Classification of singularities at infinity of polynomials of degree 4 in two variables

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Introduction

During the last years there is an increasing interest in the behaviour of polynomials at infinity. In studying the family of levels curves

$$f(x, y) = t$$

one wants to know e.g the topological type of generic fibres, the set of bifurcation values, the change of topology of the fibre near bifurcation values, the monodromy at infinity, the intersection form on fibres.

For classes of polynomials (e.g. tame polynomials) the changes in the topology depend only on the affine singular points and their singularity types. But in general also other effects (due to the non propernesss of the polynomial function) can change the fibres. One calls this 'singular behaviour at infinity'.

Examples always played an important role in the investigation of germs of isolated hypersurface singularities. From there several effects, such as characterizations of simpelness, strange duality, etc were discovered. This is the reason that we started also in the global case to study examples in a systematic way and to classify polynomials of low degree d and a small number of variables n. In this note we give the classifcation in the case $n = 2$ and $d = 4$, continuing the work of C.T.C Wall [Wal2] in case $n = 2$ and $d = 3$.

We recall first a usal technique for studying singularities at infinity.

Let $f(x, y)$ be a polynomial of degree d. We can write: $f = f_0 + f_1 + \cdots + f_d$ with $f_d \neq 0$, the decomposition in graded pieces.

We define $\bar{f}(x, y, z) = z^d f(\frac{x}{z}, \frac{y}{z}) = f_d + z f_{d-1} + z^2 f_{d-2} + \cdots + z^{d-1} f_1 + z^d f_0$ and

$$F(x, y, z; t) = \bar{f}(x, y, z) - t z^d$$

The equation $F(x, y, z; t) = 0$ defines for each t a curve in $P^2(\mathbb{C})$.

Consider $\bar{X} = \{(x, t) \in P^2(\mathbb{C}) \times \mathbb{C} \mid F(x, y, z; t) = 0\}$. $\bar{X}$ is the closure of the graph X of f.

Let $t : \bar{X} \to \mathbb{C}$ be the projection on the second factor; the map t is an extention of f to the compactification $\bar{X}$ of X. Hà-Lê [HL] used the statification theory of this algebraic map to show that there is only a finite set of points in $\mathbb{C}$, above which $t : \bar{X} \to \mathbb{C}$ fails to be a (stratified) local trivial fibre bundle. The smallest set in $\mathbb{C}$, above which $f : \mathbb{C}^2 \to \mathbb{C}$ fails to be a local trivial fibre bundle is called the bifurcation set of f and is denoted by $B = B_f$. The fibres $\bar{X}_t$ of t are defined by $F(x, y, z; t) = 0$ with t fixed; the fibers of f are denoted by X_t and are defined by $f(x, y) = t$.

Remark that the intersection of $\bar{X}$ with infinity ($z = 0$) is given by the equation

$$f_d(x, y) = 0,$$

which defines a finite set of points A in $\mathbb{P}^1(\mathbb{C})$. This intersection is independent of t and $\bar{X}$ intersects infinity in the space $A \times \mathbb{C}$.

We consider the family of functions

$$F_t(x, y, z) := F(x, y, z, t)$$

near each of the points a in A.

In case that f has only isolated singularities (in the affine plane) these are families of isolated hypersurface singularities. For generic values of t there is a well-defined μ-class with Milnor number μ_a^∞. For a finite number of bifurcation values t this type can change and the Milnor number can drop with a value $\lambda_a^t = \mu(\bar{X}_t, a) - \mu_a^\infty$. In case that f has non-isolated singularities then the family of isolated singularties will jump at that critical value to a non-isolated germ at the corresponding intersection point at infinity.

Let $\mu_p(f)$ be the local Milnor number of f at the point $p \in \mathbb{C}$. (For regular points, this is zero).

Definition: *global Milnor number:*

$$\mu(f) = \sum \mu_p(f)$$

Definition: *total jump:*

$$\lambda(f) = \sum_{t \in B} \sum_{a \in A} \lambda_a^t(f)$$

We now recall the global bouquet theorem as studied by Broughton [Br], Hà-Lê [HL] and in general form by Siersma-Tibar [SiTi]:

Theorem: *Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial with isolated singularities . Then all fibres of f have the homotopy type of a bouquet of 1-spheres and the number γ of spheres in a general fibre is equal to the sum $\mu(f) + \lambda(f)$, where $\mu(f)$ is the total Milnor number of f and $\lambda(f)$ is the sum of the jumps at infinity.*
In particular $\lambda(f)$ is invariant under diffeomorphisms of C^n.
Moreover let $\mu^b(f)$ denote the sum of the Milnor numbers of all the singularities of the fibre X_b and let $\lambda^b(f)$ denote the sum of all Milnor numbers at infinity at $A \times \{b\}$. Then

$$\chi(X_t) - \chi(X_b) = (-1)^{n-1}(\lambda^b(f) + \mu^b(f)),$$

where X_t is a general fibre of f.

Remark: There is a second formula (cf. eg [HL]), which computes the first Betti-numbers of the fibres For the generic fibre this formula is:

$$\gamma = b_1(X_t) = d^2 - 3d + 1 - \sum_{a \in A} (\mu_a^\infty - 1)$$

where the sum runs over all points of the intersection with infinity.

List of isolated singularities: (cf [AVG], [Sie1])

type	f	*conditions*
A_k	$x^2 + y^{k+1}$	$k \geq 0$
D_k	$x^2 y + y^{k-1}$	$k \geq 4$
E_6	$x^3 + y^4$	
E_7	$x^3 + xy^3$	
E_8	$x^3 + y^5$	
$X_9 = \tilde{E}_7$	$x^4 + y^4 + ax^2y^2$	$a^2 \neq 4$
X_{10}	$x^4 + y^5 + ax^2y^2$	$a \neq 0$
$J_{10} = \tilde{E}_8$	$x^3 + y^6 + axy^4$	$4a^3 + 27 \neq 0$

List of non-isolated singularities: (cf [Sie2], [dJ])

type	f	*transversal type*
A_∞	y^2	A_1
D_∞	xy^2	A_1
$J_{k,\infty}$ $(k \geq 2)$	$y^3 + x^k y^2$	A_1
$T_{\infty,k,2}$ $(k \geq 4)$	$x^2y^2 + y^k$	A_1
$T_{\infty,\infty,2}$	x^2y^2	A_1
$F_1 A_2$	xy^3	A_2
$\mathbf{T}A_k$	y^{k+1}	A_k

In this list the singular locus is a smooth 1-dimensional subvariety, the x-axis. These singularities are called line singularities. The transversal type is the type of the singularity if one restricts f to $H_c = \{x = c\}$ with $c \neq 0$.
A complete study of the local behaviour of non-isolated singularities of functions of two variables is given in [Sch], [Sie2] and [dJ].

Conventions: We will use the following notations:
$f \simeq g$: in case f is (algebraic) diffeomorfic to g.
$f \cong g$: in case f is diffeomorfic to g by an affine linear transformation.
We both use transformations in the source and target spaces. Examples are translations $\mathbb{T}_{\alpha,\beta}$:

$$\begin{cases} x := & x - \alpha \\ y := & y - \beta \end{cases}$$

and similar ones for the target. Since we are studying all fibres $f(x, y) = t$, we can assume that the constant term $f_0 = 0$ and consider the actions modulo constants.

In all cases we consider first the factorisation of f_d in its linear parts and put f_d into a normal form using linear transformations of the source.

We use the following notation:

$\mathbf{A^2BC}$ means that f_d has one double factor and two single factors at infinity, etc. The notation $\mathbf{A^3B}_{\gamma,\lambda}$ means that the singularities in this class have first betti number γ and total jump λ. In case $\lambda = 0$ we omit the second number, e.g $\mathbf{A^3B}_6$. For jumps to non-isolated singularities we use lower case stars, e.g. $\mathbf{A^2B}_{1,\star}$. The type is well-defined up to affine linear transformations.

For each singularity class we select one of its members with few monomials as its 'typical' form.

We next mention the types of the singular point at infinity. The notation $A_2 \to A_3$ means that the singularity (at infinity) jumps from A_2 to A_3 for some bifurcation value.

The classification in degree 1 and 2 are classical. Degree 3 already has been studied by Wall [Wal2]. But we have redone this part to conform it to our notations.

Remark: We found the following interesting behaviour in the list. The singularity types $\mathbf{A^2BC}_{5,\lambda}$ occur for $\lambda = 1, 2, 3, 4$ all with the same homotopy type of the general fibre. So one should consider the family

$$f = x^2(x^2 - y^2) + Kx^3 + Px^2 + Qxy + Ax = t$$

depending on K, P, Q, A and t and study its behaviour. Since $\mu + \lambda$ is constant, some critical point is expected to go to infinity if λ increases. What is the effect on monodromy, etc ? The same feature occurs in other cases, and in all those examples $\lambda \geq 1$.

Two variables, degree 1

Case A: $f = x$

type	$f_1 = x$				
A	$f = x$				
	'typical' form	$(0:1:0)$	μ	λ	γ
$\mathbf{A}_0$	x	A_0	0	0	0

Two variables, degree 2

Case AB: $f_2 = xy$

type	$f_2 = xy$				
AB	$f = xy$				
	'typical' form	two points at ∞	μ	λ	γ
$\mathbf{AB}_1$	xy	A_0, A_0	1	0	1

Case $\mathbf{A^2}$: $f_2 = x^2$

type	$f_2 = x^2$						
$\mathbf{A^2}$	$f = x^2 + By$						
	'typical' form	$(0:1:0)$	$\neq 0$	$= 0$	μ	λ	γ
$\mathbf{A^2}_0 \simeq \mathbf{A}_0$	$x^2 + y$	A_0	B		0	0	0
$\mathbf{A^2}_\star$	x^2	$A_1 \to A_\infty$		B	∞	$\star$	
All jumps occur at $t = 0$							

Two variables, degree 3

With homogeneous coordinates we can choose 3 point freely at the line of infinity which determine the whole coodinate system. Therefore f_3 can be transformed into one of the following cases:

$$f_3 = xy(x+y)$$

$$f_3 = x^2 y$$

$$f_3 = x^3$$

Case ABC: $f_3 = xy(x+y)$

We get that

$$f(x,y) = xy(x+y) + Kx^2 + Lxy + My^2 + Px + Qy$$

By using translations $\mathbb{T}_{M,K}$ we can assume $K = M = 0$.
The intersection points with infinity are $(1:0:0)$, $(0:1:0)$ and $(1:-1:0)$. In all the cases we find that

the intersection points are nonsingular, so we have type A_0, which is always the case for non multiple factors in f_d.

type	$f_3 = xy(x+y)$				
ABC	$f = xy(x+y) + Lxy + Px + Qy$				
	'typical' form	three points at ∞	μ	λ	γ
$\mathbf{ABC}_4$	$x^3 + y^3$	A_0, A_0, A_0	4	0	4

Case $\mathbf{A^2B}$: $f_3 = x^2y$

Let's look at the general form:

$$f(x,y) = x^2y + Kx^2 + Lxy + My^2 + Px + Qy$$

By using translations $\mathbb{T}_{\frac{L}{2},K}$ we can assume that $L = K = 0$: The intersection points with infinity are $(0:1:0)$ and $(1:0:0)$.
In the case of $(1:0:0)$ we find that the intersection point is always nonsingular, so type A_0.
The point $(0:1:0)$ in singular iff $M = 0$.
Let's assume that $M = 0$, then the corresponding hypersurface $F(x,y,z,t) = 0$ is given by:

$$F = x^2y + Pxz^2 + Qyz^2 - tz^3$$

On the chart $y = 1$, we have

$$F = x^2 + Pxz^2 + Qz^2 - tz^3$$

If $Q \neq 0$ we have $F \simeq x^2 + z^2$ (A_1).
If $Q = 0$ we can use the translation

$$\begin{cases} x := & x - \frac{1}{2}Pz^2 \\ z := & z \end{cases}$$

to get

$$F \simeq x^2 - tz^3 - \frac{1}{4}P^2z^4$$

So we find that if $t \neq 0$, then we have type A_2.
If $t = 0$ and $P \neq 0$, this gives type A_3.
If $t = P = 0$, we have type A_∞.

type	$f_3 = x^2y$							
$\mathbf{A^2B}$	$f = x^2y + My^2 + Px + Qy$							
	'typical' form	$(0:1:0)$	$(1:0:0)$	$\neq 0$	$= 0$	μ	λ	γ
$\mathbf{A^2B}_3$	$x^2y + y^2$	A_0	A_0	M		3	0	3
$\mathbf{A^2B}_2$	$x^2y + y$	A_1	A_0	Q	M	2	0	2
$\mathbf{A^2B}_{1,1}$	$x^2y + x$	$A_2 \to A_3$	A_0	P	M, Q	0	1	1
$\mathbf{A^2B}_{1,\star}$	x^2y	$A_2 \to A_\infty$	A_0		M, Q, P	∞	$\star$	1
All jumps occur at $t = 0$.								

Case $\mathbf{A^3}$: $f_3 = x^3$

Let's look at the general form

$$f(x,y) = x^3 + Kx^2 + Lxy + My^2 + Px + Qy$$

By using the translation $\mathbb{T}_{\frac{K}{3},0}$ we can assume that $K = 0$.
The intersection point at infinity is $(0:1:0)$.
So the corresponding hypersurface $F(x, y, z, t) = 0$ is given by

$$F = x^3 + Lxyz + My^2z + Pxz^2 + Qyz^2 - tz^3$$

On chart $y = 1$ we have

$$F = x^3 + Lxz + Mz + Pxz^2 + Qz^2 - tz^3$$

So the intersection point is singular iff $M = 0$.
If $L \neq 0$, we have corank 0, so type A_1. So we can assume that $L = 0$.
If $Q \neq 0$, we have type A_2, because of the influence of x^3.

Consider the case $M = L = Q = 0$.
We have

$$F = x^3 + Pxz^2 - tz^3$$

If $P \neq 0$, we have type D_4 with a jump to D_∞ if $t^2 = -\frac{4P^3}{27}$. So there are two jumps.

In the case $P = 0, t \neq 0$, we have D_4. If $P = t = 0$, we have $F = x^3$ this type is called $\mathbf{T}A_2$. So if $P = 0$, we get a jump from D_4 to $\mathbf{T}A_2$ at $t = 0$.

type	$f_3 = x^3$						
$\mathbf{A^3}$	$f = x^3 + Lxy + My^2 + Px + Qy$						
	'typical' form	$(0:1:0)$	$\neq 0$	$= 0$	μ	λ	γ
$\mathbf{A^3}_2$	$x^3 + y^2$	A_0	M		2	0	2
$\mathbf{A^3}_1 \simeq \mathbf{AB}_1$	$x^3 + xy$	A_1	L	M	1	0	1
$\mathbf{A^3}_0 \simeq \mathbf{A}_0$	$x^3 + y$	A_2	Q	M, L	0	0	0
$\mathbf{A^3}_\star$	$x^3 + x$	$D_4 \to D_\infty$	P	M, L, Q	∞	$\star$	
$\mathbf{A^3}_{\star\star}$	x^3	$D_4 \to \mathbf{T}A_2$		M, L, Q, P	∞	$\star$	
All jumps occur if $t^2 = -\frac{4P^3}{27}$ Remark that in the last case $P = 0$, so the jump occurs at $t = 0$							

Two variables, degree 4

Like degree 3 we can use linear transformations to transform f_4 over $\mathbb{C}$ into one of the following cases:

$$f_4 = x^4 + y^4 + ax^2y^2, \quad \text{with } a^2 \neq 4$$

$$f_4 = x^2(x^2 - y^2)$$

$$f_4 = x^2y^2$$

$$f_4 = x^3y$$

$$f_4 = x^4$$

Case ABCD: $f_4 = x^4 + y^4 + ax^2y^2$

All the intersectionpoints at infinity are regular (type A_0).

type	$f_4 = x^4 + y^4 + ax^2y^2$				
ABCD	$f = f_4 + Lxy^2 + Mxy^2 + Px^2 + Qxy + Ry^2 + Ax + By$				
	'typical' form	four points at ∞	μ	λ	γ
$\mathbf{ABCD}_9$	$x^4 + y^4$	A_0, A_0, A_0, A_0	9	0	9

Case $\mathbf{A^2BC}$: $f_4 = x^2(x^2-y^2)$

First we consider

$$f_3 = Kx^3 + Lx^2y + Mxy^2 + Ny^3$$

By using translations $\mathbb{T}_{\frac{M}{2},\frac{L}{2}}$ we can assume that $L = M = 0$. Remark that we could have put K instead of M zero.

So we get

$$f = x^2(x^2-y^2) + Kx^3 + Ny^3 + Px^2 + Qxy + Ry^2 + Ax + By$$

The intersection points at infinity are $(0:1:0)$, $(1:1:0)$ and $(1:-1:0)$.
The corresponding hypersurface $F(x,y,z,t) = 0$ is given by

$$F = x^2(x^2-y^2) + Kx^3z + Ny^3z + Px^2z^2 + Qxyz^2 + Ry^2z^2 + Axz^3 + Byz^3 - tz^4$$

On chart $x = 1$ we get that the points $(1:1:0)$ and $(1:-1:0)$ are always regular (A_0).

To study $(0:1:0)$ we consider the chart $y = 1$:

$$F = x^4 - x^2 + Kx^3z + Nz + Px^2 + Qxz^2 + Rz^2 + Axz^3 + Bz^3 - tz^4$$

So the point $(0:1:0)$ is singular iff $N = 0$. If $R \neq 0$ we have type A_1.
Next we assume that $N = R = 0$.
We have now

$$F = x^4 - x^2 + Kx^3z + Px^2z^2 + Qxz^2 + Axz^3 + Bz^3 - tz^4$$

If $B \neq 0$, we have type A_2.
In the case $B = 0$ we have

$$F = -(x - \frac{1}{2}Qz^2)^2 + z^4(-t + \frac{1}{4}Q^2) + Axz^3 + Px^2z^2 + x^4 + Kx^3z$$

So if $t \neq \frac{Q^2}{4}$, we have type A_3.
If $B = 0$ and $t = \frac{Q^2}{4}$, we get a jump to A_4 iff $AQ \neq 0$.
So let's assume that $AQ = 0$. We split this into two cases:
1) $Q \neq 0$
2) $Q = 0$

1) $Q \neq 0$, so $A = 0$. In this case we have type A_3 with a jump to A_5 at $t = \frac{Q^2}{4}$ iff $PQ^2 \neq 0$ which is equivalent to $P \neq 0$.
If $P = 0$ and $K \neq 0$, we have type A_3 with a jump to A_6 when $t = \frac{Q^2}{4}$ and if $P = K = 0$ we have type A_3 with a jump at $t = \frac{Q^2}{4}$ to A_7, because $Q \neq 0$.

2) $Q = 0$. We get

$$F = -x^2 + Axz^3 + x^4 + Px^2z^2 + Kx^3z - tz^4$$

So if $A \neq 0$, this gives type A_3 with a jump to A_5 at $t = 0$.
In case $A = 0$, we have

$$F = x^2(-1 + x^2 + Pz^3 + Kxz) - tz^4$$

so we have type A_3 with a jump to A_∞ when $t = 0$.

type	$f_4 = x^2(x^2 - y^2)$								
$\mathbf{A^2BC}$	$f = x^2(x^2 - y^2) + Kx^3 + Ny^3 + Px^2 + Qxy + Ry^2 + Ax + By$								
	'typical' form	$(0:1:0)$	$(1:-1:0)$	$(1:1:0)$	$\neq 0$	$= 0$	μ	λ	γ
$\mathbf{A^2BC}_8$	$x^4 - x^2y^2 + y^3$	A_0	A_0	A_0	N		8	0	8
$\mathbf{A^2BC}_7$	$x^4 - x^2y^2 + y^2$	A_1	A_0	A_0	R	N	7	0	7
$\mathbf{A^2BC}_6$	$x^4 - x^2y^2 + y$	A_2	A_0	A_0	B	N, R	6	0	6
$\mathbf{A^2BC}_{5,1}$	$x^4 - x^2y^2 + Qxy + Ax$	$A_3 \to A_4$	A_0	A_0	Q, A	N, R, B	4	1	5
$\mathbf{A^2BC}_{5,2}$	$x^4 - x^2y^2 + Qxy + Px^2$	$A_3 \to A_5$	A_0	A_0	Q, P	N, R, B, A	3	2	5
$\mathbf{A^2BC}_{5,3}$	$x^4 - x^2y^2 + Qxy + Kx^3$	$A_3 \to A_6$	A_0	A_0	Q, K	N, R, B, A, P	2	3	5
$\mathbf{A^2BC}_{5,4}$	$x^4 - x^2y^2 + Qxy$	$A_3 \to A_7$	A_0	A_0	Q	N, R, B, A, P, K	1	4	5
$\mathbf{A^2BC}^\star_{5,2}$	$x^4 - x^2y^2 + x$	$A_3 \to A_5$	A_0	A_0	A	N, R, B, Q	3	2	5
$\mathbf{A^2BC}_{5,\star}$	$x^4 - x^2y^2$	$A_3 \to A_\infty$	A_0	A_0		N, R, B, Q, A	∞		5
All jumps occur if $t = \frac{Q^2}{4}$. Remark that in the last two cases $Q = 0$, so the jump occurs at $t = 0$.									

NB: The two cases $\mathbf{A^2BC}_{5,3}$ and $\mathbf{A^2BC}^\star_{5,3}$ are connected !

Case $\mathbf{A^2B^2}$: $f_4 = x^2y^2$

First we consider

$$f_3 = Kx^3 + Lx^2y + Mxy^2 + Ny^3$$

By using translations $\mathbb{T}_{\frac{M}{2},\frac{L}{2}}$ we can assume $L = M = 0$

There are two intersection points with infinity $(1:0:0)$ and $(0:1:0)$.

If $K = 0$ then $(1:0:0)$ is a singular point and
if $N = 0$ then $(0:1:0)$ is a singular point.

Consider the general form:

$$f = x^2y^2 + Kx^3 + Ny^3 + Px^2 + Qxy + Ry^2 + Ax + By$$

on which we have still the possibility of permuting our coordinates x and y and multiplication with constants.

The corresponding hypersurface $F(x, y, z, t) = 0$ is given by:

$$F = x^2y^2 + Kx^3z + Ny^3z + Px^2z^2 + Qxyz^2 + Ry^2z^2 + Axz^3 + Byz^3 - tz^4$$

First we consider the case that $N \neq 0$, where $(0:1:0)$ is nonsingular, but later we'll use the same computations in case $N = 0$.

On the chart $x = 1$ we have:

$$F = y^2 + Kz + Ny^3z + Pz^2 + Qyz^2 + Ry^2z^2 + Az^3 + Byz^3 - tz^4$$

We are going to discuss the type of F (depending on t) at $(0, 0)$:

0) If $K \neq 0$ we have the generic case: type A_0 : no jumps.
1) If $K = 0$ and $P \neq 0$: type A_1 : no jumps.
2) If $K = P = 0$ and $A \neq 0$: type A_2: no jumps.
3) Consider now $K = P = A = 0$:

$$F = y^2 + Ny^3z + Qyz^2 + Ry^2z^2 + Byz^3 - tz^4$$

If $t \neq -Q^2/4$ we have type A_3, but if $t = -Q^2/4$ we get a jump to type A_4 if $BQ \neq 0$, see the next computation:

Set

$$y := y - Q^2/2$$

then we get

$$F \simeq y^2 + Ny^3z - \frac{3}{2}NQy^2z^3 + \frac{3}{4}NQ^2yz^5 - \frac{1}{8}NQ^3z^7 + Ry^2z^2 - QRyz^4 + \frac{1}{4}RQ^2z^6 + Byz^3 - \frac{1}{2}BQz^5$$

The next step splits into two cases:
4a) $K = P = A = B = 0$:

$$F = y^2 + Ny^3z + Qyz^2 + Ry^2z^2 - tz^4$$

If $t \neq -Q^2/4$ we have type A_3, but if $t = -Q^2/4$ we get a jump to type A_5 if $QR \neq 0$ (use the same substitution as above with $B = 0$).
4b) $K = P = Q = A = 0$:

$$F = y^2 + Ny^3z + Ry^2z^2 + Byz^3 - tz^4$$

If $t \neq 0$ we have type A_3, but if $t = 0$ we get a jump to type A_5 if $B \neq 0$.

5a) Next combine the two conditions from 4): $K = P = Q = A = B = 0$:

$$F = y^2 + Ny^3z + Ry^2z^2 - tz^4$$

If $t \neq 0$ we have type A_3, but if $t = 0$ we get a jump to type A_∞

5b) The case 4a) degenerates to: $K = P = R = A = B = 0$:

$$F = y^2 + Ny^3z + Qyz^2 - tz^4$$

If $t \neq -Q^2/4$ we have type A_3, but if $t = -Q^2/4$ we get a jump to typeA_6 if $NQ \neq 0$. (This is the first case where $N \neq 0$ is really important).

6) $K = P = Q = R = A = B = 0$:

$$F = y^2 + Ny^3z - tz^4$$

If $t \neq 0$ we have type A_3, but if $t = 0$ we get a jump to type A_∞ (also if $N = 0$, but then the situation at the other point is changing drastically).

In the case $N = 0$ we have that both points are singular. The computations follow the above scheme and the answers are listed in the table below (second part):

type	$f_4 = x^2y^2$							
$\mathbf{A^2B^2}$	$f = x^2y^2 + Kx^3 + Ny^3 + Px^2 + Qxy + Ry^2 + Ax + By$							
	'typical' form	$(1:0:0)$	$(0:1:0)$	$=0$	$\neq 0$	μ	λ	γ
$\mathbf{A^2B^2}_7$	$x^2y^2 + x^3 + y^3$	A_0	A_0		N, K	7	0	7
$\mathbf{A^2B^2}_6$	$x^2y^2 + x^2 + y^3$	A_1	A_0	K	N, P	6	0	6
$\mathbf{A^2B^2}_5$	$x^2y^2 + x + y^3$	A_2	A_0	K, P	N, A	5	0	5
$\mathbf{A^2B^2}_{4,1}$	$x^2y^2 + y^3 + Qxy + Ry^2 + By$	$A_3 \to A_4$	A_0	K, P, A	N, Q, B, R	3	1	4
$\mathbf{A^2B^2}_{4,2}$	$x^2y^2 + y^3 + Qxy + Ry^2$	$A_3 \to A_5$	A_0	K, P, A, B	N, Q, R	2	2	4
$\mathbf{A^2B^2}^{\star}_{4,2}$	$x^2y^2 + y^3 + Ry^2 + By$	$A_3 \to A_5$	A_0	K, P, A, Q	N, R, B	2	2	4
$\mathbf{A^2B^2}_{4,\star}$	$x^2y^2 + y^3 + Ry^2$	$A_3 \to A_\infty$	A_0	K, P, A, B, Q	N, R	∞	$\star$	4
$\mathbf{A^2B^2}_{4,3}$	$x^2y^2 + y^3 + Qxy$	$A_3 \to A_6$	A_0	K, P, R, A, B	N, Q	1	3	4
$\mathbf{A^2B^2}_{4,\star\star}$	$x^2y^2 + y^3$	$A_3 \to A_\infty$	A_0	K, P, R, A, B, Q	N	∞	$\star$	4
$\mathbf{A^2B^2}^{+}_5$	$x^2y^2 + x^2 + y^2$	A_1	A_1	K, N	P, R	5	0	5
$\mathbf{A^2B^2}^{+}_4$	$x^2y^2 + x + y^2$	A_2	A_1	K, N, P	A, R	4	0	4
$\mathbf{A^2B^2}^{+}_{3,1}$	$x^2y^2 + Qxy + Ry^2 + By$	$A_3 \to A_4$	A_1	K, N, P, A	B, Q, R	2	1	3
$\mathbf{A^2B^2}^{+}_{3,2}$	$x^2y^2 + Qxy + Ry^2$	$A_3 \to A_5$	A_1	K, N, P, A, B	Q, R	1	2	3
$\mathbf{A^2B^2}^{+,\star}_{3,2}$	$x^2y^2 + Ry^2 + By$	$A_3 \to A_5$	A_1	K, N, P, A, Q	B, R	1	2	3
$\mathbf{A^2B^2}^{+}_{3,\star}$	$x^2y^2 + Ry^2$	$A_3 \to A_\infty$	A_1	K, N, P, A, B, Q	R	∞	$\star$	3
$\mathbf{A^2B^2}^{+}_3$	$x^2y^2 + x + y$	A_2	A_2	K, N, P, R	A, B	3	0	3
$\mathbf{A^2B^2}^{+}_{2,1}$	$x^2y^2 + Qxy + By$	$A_3 \to A_4$	A_2	K, N, P, R, A	B, Q	1	1	2
$\mathbf{A^2B^2}^{+}_{2,2}$	$x^2y^2 + By$	$A_3 \to A_5$	A_2	K, N, P, R, A, Q	B	0	2	2
$\mathbf{A^2B^2}^{+}_{1,\star}$	$x^2y^2 + Qxy$	$A_3 \to A_\infty$	$A_3 \to A_\infty$	K, N, P, R, A, B		∞	$\star$	1
All jumps occur if $t = -Q^2/4$								

Case $\mathbf{A^3B}$: $f_4 = x^3y$

First we consider

$$f_3 = Kx^3 + Lx^2y + Mxy^2 + Ny^3$$

By using translations $\mathbb{T}_{\alpha,\beta}$ we could assume $K = L = 0$, but we can also arrange that $K = B = 0$. We make this last choice and get easier computations later.
The intersection points at infinity are $(1:0:0)$ and $(0:1:0)$.
We find that $(1:0:0)$ is always regular (A_0) and $(0:1:0)$ is singular iff $N = 0$.
Consider the general form

$$f = x^3y + Lx^2y + Mxy^2 + Px^2 + Qxy + Ry^2 + Ax$$

The corresponding hypersurface $F(x, y, z, t) = 0$ is given by

$$F = x^3y + Lx^2yz + Mxy^2z + Px^2z^2 + Qxyz^2 + Ry^2z^2 + Axz^3 - tz^4$$

On chart $y = 1$ we have

$$F = x^3 + Lx^2z + Mxz + Px^2z^2 + Qxz^2 + Rz^2 + Axz^3 + -tz^4$$

If $M \neq 0$ we have type A_1.
So consider the case $N = M = 0$ and $R \neq 0$. We have

$$F = x^3 + Lx^2z + Px^2z^2 + Qxz^2 + Rz^2 + Axz^3 - tz^4$$

We find that $F \simeq x^3 + z^2$, we we have type A_2.

Consider $N = M = R = 0$. We have

$$f = x(x^2 + Lx + Q)y + Px^2 + Ax$$

$$F = x^3 + Lx^2z + Qxz^2 + Px^2z^2 + Axz^3 - tz^4$$

We have 3 cases; related to the behaviour of the three roots of $x(x^2 + Lx + Q)$:
1) three different roots: $L^2 - 4Q \neq 0$ and $Q \neq 0$
2) two different roots: $L^2 - 4Q = 0$ or $Q = 0$ but not both zero
3) one root: $L = Q = 0$

In the first case, we always have type D_4.
In the second case, we can assume that the double root corresponds to $x = 0$, which implies $Q = 0$.

$$F = x^3 + Lx^2z + Px^2z^2 + Axz^3 - tz^4$$

If $t \neq 0$ we have type D_5.

In case $t = 0$ and $A \neq 0$, then we have type D_6.
But if $A = 0$ then we have

$$F \simeq x^2(x + Lz + Pz^2)$$

So we have type D_∞ (since $L \neq 0$).

In the third case, we have

$$F = x^3 + Px^2z^2 + Az^3x - tz^4$$

If $t \neq 0$, we have E_6 .
If $t = 0$ and $A \neq 0$, we have E_7.
If $t = A = 0$ and $P \neq 0$, we have a non isolated point. Type $J_{2,\infty}$.
If $t = A = P = 0$, we have $F = x^3$ that is type $\mathbf{T}A_2$.

type	$f_4 = x^3y$							
$\mathbf{A^3B}$	$f = x^3y + Lx^2y + Ny^3 + Mxy^2 + Px^2 + Qxy + Ry^2 + Ax$							
	'typical' form	$(0:1:0)$	$(1:0:0)$	$\neq 0$	$= 0$	μ	λ	γ
$\mathbf{A^3B}_7$	$x^3y + y^3$	A_0	A_0	N		7	0	7
$\mathbf{A^3B}_6$	$x^3y + xy^2$	A_1	A_0	M	N	6	0	6
$\mathbf{A^3B}_5$	$x^3y + y^2$	A_2	A_0	R	N, M	5	0	5
$\mathbf{A^3B}_3$	$x^3y + xy$	D_4	A_0	$L^2 - 4Q, Q$	N, M, R	3	0	3
$\mathbf{A^3B}_{2,1}$	$x^3y + x^2y + y$	$D_5 \to D_6$	A_0	L, A	N, M, R, Q	1	1	2
$\mathbf{A^3B}_{2,\star}$	$x^3y + x^2y$	$D_5 \to D_\infty$	A_0	L	N, M, R, Q, A	∞	$\star$	2
$\mathbf{A^3B}_{1,1}$	$x^3y + x$	$E_6 \to E_7$	A_0	A	N, M, R, Q, L	0	1	1
$\mathbf{A^3B}_{1,\star}$	$x^3y + x^2$	$E_6 \to J_{2,\infty}$	A_0	P	N, M, R, Q, L, A	∞	$\star$	1
$\mathbf{A^3B}_{1,\star\star}$	x^3y	$E_6 \to \mathbf{T}A_2$	A_0		N, M, R, Q, L, A, P	∞	$\star$	1
All jumps other occur at $t = 0$								

Case $\mathbf{A^4}$: $f_4 = x^4$

In this case we will use a different approach. We will alter the normal form of f several times in order to get easier conditions on the constants.
Let's begin with the general case.

$$f = x^4 + Kx^3 + Lx^2y + Mxy^2 + Ny^3 + Px^2 + Qxy + Ry^2 + Ax + By$$

By using the translation $\mathbb{T}_{\frac{K}{4},0}$ we can assume $K = 0$.
The intersection point at infinity is $(0 : 1 : 0)$ and it is singular iff $N = 0$. If $N = 0$ and $M \neq 0$ we have type A_1.

So let's consider $N = M = 0$ and $R \neq 0$.

By using the translation $\mathbb{T}_{0,\frac{B}{2R}}$ we can put $B = 0$.
So the corresponding hypersurface $F(x, y, z, t) = 0$ on the chart $y = 1$ is given by

$$F = x^4 + L_1x^2z + R_1z^2 + P_1x^2z^2 + Q_1xz^2 + A_1z^3x - tz^4.$$

Remark that $R_1 = R \neq 0$, $L_1 = L$ and $Q_1 = Q$.
In the case $L_1^2 \neq 4R_1$ we have type A_3.

So let's assume that $L_1^2 = 4R_1$,

$$f = (x^2 - cy)^2 + P_1x^2 + Q_1xy + A_1x$$

where $c = L_1/2$. The non-linear substitution $y := c^{-1}y - x^2$ changes the polynomial to one of degree less than 4, which is considered before. (See the information in the list)

We continue now the affine classification of this case in degree 4. Remember $L_1^2 = 4R_1$, We have type A_4 iff $Q_1 \neq 0$.
In the case $Q_1 = 0$ and $P_1 \neq 0$, we have type A_5.
If $Q_1 = P_1 = 0$ and $A_1 \neq 0$, we have type A_6.
If $A_1 = 0$ and $t \neq 0$, we have type A_7 and if $t = 0$ as well, then we have type A_∞.

The other case is $M = N = R = 0$.
We have the following hypersurface:

$$\begin{aligned} F &= Lx^2z + Qxz^2 + Bz^3 + x^4 + Px^2z^2 + Axz^3 - tz^4 \\ &= z(Bz^2 + Lx^2 + Qxz) + x^4 + Px^2z^2 + Axz^3 - tz^4. \end{aligned}$$

We now split into two different cases, namely:
1) $L \neq 0$
2) $L = 0$

1) In the case $Q^2 \neq 4BL$ we have type D_4.
We can from now on assume that $Q^2 = 4BL$.
So we have $f = x^4 + Lx^2y + Px^2 + Qxy + Ax + By$.
By using the transformation $\mathbb{T}_{0,\frac{P}{L}}$ we find

$$f = x^4 + L_2x^2y + Q_2xy + A_2x + B_2y.$$

Remark that $L_2 = L \neq 0$, $Q_2 = Q$ and $B_2 = B$. So we have

$$F = z(L_2x^2 + B_2z^2 + Q_2xz) + x^4 + A_2xz^3 - tz^4.$$

This results into type D_5 iff

$$t \neq \frac{B}{L^2} - \frac{A_2 Q}{2L}.$$

If $t = \frac{B}{L^2} - \frac{A_2 Q}{2L}$, and $A_2 L^2 - 2QB \neq 0$ we have D_6 and in the other case we have D_∞.

2) If $L = 0$ and $Q \neq 0$, we can use the transformation

$$\begin{cases} x := x \\ \\ y := \frac{-P}{Q}x + y + \frac{PB - AQ}{Q^2} \end{cases}$$

on $f = x^4 + Px^2 + Qxy + Ax + By$ to get

$$F = z^2(Qx + Bz) + x^4 - tz^4.$$

This is type D_5, because of the influence of x^4.
The remaining case is $L = Q = 0$.
If $B \neq 0$, we can use the transformation

$$\begin{cases} x := x \\ y := y - \frac{A}{B}x \end{cases}$$

This results into the following form

$$F = Bz^3 + x^4 + Px^2z^2 - tz^4,$$

so we have type E_6.

If $L = Q = B = 0$, we find

$$F = x^4 + Px^2z^2 + Axz^3 - tz^4.$$

The polynomial is homogeneous of degree 4. In general there are four different solutions of the equation $F = 0$. In case we have a double root, the following system has to be solved.

$$\begin{cases} x^4 + Px^2 + Ax - t = 0 \\ 4x^3 + 2Px + A = 0 \end{cases}$$

So in case $A \neq 0$ and $27A^2 + 8P^3 \neq 0$, we have $\tilde{E}_7$ with jumps to $T_{\infty,4,2}$. We have three different jumps (t_1, t_2 and t_3). This is a result of the fact that $t = \frac{2Px^2 + 3Ax}{4}$ and there are three different solutions to $4x^3 + 2Px + A = 0$.

When $PA \neq 0$ and $27A^2 + 8P^3 = 0$, we get $\tilde{E}_7$ with jumps to F_1A_2 if $t^3 = \frac{-27A^4}{8^4}$.
If $A = 0$ and $P \neq 0$ then we have a very special situation. Here occurs two different jumps. We have $\tilde{E}_7$ with a jump to $T_{\infty,\infty,2}$ when $t = \frac{P^2}{4}$. And we have $\tilde{E}_7$ with a jump to $T_{\infty,4,2}$ when $t = 0$.
The remaining case is $P = A = 0$. In this case we have $\tilde{E}_7$ with a jump to $\mathbf{T}A_3$ if $t = 0$.

type	$f_4 = x^4$						
$\mathbf{A^4}$	$f = x^4 + Lx^2y + Mxy^2 + Ny^3 + Px^2 + Qxy + Ry^2 + Ax + By$						
	‘typical’ form	$(0:1:0)$	$\neq 0$	$= 0$	μ	λ	γ
$\mathbf{A^4}_6$	$x^4 + y^3$	A_0	N		6	0	6
$\mathbf{A^4}_5$	$x^4 + xy^2$	A_1	M	N	5	0	5
$\mathbf{A^4}_3$	$x^4 + y^2$	A_3	$R, (L_1^2 - 4R_1)$	M, N	3	0	3
$\mathbf{A^4}_2 \simeq \mathbf{A^3}_2$	$(x^2+y)^2 + xy$	A_4	R, Q_1	$M, N, (L_1^2 - 4R_1)$	2	0	2
$\mathbf{A^4}_1 \simeq \mathbf{AB}_1$	$(x^2+y)^2 + x^2$	A_5	R, P_1	$M, N, (L_1^2 - 4R_1), Q_1$	1	0	1
$\mathbf{A^4}_0 \simeq \mathbf{A}_0$	$(x^2+y)^2 + x$	A_6	R, A_1	$M, N, (L_1^2 - 4R_1), Q_1, P_1$	0	0	0
$\mathbf{A^4}_\star \simeq \mathbf{A^2}_\star$	$(x^2+y)^2$	$A_7 \to A_\infty$	R	$M, N, (L_1^2 - 4R_1), Q_1, P_1, A_1$	∞		
$\mathbf{A^4}^+_2 \simeq \mathbf{A^2B}_2$	$x^4 + x^2y + xy$	D_4	$(Q^2 - 4BL), Q, L$	N, M, R	2	0	2
$\mathbf{A^4}^+_{1,1} \simeq \mathbf{A^2B}_{1,1}$	$x^4 + x^2y + x$	$D_5 \xrightarrow{\star} D_6$	$L, (A_2L^2 - 2QB)$	$N, M, R, (Q^2 - 4BL)$	0	1	1
$\mathbf{A^4}^+_{1,\star} \simeq \mathbf{A^2B}_{1,\star}$	$x^4 + x^2y$	$D_5 \xrightarrow{\star} D_\infty$	L	$N, M, R, (Q^2 - 4BL),$ $(A_2L^2 - 2QB)$	∞	$\star$	1
$\mathbf{A^4}^+_1 \simeq \mathbf{AB}_1$	$x^4 + xy$	D_5	Q	N, M, R, L	1	0	1
$\mathbf{A^4}^+_0 \simeq \mathbf{A}_0$	$x^4 + y$	E_6	B	N, M, R, L, Q	0	0	0
$\mathbf{A^4}_{\star 1}$	$x^4 + Px^2 + Ax$	$\tilde{E}_7 \xrightarrow{*} T_{\infty,4,2}$	$A, (27A^2 + 8P^3)$	N, M, R, L, Q, B	∞		
$\mathbf{A^4}_{\star 2}$	$x^4 + Px^2 + Ax$	$\tilde{E}_7 \xrightarrow{\triangle} F_1A_2$	A, P	$N, M, R, L, Q, B, (27A^2 + 8P^3)$	∞		
$\mathbf{A^4}_{\star 3}$	$x^4 + Px^2$	$\tilde{E}_7 \xrightarrow{\diamond} T_{\infty,\infty,2}$	P	N, M, R, L, Q, B, A	∞		
$\mathbf{A^4}_{\star 4}$	$x^4 + Px^2$	$\tilde{E}_7 \to T_{\infty,4,2}$	P	N, M, R, L, Q, B, A	∞		
$\mathbf{A^4}_{\star 5}$	x^4	$\tilde{E}_7 \to \mathbf{T}A_3$		N, M, R, L, Q, B, A, P	∞		
The jumps marked with a $\star$ occur if $t = \frac{B}{L^2} - \frac{A_2Q}{2L}$. The jump marked with a $\diamond$ occurs if $t = \frac{P^2}{4}$. The jump marked with a $*$ occurs at $t \in \{t_1, t_2, t_2\}$. The jump marked with a $\triangle$ occurs when $t^3 = \frac{-27A^4}{8^4}$. All other jumps occur if $t = 0$.							

List of polynomials with $\mu(f) = 0$.

Only the following polynomials occur in the list (up to diffeomorphism $\cong$):

$\mathbf{A}_0$	x	
$\mathbf{A^2B}_{1,1}$	$x^2y + Px$	$P \neq 0$
$\mathbf{A^2B2}^+_{2,2}$	$x^2y^2 + By$	$B \neq 0$
$\mathbf{A^3B}_{1,1}$	$x^3y + Px^2 + Ax$	$A \neq 0$

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