PERVERSE SHEAVES

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1. T-STRUCTURES

$\mathcal{A}$ = abelian category $\rightsquigarrow D^b(\mathcal{A})$: triangulated category.

Question: How to recover $\mathcal{A}$ inside $D^b(\mathcal{A})$?

Truncation: $E \in D^b(\mathcal{A})$ and $n \in \mathbb{Z}$, we denote:

$\tau_{\leq n}(E) = (\ldots \to E^{n-1} \to \ker(E^n \to E^{n+1}) \to 0)$

similarly for $\tau_{\geq n}(E)$.

Then we have a distinguished triangle:

$\tau_{\leq n}(E) \to E \to \tau_{\geq n+1}(E) \to +1$,

and

$\mathcal{A} \simeq \{E|H^i(E) = 0, \forall i \neq 0\}$

Definition 1.1. If $D$ is a triangulated category, a $t$-structure on $D$ is a pair of full additive subcategories $(D_{\leq 0}, D_{\geq 0})$. We denote by $D_{\leq -n} = D_{\leq 0}[-n]$ and $D_{\geq n} = D_{\geq 0}[n]$. We impose the conditions:

1. $D_{\geq -1} \subset D_{\geq 0} \subset D_{\geq 1} \ldots$
2. (Semi-orthogonality) $\text{Hom}(D_{\leq 0}, D_{\geq 1}) = 0$.
3. (decomposability) $\forall E \in D$, $\exists$ distinguished triangle:

   $E' \to E \to E'' \to +1$

   with $E' \in D_{\leq 0}$ and $E'' \in D_{\geq 1}$

The classical situation provides an example of $t$-structure on $D(\mathcal{A})$.

Definition 1.2. The category $C = D_{\leq 0} \cap D_{\geq 0}$ is called the heart of the $t$-structure.

Proposition 1.3. The decomposition of objects is canonical.

Proof. If we have two distinct decompositions of $E$: $(E', E'')$ and $(F', F'')$. Then we have a long exact sequence:

$\ldots \to \text{Hom}(E', F''[-1]) \to \text{Hom}(E', F') \to \text{Hom}(E', E) \to \text{Hom}(E', F'') \to \ldots$

The two exterior are zero by semi-orthogonality condition. $\square$

Corollary 1.4. For all $n \in \mathbb{Z}$, we have functors: $\tau_{\leq n} : D \to D_{\leq n}$ and $\tau_{\geq n} : D \to D_{\geq n}$, s.t. $(i_{\leq n}, \tau_{\leq n})$ and $(\tau_{\geq n}, i_{\geq n})$ are adjoint pairs.

Remark 1.5. $\text{Hom}(D_{\leq n}, D_{\geq n+1}) = 0$ by definition. We have actually: $D_{\leq n}$ is orthogonal of $D_{\geq n+1}$.

Theorem 1.6. (BDD) $C = D_{\leq 0} \cap D_{\geq 0}$ is an abelian category, and all short exact sequences $0 \to E' \to E \to E'' \to 0$ in $C$ give rise to a distinguished triangle $E' \to E \to E'' \to +1$ in $D$.

Proof. $\bullet C$ is closed under $\oplus$.
• let $E,F \in \mathcal{C}$, $f : E \to F$ complete into $E \to F \to G \to +1$ distinguished. Then $\ker(f) = H^{-1}(G) = \tau^{\leq -1}\tau^{\geq -1}(G)$, and $\text{coker}(f) = H^0(G) = \tau^{\geq 0}\tau^{\leq 0}(G)$.

$D^{\leq n}$ is stable by extension. Then for all $W$ in $\mathcal{C}$ we have to show that

$$0 \to \text{Hom}(W, \ker(f)) \to \text{Hom}(W, \ker E) \to \text{Hom}(W, F)$$

is exact. We use the above distinguished triangle to see that:

$$\text{Hom}(W, F[-1]) = 0 \to \text{Hom}(W, G[-1]) \to \text{Hom}(W, E) \to \text{Hom}(W, F)$$

The first term is $\text{Hom}(W, \ker)$. (same for coker, while the proof for images and co-images uses the octahedron axiom)

\[ \square \]

We can in fact defined truncation functors: $\tau^{[a,b]} : D \to D^{\geq a} \cap D^{\leq b}$.

**Proposition 1.7.** $H^0 = \tau^{\leq 0} \circ \tau^{\geq 0}$ is a cohomological functor from $D$ to $\mathcal{C}$, i.e. for any distinguished triangle, we have a long exact sequence.

## 2. Exactness

Let $F : D_1 \to D_2$ be a triangulated functor between triangulated endowed with $t$-structures. (giving $\mathcal{C}_1$ and $\mathcal{C}_2$).

**Definition 2.1.** $pF : \mathcal{C}_1 \to D_1 \to D_2 \to \mathcal{C}_2$

**Definition 2.2.** $F$ is left $t$-exact if $F(D^{\geq 0}) \subset D^{\geq 0}_2$ (and right...)

**Proposition 2.3.** If $F$ is left $t$-exact then $pF$ is left exact (same for right).

**Proposition 2.4.** $F : D_1 \leftrightarrow D_2 : G$ adjoint pair, then $F$ right $t$-exact is equivalent to $G$ left $t$-exact.

## 3. Perverse $t$-structure

$X$: analytic space (algebraic variety/ $\mathbb{C}$).

$D^b(\mathbb{C}_X) = \text{bounded derived cat of } \mathbb{C}_X$-modules.

$D^n_\epsilon(\mathbb{C}_X) = D^b(\mathbb{C}_X)$ the category of $E \in D^b(\mathbb{C}_X)$, such that for all $i$, $H^i(E) \in \text{Sh}(X)$ is constructible.

Recall a sheaf $F$ on $X$ is constructible (algebraically) if there exists an alg. stratification of $X$ such that $F$ is a local system on each stratum.

Six functor formalism for $f : X \to Y$:

- $f^* : D^b(\mathbb{C}_Y) \to D^b(\mathbb{C}_X)$;
- $f_* = Rf_* : D^b(\mathbb{C}_X) \to D^b(\mathbb{C}_Y)$;
- $f_! : D^b(\mathbb{C}_X) \to D^b(\mathbb{C}_Y)$;
- $f^! : D^b(\mathbb{C}_Y) \to D^b(\mathbb{C}_X)$;
- $\otimes$ and $R\text{Hom}(\cdot, \cdot)$.

Adjoint pairs $(f^*, f_*)$ and $(f_!, f^!)$.

Verdier duality: $\pi : X \to \text{pt}$, $\omega_X = \pi^!(\mathbb{C})$.

$$D_X = (E \mapsto R\text{Hom}(E, \omega_X))$$

**Proposition 3.1.**

1. $D_X^2 \simeq Id_{D^b(\mathbb{C}_X)}$
2. $f_* \circ D_X = D_Y \circ f_*$
3. $f^! \circ D_Y = D_X \circ f^*$ on $D^\epsilon_\epsilon$. 
Definition 3.2. (Perverse $t$-structure) $F$ in $^\nu D_D^{\leq 0}(X)$ if $\dim(\text{supp}(H^i(F))) \leq -i$ for all $i$.

$F$ in $^\nu D_D^{\geq 0}(X)$ if $\dim(\text{supp}(H^i(DD\chi(F)))) \leq -i$ for all $i$.

$Perv(X)$ is the heart of this $t$-structure.

Theorem 3.3. (BBD) This is a $t$-structure.

The proof goes by constructing the $t$-structure obtained by gluing pieces.

Definition 3.4. A recollement is the datum of: $D_Z \xrightarrow{i_*} D \xrightarrow{j^*} D_U$ that gives two triples of adjoints $(i^*, i_*, i^!)$, $(j_!, j^*, j_*^!)$ such that:

1. $j^* \circ i_* = 0$
2. $j_! j^* \rightarrow id \rightarrow i_* i^*$ distinguished
3. $i_* i^! \rightarrow id \rightarrow j_* j^*$ distinguished
4. $i_*, j_!, j_*$ are fully faithful.

Claim: From $t$-structures on $D_Z$ and $D_U$, then $D_D^{\leq 0} = \{E/i^* E \in D_D^{\leq 0}, j^* E \in D_U^{\leq 0}\}$ is a $t$-structure on $D$. 