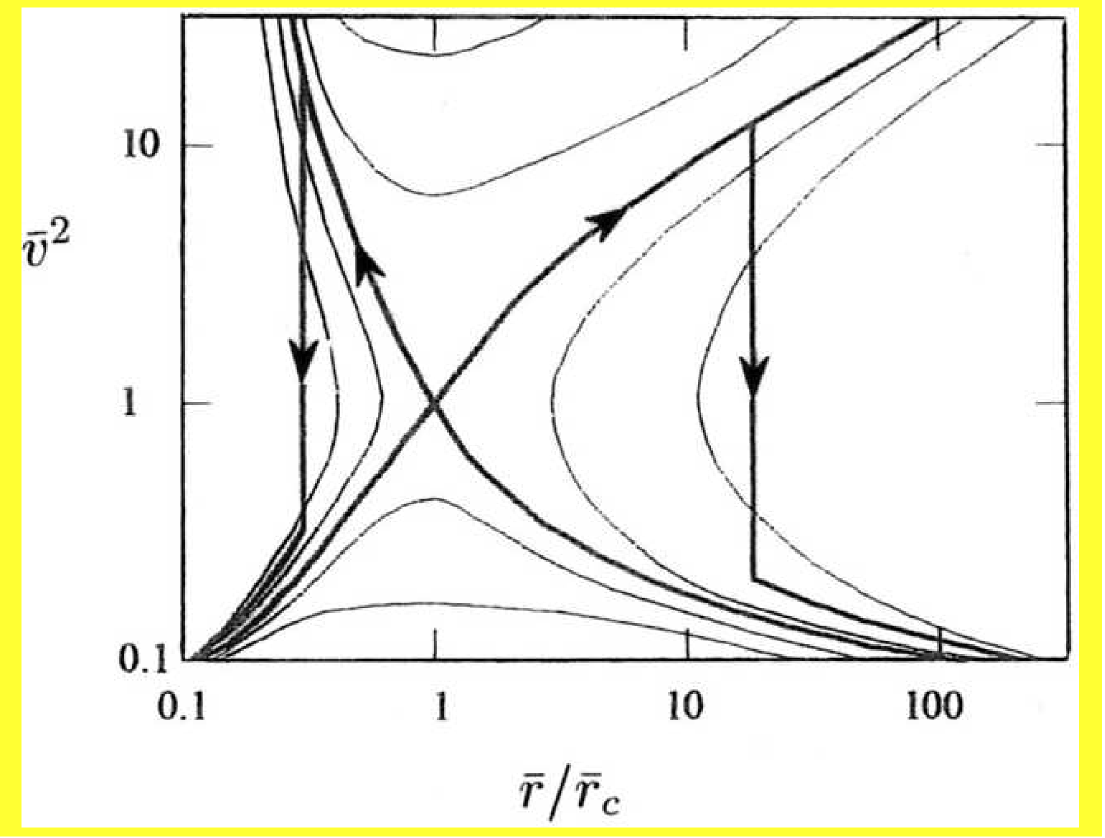
  - uses constant isothermal sound speed $p/\rho \equiv v_{\mathrm{th}}^2$.

- Scale $\bar{v} \equiv v/v_{\rm th}$ (Mach number) and $\bar{r} \equiv r/R_\odot$ to get

$$F(\bar{v},\bar{r}) \equiv \tfrac{1}{2}\bar{v}^2 - \ln \bar{v} - 2\,\ln\left(\frac{\bar{r}}{\bar{r}_c}\right) - 2\,\frac{\bar{r}_c}{\bar{r}} + \tfrac{3}{2} = C\,, \qquad \bar{r}_c \equiv \tfrac{1}{2}\frac{GM_\odot}{R_\odot v_{\rm th}^2}\,.$$

  – Implicit relation determining $\bar{v}(\bar{r})$,

  – unique solution with transonic acceleration:



Solar wind modeling

- Generalization to 1.5D magnetized wind possible analytically:

  – appropriate for equatorial plane including rotation.

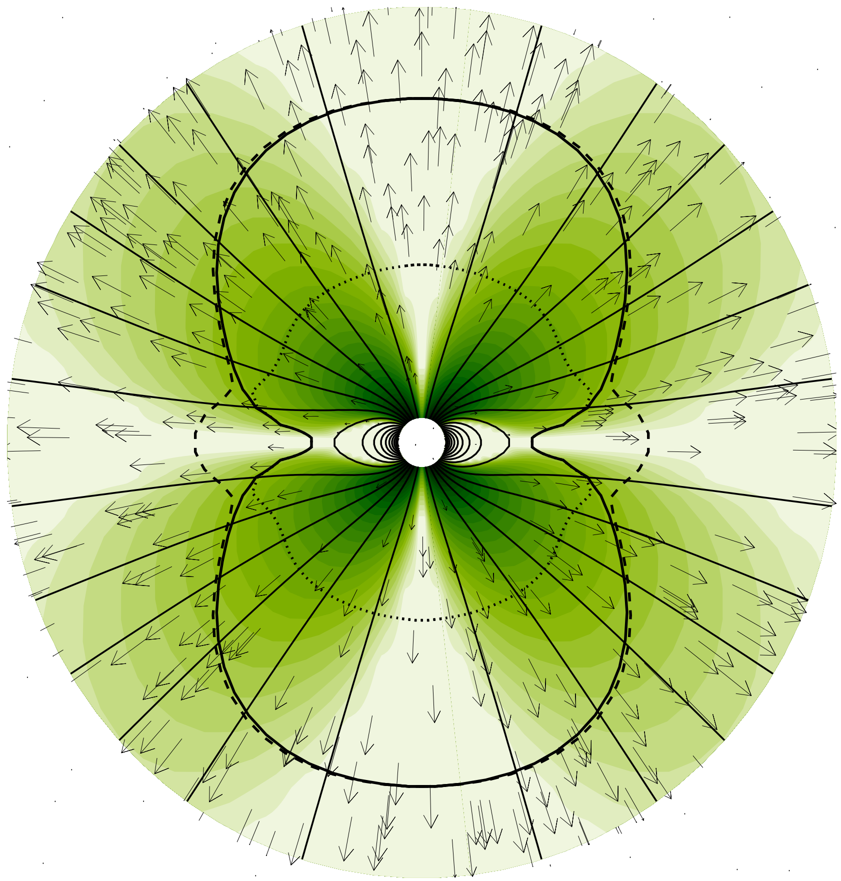
- More advanced models solve for numerical MHD steady-state:

  – MHD: 3 Mach numbers, critical transitions as hourglass curves,

  – Axisymmetric magnetized wind with a ‘wind’ and a ‘dead’ zone.

  [simulation by Keppens & Goedbloed, Ap. J. **530**, 1036 (2000) , using VAC]

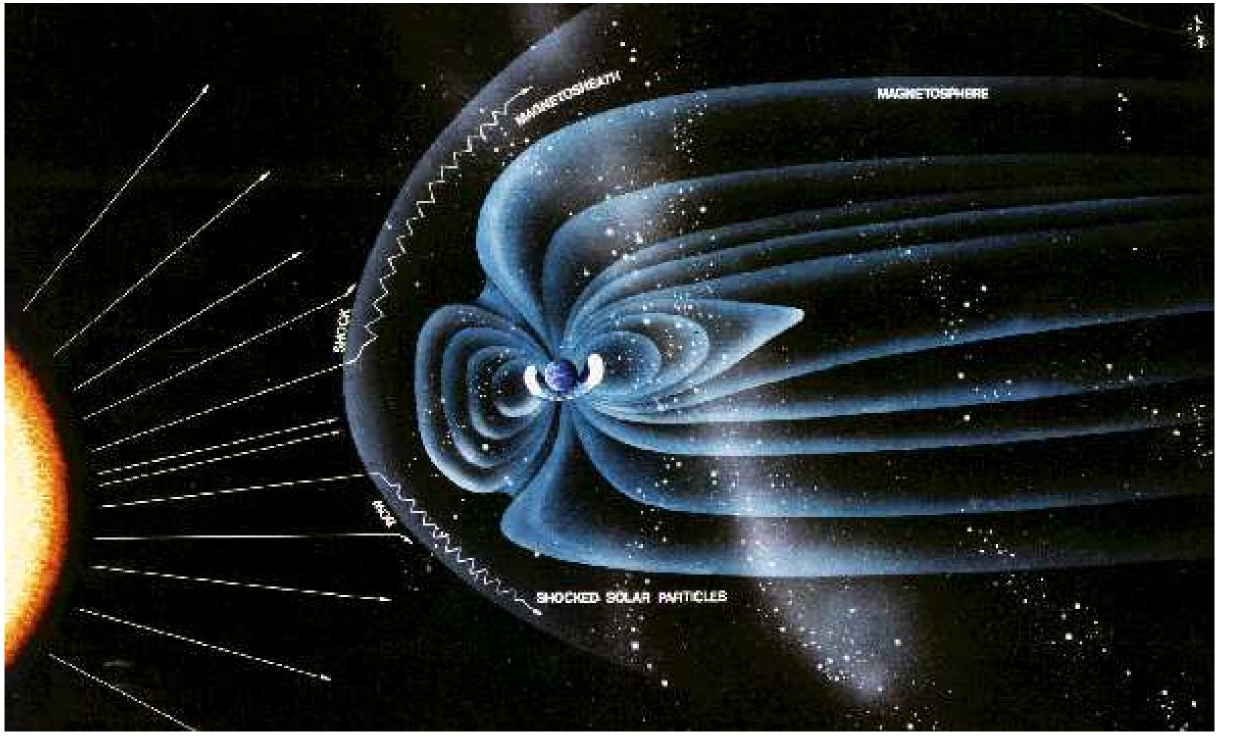
  www.rijnh.nl/n3/n2/f1234.htm



Interaction of solar wind and planetary magnetic field yields:

Magnetosphere

- Large-scale magnetic structure with
  - bow shock due to impinging supersonic solar wind (day-side),
  - magnetopause (contact discontinuity) and inner magnetosphere,
  - night-side stretched into magnetotail with equatorial current sheet.
- Size estimate from magnetic pressure $\sim (R/r)^6$ dipole field versus ram pressure $(\frac{1}{2}\rho v^2)_{\rm sw}$ wind:
  $\sim 10\,R_{\rm E}$ for Earth,
  $\sim 60\,R_{\rm J}$ for Jupiter.



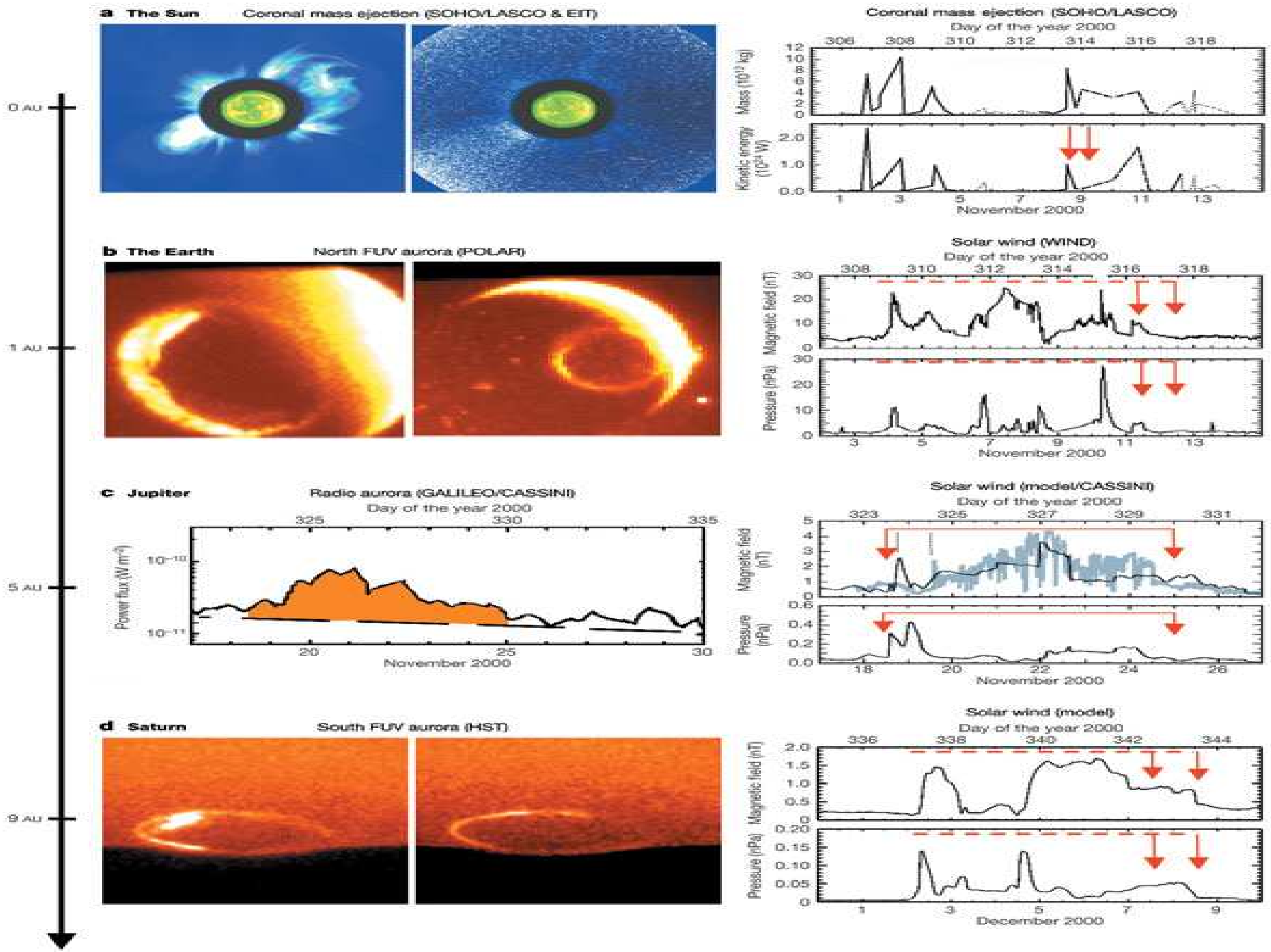
Space weather modeling

- Modern shock-capturing MHD simulations:
  - trigger (flux emergence, cancellation, shearing) + evolution of CMEs,
  - Mikic *et al.*, SAIC San Diego: **CME by flux cancellation** (Mikic-flx2d.anim.qt)
- Compute impact effect on Earth's magnetosphere *faster than real time*
  - computing challenge (few days), significant range of scales
  - Gombosi, Toth *et al.*, Univ. of Michigan:
    Centre for Space Environment movie (Toth-CSEM2004-Zoom.mov)
- **Space weather affects all planets!** Near-alignment of Earth, Jupiter, Saturn (2000)

  $\Rightarrow$ *Series of CMEs (seen by SOHO) leading to interplanetary shock (overtaking and merging shocks), detected as auroral storms on Earth (Polar orbiter), observed in Jovian radio activity as measured by Cassini (fly-by on its way to Saturn), seen by Hubble as auroral activity on Saturn.*

  $\Rightarrow$ *MHD model (using VAC code) used to simulate time evolution.*

- **First observation of CME event traced all the way from Sun to Saturn,** Prangé *et al.*, Nature, **432** (4), 78 (2004). *Right:* comparison with VAC simulations, with input from WIND spacecraft.



Sun (SOHO) LASCO & EIT

Earth (POLAR) aurora image

Jupiter (Casini) radio signal

Saturn (HST) aurora image

Perspective

- **Space missions produce(d) numerous observations**
  - *SOHO* (1995): solar phenomena from core to beyond the Earth's orbit.
  - *Cluster satellites* (1996, 2000): 3D spatial structure of Earth's magnetosphere.
  - *Ulysses* (1990) in situ investigations of inner heliosphere.
  - *Solar Orbiter* (2012–2017) highest resolution and images of Sun's polar regions.
- **Observed dynamics demonstrates:**
  - Validity of *magnetic flux conservation* and *dynamics of magnetic flux tubes*.
  - Observed magnetic flux tubes occur in *large numbers.*
- **Many unsolved problems remain:**
  - quantitative theory of *solar dynamo,*
  - theory of *coronal heating,*
  - prediction of *solar flares,*
  - theory of *solar wind generation, heating, interaction with magnetospheres,*
  - prediction of *space weather.*

Challenging the MHD model

- there are large populations of accelerated particles in the solar atmosphere
- in the corona and the solar wind electrons and ions behave differently
- the photospheric plasma is
  - *strongly collisional*
  - very *weakly ionized* $n_{i0}/n_{n0} \sim 10^{-4}$
  - ions (and electrons!) are both *un-magnetized*
  - intrinsically *multi-component*
- NLFFF algorithms can model test fields but deliver less consistent, less acceptable fields when applied on actual observations (cf. Schrijver, SPD Boulder 2009)

  $\Rightarrow$ big problems: it can be shown that the assumptions in the models are not consistent with the observations!

$\Rightarrow$ 3D, time accurate 'MHD$^+$' models are needed (*multi-fluid*, *kinetic*,. . .)