LINES ON BRIESKORN-PHAM SURFACES

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Abstract. By using toric modifications and a result of Gonzalez-Sprinberg and Lejeune-Jalabert, we answer the following questions completely. On which Brieskorn-Pham surface there exist smooth curves passing through the singular point? If there exist, how "many" and what are the defining equations?

1. Introduction

Let $(X,0)$ be an analytic space germ embedded in $(\mathbb{C}^n,0)$, and $(L,0)$ be a smooth curve germ in $(\mathbb{C}^n,0)$. Since L is locally biholomorphic to a line, we often say that L is a line. If for a singular point $O \in X_{\text{sing}}$, there exists a line (smooth curve) L in $\mathbb{C}^n$ such that $O \in L$ and $L \setminus \{O\} \subset X_{\text{reg}}$, we say that X *contains* (*or has*) *a line* passing through O.

On a singular surface X in $\mathbb{C}^3$ one can not always find a smooth curve passing through (not contained in) the singular locus of X. Gonzalez-Sprinberg and Lejeune-Jalabert proved a criterion for the existence of smooth curve on any (two dimensional) surface. We quote this result here for the convenience of the reader. Let $\pi : \tilde{X} \longrightarrow (X,0)$ be the minimal resolution of a singular surface $(X,0)$. Let $\mathcal{L}$ be the set of lines on $(X,0)$. For an exceptional divisor E, let $\mathcal{L}_E$ denote the set of lines on $(X,0)$ with the strict transform intersecting E transversally. Let $\mathfrak{m}$ be the maximal ideal of the local ring $\mathcal{O}_{X,0}$. The cycle $\mathcal{Z}_X$ on $\tilde{X}$ defined by the ideal sheaf $\mathfrak{m}\mathcal{O}_{\tilde{X}}$ is called the *maximal cycle of* π.

Gonzalez-Sprinberg and Lejeune-Jalabert Theorem ([1, 2]) (1) *There exists lines on* $(X,0)$ *if and only if the maximal cycle* $\mathcal{Z}_X$ *has at least one reduced component;*

(2) *The set of all the lines on* $(X,0)$ *is a disjoint union of the* $\mathcal{L}_E$*'s with each* E *a reduced component in* $\mathcal{Z}_X$. *And* $\mathcal{L}_E$ *is called the family of lines corresponding to* E.

In [3, 4], lines on surfaces with simple singularities are classified by using the classification machinery. All the surfaces with simple or simple elliptic singularities passing through x-axis are equivalent to (under the coordinate transformation preserving the x-axis) some surfaces defined by explicit equations.

Remark that by the aforementioned theorem many singular surfaces have lines. Among the examples are minimal surfaces, sandwiched surfaces and so on.

In this paper we are interested in the following questions on the so-called Brieskorn-Pham type surfaces, denoted by $G(p,q,r)$, and defined in $\mathbb{C}^3$ by

$$x^p + y^q + z^r = 0.$$

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For what kind of (p, q, r), there are lines on $G(p, q, r)$? And for what kind of (p, q, r), there are no lines on $G(p, q, r)$? Furthermore, if there are lines, how "many"? What are the defining equations for the lines? We give complete answers to these questions. The method we use here is the toric modifications of surface singularities. We refer the reader to the book [6] for notions and notations.

There is an invariant, called the *torsion number* [3], attached to each line. This torsion number is defined as follows. For a pair of analytic space germs $(\Sigma, 0) \subset (X, 0)$, embedded in $(\mathbb{C}^n, 0)$, defined by the ideals $\mathfrak{g} \supset \mathfrak{h}$ of $\mathcal{O} := \mathcal{O}_{\mathbb{C}^n,0}$ respectively. Denote $\mathcal{O}_X := \mathcal{O}/\mathfrak{h}$, $\mathcal{O}_\Sigma := \mathcal{O}/\mathfrak{g}$. The image ideal of $\mathfrak{g}$ in $\mathcal{O}_X$ is denoted by $\bar{\mathfrak{g}}$. The $\mathcal{O}_\Sigma$-module $M := \bar{\mathfrak{g}}/\bar{\mathfrak{g}}^2 \simeq \mathfrak{g}/(\mathfrak{g}^2 + \mathfrak{h})$ is called the conormal module of $\bar{\mathfrak{g}}$. If the torsion submodule $T(M)$ of M has finite length, then the length is called the torsion number of (Σ, X) and is denoted by $\lambda := \lambda(\Sigma, X)$.

For two integers $m > n$, let $m/n := [m_1, m_2, \dots, m_k]$ be the continued fraction representation of m/n. Define $\Phi(m/n) := \max\{i \mid m_1 = \cdots = m_i = 2, 1 \le i \le k\}$. Denote the number of families of lines on $G(p, q, r)$ by $\varrho(p, q, r) := \sharp\{\mathcal{L}_E \neq \emptyset\}$. By the above mentioned theorem, $\varrho(p, q, r) = \sharp\{$ reduced components in $\mathcal{Z}_X\}$.

2. The Newton boundary and the dual Newton diagram

Let $s := \gcd(p, q, r)$ and $p_1 = \gcd(q, r)/s, q_1 = \gcd(r, p)/s, r_1 = \gcd(p, q)/s$. Let $\tilde{p}, \tilde{q}, \tilde{r}$ be the integers such that $p/s = \tilde{p}q_1r_1, q/s = p_1\tilde{q}r_1, r/s = p_1q_1\tilde{r}$. Then $\tilde{p}, \tilde{q}, \tilde{r}$ are pairwisely coprime.

The Newton boundary of $G(p, q, r)$ is a triangle with vertices $A(p, 0, 0), B(0, q, 0), C(0, 0, r)$ on the plane $x/p + y/q + z/r = 1$. The dual Newton diagram Γ^* of $G(p, q, r)$ consists of $E_1 = {}^{\mathrm{T}}(1, 0, 0)$, $E_2 = {}^{\mathrm{T}}(0, 1, 0)$, $E_3 = {}^{\mathrm{T}}(0, 0, 1)$ and $P = {}^{\mathrm{T}}(p_1\tilde{q}\tilde{r}, \tilde{p}q_1\tilde{r}, \tilde{p}\tilde{q}r_1)$, where and in the following we use "T" to indicate that the vector is a column vector.

According to [6, III (6.3)], in order to obtain a *canonical toric resolution* of the surface $G(p, q, r)$, we need to make a *canonical regular simplicial subdivision* Σ^* of Γ^*, which we describe roughly as follows (see [6, 7] for details). For two vectors Q_1 and Q_2 in $\mathbb{Z}^3$, denote by $\det(Q_1, Q_2)$ the greatest common divisor of the absolute values of the 2×2 minors of the 3×2 matrix (Q_1, Q_2). Obviously, $\det(E_1, P) = \tilde{p}$. If $\tilde{p} > 1$, E_1P is not regular and we need to subdivide it. By [6, II(2.1)], there exists a unique integer $\tilde{p} > \tilde{p}_1 > 0$ such that the vector $R_1 := (P + \tilde{p}_1E_1)/\tilde{p}$ is integral. Then $\det(R_1, E_1) = 1$ and $\det(R_1, P) = \tilde{p}_1 < \tilde{p}$. If $\tilde{p}_1 > 1$, repeat the process for R_1P, until we obtain a regular subdivision of E_1P. In this way, we obtain the canonical regular simplicial subdivisions of E_1P, E_2P, E_3P. Let $R_i = {}^{\mathrm{T}}(r_{1,i}, r_{2,i}, r_{3,i})$ $(1 \le i \le \gamma)$ be the points being added to E_1P (see Figure 1). Similarly one adds $S_j = {}^{\mathrm{T}}(s_{1,j}, s_{2,j}, s_{3,j})$ $(1 \le j \le \sigma)$ to E_2P and $T_k = {}^{\mathrm{T}}(t_{1,k}, t_{2,k}, t_{3,k})$ $(1 \le k \le \tau)$ to E_3P. Then Σ^* is a regular simplicial subdivision of Γ^* such that its restriction to E_iP $(i = 1, 2, 3)$ is the canonical regular simplicial subdivision of E_iP described above.

The dual resolution graph $\mathcal{G}$ of $G(p, q, r)$ is star-shaped with the center P, the exceptional divisor determined by P. In the following we use the capital letters to denote the points on Γ^* as well as the exceptional divisors corresponding to them by abusing the symbol. The arms in $\mathcal{G}$ consist of $sp_1 = \gcd(q, r)$ copies of PR_1, $sq_1 = \gcd(r, p)$ copies of PS_1 and $sr_1 = \gcd(p, q)$ copies of PT_1 (loc. cit.).

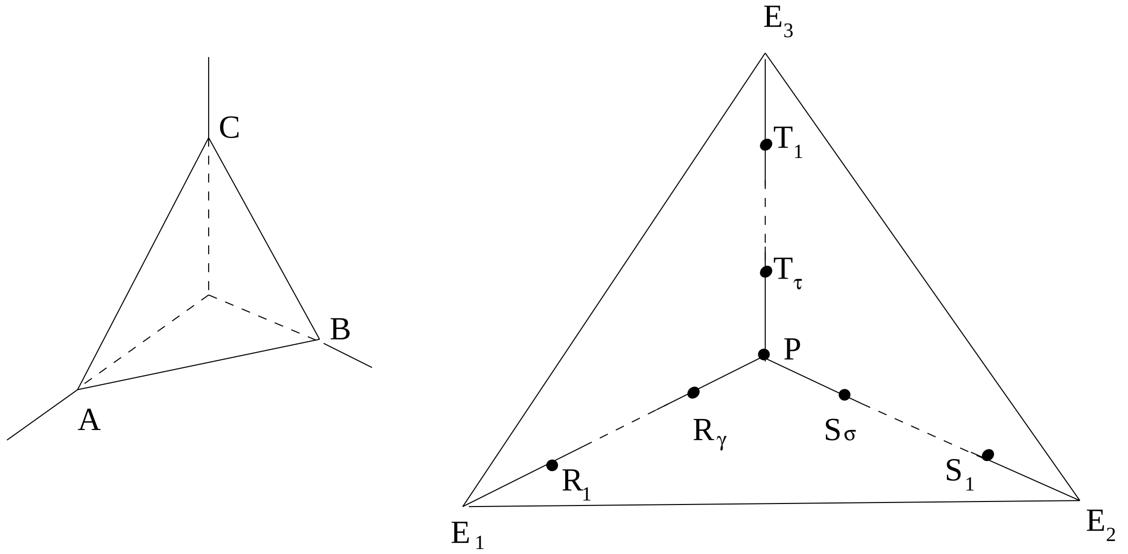

FIGURE 1. Newton boundary and dual Newton diagram of Brieskorn-Pham surface

Lemma 1. *With the notations as above, we have*

$$r_{l,i} \le r_{l,i+1} \le \theta_l, \quad s_{l,j} \le s_{l,j+1} \le \theta_l, \quad t_{l,k} \le t_{l,k+1} \le \theta_l, \quad l = 1,2,3,$$

where $\theta_1 := p_1\tilde{q}\tilde{r}$, $\theta_2 := \tilde{p}q_1\tilde{r}$, $\theta_3 := \tilde{p}\tilde{q}r_1$ *are the coordinates of* P.

Proof. We prove the lemma for the R_i's. Let π be the toric modification associated with the above regular simplicial cone subdivision Σ^*. Let $(\pi^*x) = E_1 + \sum r_{1,i}E(R_i) + p_1\tilde{q}\tilde{r}E(P) + D$, where D is the sum of the divisors which do not intersect with $E(R_i)$'s. Note that the exceptional divisor $E(R_i)$ is a disjoint union of $E(R_i^{(\alpha)})$ $(1 \le \alpha \le sp_1)$. Each $E(R_i^{(\alpha)})$ possesses the following properties: 1) $E(R_i^{(\alpha)})$ is isomorphic to $\mathbb{P}^1$, 2) the self intersection number of $E(R_i^{(\alpha)})$ is not bigger than -2 by [7] and 3) $E(R_i^{(\alpha)})$ only intersects with $E(R_{i-1}^{(\alpha)})$ and $E(R_{i+1}^{(\alpha)})$ (here $R_0^{(\alpha)} := E_1^{(\alpha)}$, and $R_{\gamma+1}^{(\alpha)} := P$), and $E(R_i^{(\alpha)}) \cdot E(R_{i+1}^{(\beta)}) = \delta_{\alpha\beta}$ ([6] P.156). Hence from $(\pi^*x)\cdot E(R_1^{(\alpha)}) = 1 + r_{1,1}E(R_1^{(\alpha)})^2 + r_{1,2} = 0$, we have $E(R_1^{(\alpha)})^2 = -(1+r_{1,2})/r_{1,1} \le -2$ which implies $r_{1,1} \le r_{1,2}$. From $(\pi^*y) \cdot E(R_1^{(\alpha)}) = 0 + r_{2,1}E(R_1^{(\alpha)})^2 + r_{2,2} = 0$, we have $r_{2,2} \ge r_{2,1}$. Similarly, we can prove the other inequalities. □

3. The existence of Lines

Theorem 2. *Let* $s = \gcd(p,q,r)$. $G(p,q,r)$ *has lines if and only if* $G(p/s, q/s, r/s)$ *has lines.*

Proof. It follows from the fact that the two surfaces have the same dual Newton diagram. □

Theorem 3. *Assume* $\gcd(p,q,r) = 1$ *and* $(\sharp):\ p \le q \le r$. $G(p,q,r)$ *has lines if and only if at least one of the following conditions holds.*

I) *Two of the three integers* p, q, r *are coprime and the other one is divisible by at least one of the coprime numbers.*
II) *The inequality* $r > pq/\gcd(p,q)$ *holds.*

Proof. We continue to use the notations in the beginning of §2. By the assumption ($\sharp$), we have the inequalities ($\sharp\sharp$) : $p_1\tilde{q}\tilde{r} \geq \tilde{p}q_1\tilde{r} \geq \tilde{p}\tilde{q}r_1$.

We first show the necessity.

(a) Assume that there exists 1 in the coordinates of P. Then by ($\sharp\sharp$) we have $\tilde{p}\tilde{q}r_1 = 1$ which implies that $\gcd(p,q) = 1$ and $pq|r$. This corresponds to a special case of (I).

(b) Let $\det(P, E_1) = \tilde{p} > 1$. The coordinates of R_1 are $r_{1,1} = (p_1\tilde{q}\tilde{r}+\tilde{p}_1)/\tilde{p} \geq q_1\tilde{r}+\tilde{p}_1/\tilde{p} > 1$ by ($\sharp\sharp$), $r_{2,1} = q_1\tilde{r} \geq \tilde{q}r_1 = r_{3,1}$. We assume that $r_{3,1} = 1$. This implies that $r_1 = \tilde{q} = 1$. Then we have $p = \tilde{p}q_1, q = p_1, r = p_1\tilde{r}$, from which we have $\gcd(p,q) = 1$ and $q|r$. This corresponds to a case of (I).

(c) In case $\det(P, E_2) = \tilde{q} > 1$, the coordinates of S_1 are $s_{2,1} = (\tilde{p}q_1\tilde{r} + \tilde{q}_1)/\tilde{q} > 1$, $s_{1,1} = p_1\tilde{r} \geq s_{3,1} = \tilde{p}r_1$. Assume that $s_{3,1} = \tilde{p}r_1 = 1$. Then $p|r$ and $\gcd(p,q) = 1$. This is also a case of (I).

(d) If $\det(P, E_3) = \tilde{r} > 1$, the coordinates of T_1 are $t_{1,1} = p_1\tilde{q} \geq t_{2,1} = \tilde{p}q_1$ and $t_{3,1} = (\tilde{p}\tilde{q}r_1 + \tilde{r}_1)/\tilde{r}$. In this case, we have only two possibilities. (d-1) Firstly, assume that $t_{2,1} = 1$ which is equivalent to $p|q$ and $\gcd(p,r) = 1$. This is a case of (I). Since $\gcd(p,q,r) = 1$, this is also a case of (II) if $p > 1$. (d-2) Secondly, assume that $t_{3,1} = 1$. This implies $\tilde{r} = \tilde{p}\tilde{q}r_1 + \tilde{r}_1$ for some $\tilde{r} > \tilde{r}_1 > 0$, which implies that $\tilde{r} > \tilde{p}\tilde{q}r_1$. This implies that $r > pq/\gcd(p,q)$, which corresponds to (II).

Note that all the coordinates of R_i ($i > 1$ in case $\tilde{p}_1 > 1$) and that of S_j ($j > 1$ in case $\tilde{q}_1 > 1$) are bigger than one. Also the first two coordinates of T_k ($k > 1$ in case $\tilde{r}_1 > 1$) are bigger than one. By lemma 1, $t_{3,k} = 1$ implies that $t_{3,\alpha} = 1$ for all $\alpha = 1,\dots,k-1$. Hence $r > pq/\gcd(p,q)$.

We show the sufficiency. Assume first that (II): $r > pq/\gcd(p,q)$. Then $\tilde{r} > \tilde{p}\tilde{q}r_1$ and $\tilde{r}_1 = \tilde{r} - \tilde{p}\tilde{q}r_1$ and we get $t_{3,1} = 1$. Next, assume that (I) is satisfied. Then by ($\sharp$), either (1) $\gcd(p,q) = 1$ and $p|r$ or $q|r$ or (2) $\gcd(p,r) = 1$ and $p|q$. (1) corresponds to a case of (a), (b), or (c) and (2) corresponds to a case of (a) or (d). For example assume that $\gcd(p,q) = 1$ and $p|r$. This implies that $r_1 = 1$ and $\tilde{p}q_1|\tilde{r}p_1q_1$. Thus $\tilde{p} = 1$ and this corresponds to (c) (respectively to (a)) if $\tilde{q} > 1$ (resp. if $\tilde{q} = 1$). The other cases are treated similarly. $\square$

Remark 4. The case that $p = 1$ is a special case of (I) and (II).

4. Equations, numbers and torsion numbers of the lines

We continue to use the notations in the beginning of §2 and to assume $p \leq q \leq r$. Gonzalez-Sprinberg and Lejeune-Jalabert theorem also gives a way to find the defining equations of the lines (as curves in $\mathbb{C}^3$) on a surface. The computation of the torsion number can be done as follows. For a line L with parameterization $x = x(t), y = y(t), z = z(t)$, one may assume that $\frac{dx}{dt}(0) \neq 0$. By the inverse function theorem, we have $t = t(x)$. Hence the line L can be defined by $y - \phi(x) = 0$ and $z - \psi(x) = 0$. We choose the following local coordinate transformation of $(\mathbb{C}^3, 0)$: $x' = x, y' = y - \phi(x), z' = z - \psi(x)$. The defining ideal of L is thus $\mathfrak{g} = (y', z')$. The defining equation of the surface containing L can be written as $h \equiv ax'y' + bx'z' \mod \mathfrak{g}^2$. Then $\lambda = \dim_{\mathbb{C}} \mathcal{O}/(x'(a,b) + \mathfrak{g})$ (see [4]).

Lemma 5. *$\varrho(p,p,p)=1$ and the line can be parameterized by*

$$x=bt,\quad y=ct,\quad z=t,\qquad \text{with}\quad b^p+c^p+1=0,$$

and the torsion number $\lambda=p-1$.

Proof. The proof is trivial and is omitted. □

Lemma 6. *1) If $p>1$, then $\varrho(p,pq,pqr)=p(r-1)+1$. Let $T_0:=E_3, T_r:=P$, and $T_1,\dots,T_{r-1}$ be the points on E_3P to make the canonical regular simplicial subdivision of E_3P. Then to each T_k correspond p reduced components $T_k^{(i)}$ $(r>k>0)$ in the maximal cycle of $G(p,pq,pqr)$. The torsion numbers for the lines in $\mathcal{L}_{T_k^{(i)}}$ (k fixed) are the same: $\lambda=k(p-1)q$;*

2) If $p>1$ and $q\nmid r$, then $\varrho(p,pq,pr)=p(1+\left[\frac{r}{q}\right])$. Let $T_0:=E_3$, and $T_1,\dots,T_{\left[\frac{r}{q}\right]}$ be the points on E_3P to make the canonical regular simplicial subdivision of E_3P. Then to each $T_k={}^{\mathrm{T}}(kq,k,1)$ correspond p reduced components $T_k^{(i)}$ $(k>0)$ in the maximal cycle of $G(p,pq,pr)$. The torsion numbers of the lines in $\mathcal{L}_{T_k^{(i)}}$ for each fixed k ($k=1,\dots,\left[\frac{r}{q}\right]$) are the same $\lambda=k(p-1)q$.

Let S_1 be on PE_2 (see Figure 1) with 1 as the last coordinate. There are p reduced components $S_1^{(i)}$ $(i=1,\dots,p)$ in the maximal cycle corresponding to S_1. The torsion numbers of the lines in $\mathcal{L}_{S_1^{(i)}}$ are the same: $\lambda=\min\left\{(p-1)r,(pq-1)\left(1+\left[\frac{r}{q}\right]\right)\right\}$.

Proof. We leave the easy proof to the reader. □

In the following, we assume that $\gcd(p,q,r)=1$ and $1<p\le q\le r$. The exact numbers of families of lines are given in Table 1. The proof is in lemma 7–11.

Lemma 7. *Let $1<p<q=r$ and $\gcd(p,q)=1$. There are q reduced components $R_1^{(i)}$ in the maximal cycle corresponding to R_1 (see figure 1). And the q families of lines on $G(p,q,q)$ can be parameterized by*

$$x=ct^{1+\left[\frac{q}{p}\right]},\quad y=t,\quad z=\omega_\beta(1+c^pt^{p-\delta})^{1/q}t,\quad \beta=1,\dots,q,$$

where $c\ne0$ is a constant, ω_β is a root of $\omega^q=-1$, and $\delta=q-p\left[\frac{q}{p}\right]$. The torsion numbers of these lines are the same $\lambda=\min\left\{q-1,(p-1)\left(1+\left[\frac{q}{p}\right]\right)\right\}$.

Proof. We omit the easy proof. □

Lemma 8. *Let $\gcd(p,q)=1$ and $pq|r$. The maximal cycle is reduced and consists of $T_{k+1}={}^{\mathrm{T}}((k+1)q,(k+1)p,1)$ $(k=0,\dots,\tilde r-1=r/pq-1)$. The lines in $\mathcal{L}_{T_{k+1}}$ can be parameterized by*

$$x=c^{kq}u^{(1+\alpha q+kpq)/p}t^{(k+1)q},\quad y=c^{kp}u^{\alpha+kp}t^{(k+1)p},\quad z=cut,$$

where $0\le\alpha<p$ such that $(1+\alpha q+kpq)/p$ is an integer, $c\ne0$ is a constant and $u=u(t)$ is a unit satisfying $1+u+u^{r-(\alpha+pk)q}v^{r-(k+1)pq}w^{r-kpq}=0$. The torsion number for the lines in $\mathcal{L}_{T_{k+1}}$ is $\lambda=(k+1)(p-1)q$.

Conditions		$\varrho(p,q,r)$
$\gcd(p,q)=1$ $1<p<q\le r$	$r=q$	q
	$r>q, pq\vert r$	$\frac{r}{pq}=1+\Phi(\frac{r}{r-pq})$
	$r>q, p\nmid r, q\vert r$	$\left[\frac{r}{pq}\right]+q=1+\Phi(\frac{r}{r-pq})+q$
	$r>q, p\vert r, q\nmid r$	$\left[\frac{r}{pq}\right]+p=1+\Phi(\frac{r}{r-pq})+p$
$r>\frac{pq}{\gcd(p,q)}$ $1<p\le q\le r$ $\gcd(p,q,r)=1$	$\gcd(p,q)=1, p\nmid r, q\nmid r$ or $\gcd(p,q)>1$	$\gcd(p,q)\left[\frac{r\gcd(p,q)}{pq}\right]$ $=\gcd(p,q)\left\{1+\Phi\left(\frac{r}{r-pq/\gcd(p,q)}\right)\right\}$

TABLE 1. Number of lines on Brieskorn-Pham surfaces

Proof. In this case $P={}^{\mathrm{T}}(\tilde{r}q,\tilde{r}p,1)$ and $\det(P,E_1)=\det(P,E_2)=1, \det(P,E_3)=\tilde{r}$. Let $0\le\alpha,\beta<p$ such that

$$M_k=\frac{1}{p}\left(E_1+\alpha T_{k+1}+\beta T_k\right)={}^{\mathrm{T}}\left(\frac{1+\alpha q+kpq}{p},\alpha+pk,1\right)$$

is integral. From the chart determined by M_k, T_{k+1}, T_k, one can do all the computations. □

Lemma 9. *Let $\gcd(p,q)=1$ and $p\nmid r, q|r$. In the maximal cycle there are q reduced components $R_1^{(i)}$ $(1\le i\le q)$ corresponding to $R_1={}^{\mathrm{T}}((r+\delta)/p, r/q, 1)$, where $0<\delta<p$ such that $(r+\delta)/p$ is an integer. The lines in $\mathcal{L}_{R_1^{(i)}}$ can be parameterized as*

$$x=ct^{(r+\delta)/p},\quad y=\omega_i\left(1+c^qt^\delta\right)^{1/q}t^{r/q},\quad z=t,$$

where $c\neq 0$ is a constant and ω_i is a root of $\omega^q=-1$, $i=1,\dots,q$. The torsion numbers are the same: $\lambda=\min\left\{(p-1)\left(\left[\frac{r}{p}\right]+1\right),(q-1)r/q\right\}$. Moreover, if $r>pq$, in the maximal cycle there are also reduced components corresponding to $T_{k+1}={}^{\mathrm{T}}((k+1)q,(k+1)p,1)$ $(k=0,\dots,\left[\frac{r}{pq}\right]-1)$. The parameterizations and the torsion numbers of the lines in $\mathcal{L}_{T_{k+1}}$ are the same as in lemma 8.

Proof. One only needs to prove the statement about $\mathcal{L}_{R_1^{(i)}}$, which can be done from the chart determined by E_1, E_2, R_1. □

Lemma 10. *Let* $\gcd(p,q) = 1$ *and* $p|r, q \nmid r$. *In the maximal cycle, there are* p *reduced components* $S_1^{(j)}$ $(1 \le j \le p)$ *corresponding to* $S_1 = {}^{\mathrm{T}}(r/p, (r+\delta)/q, 1)$, *where* $0 < \delta < q$ *such that* S_1 *is integral. The lines in* $\mathcal{L}_{S_1^{(j)}}$ *can be parameterized by*

$$x = \omega_j (1 + c^q t^\delta)^{1/p} t^{r/p}, \quad y = ct^{(r+\delta)/q}, \quad z = t,$$

where $c \neq 0$ *is a constant and* ω_j *is a root of* $\omega^p = -1$, $j = 1, \dots, p$. *The torsion numbers are the same:* $\lambda = \min\left\{(q-1)\left(\left[\frac{r}{q}\right]+1\right), (p-1)r/p\right\}$. *Moreover, if* $r > pq$, *there are also* $\left[\frac{r}{pq}\right]$ *reduced components in the maximal cycle corresponding to* $T_{k+1} = {}^{\mathrm{T}}((k+1)q, (k+1)p, 1)$ $(k = 0, \dots, \left[\frac{r}{pq}\right] - 1)$. *The parameterizations and the torsion numbers of the lines in* $\mathcal{L}_{T_{k+1}}$ *are the same as in lemma 8.*

Proof. Similar to that of lemma 9. □

Lemma 11. *Let* $\gcd(p,q,r) = 1$, $1 < p \le q \le r$ *and* $r > pq/\gcd(p,q)$. *Assume that either* $\gcd(p,q) = 1, p \nmid r, q \nmid r$ *or* $\gcd(p,q) > 1$. *Then to each*

$$T_{k+1} = {}^{\mathrm{T}}\left(\frac{(k+1)q}{\gcd(p,q)}, \frac{(k+1)p}{\gcd(p,q)}, 1\right), \quad (k = 0, \dots, \left[\frac{r\gcd(p,q)}{pq}\right] - 1)$$

correspond $\gcd(p,q)$ *reduced components* $T_{k+1}^{(\gamma)}$ $(\gamma = 1, \dots, \gcd(p,q))$ *in the maximal cycle. And they are the only reduced components in the maximal cycle. For fixed* k, *the lines in* $\mathcal{L}_{T_{k+1}^{(\gamma)}}$ *can be parameterized by*

$$x = c^{k\bar{q}} u_\gamma^{(1+(k\bar{p}+\alpha)\bar{q})/\bar{p}} t^{(k+1)\bar{q}}, \quad y = c^{k\bar{p}} u_\gamma^{k\bar{p}+\alpha} t^{(k+1)\bar{p}}, \quad z = cu_\gamma t, \quad \gamma = 1, \dots, r_1,$$

where u_γ *is a unit satisfying* $1 + u^{r_1} + c^{r-k\bar{p}q} u^{r-(k\bar{p}+\alpha)q} t^{r-(k+1)\bar{p}q} = 0$, *and* $r_1 = \gcd(p,q), \bar{p} = p/r_1, \bar{q} = q/r_1$, $0 \le \alpha < \bar{p}$ *such that* $(1 + (k\bar{p}+\alpha)\bar{q})/\bar{p}$ *is an integer. And the torsion numbers of the lines in* $\mathcal{L}_{T_{k+1}^{(\gamma)}}$ *are:* $\lambda = (k+1)(p-1)q/\gcd(p,q)$.

Proof. Notice that all the coordinates of R_i and S_j are greater than 1 and T_{k+1} is as in the statement of the lemma $(k = 0, \dots, \left[\frac{r\gcd(p,q)}{pq}\right] - 1)$. Denote $T_0 := E_3$. Let $0 \le \alpha, \beta < \bar{p}$ be the integers such that

$$M_k := \frac{1}{\bar{p}}(E_1 + \alpha T_{k+1} + \beta T_k) = {}^{\mathrm{T}}\left(\frac{1 + (k\bar{p}+\alpha)\bar{q}}{\bar{p}}, k\bar{p}+\alpha, 1\right)$$

is integral. Then the simplex determined by M_k, T_{k+1}, T_k is regular. From this chart, one can obtain the given parameterization of the lines in $\mathcal{L}_{T_{k+1}^{(\gamma)}}$ by a calculation. □

Corollary 12. *If any two of the* p,q,r *are coprime and* $q = p + \alpha, r = q + \beta$ *with* $0 < \alpha < p$, $0 < \beta < q$, *there do not exist smooth curves on* $G(p,q,r)$.

Proof. We give an elementary proof. Let C be a smooth curve in $(\mathbb{C}^3, 0)$, parameterized by

$$x = x(t) = \sum_{k=1}^{\infty} a_k t^k, y = y(t) = \sum_{k=1}^{\infty} b_k t^k, z = z(t) = \sum_{k=1}^{\infty} c_k t^k, \quad |t| < \epsilon.$$

Then $C \subset G(p,q,r)$ if and only if

$$[x(t)]^p + [y(t)]^q + [z(t)]^r = 0 \quad \text{for all } |t| < \epsilon.$$

From this and the assumption on p,q,r, one can deduce that $a_1 = b_1 = c_1 = 0$. □

Example 13. 1) On E_8 surface $G(2,3,5)$ there do not exist lines.

2) On the surfaces $G(2,3,7)$, there exists one family of lines. A regular subdivision of the dual Newton diagram is given in Figure 2. Where $P = {}^{\mathrm{T}}(21,14,6)$, $Q = {}^{\mathrm{T}}(11,7,3)$,

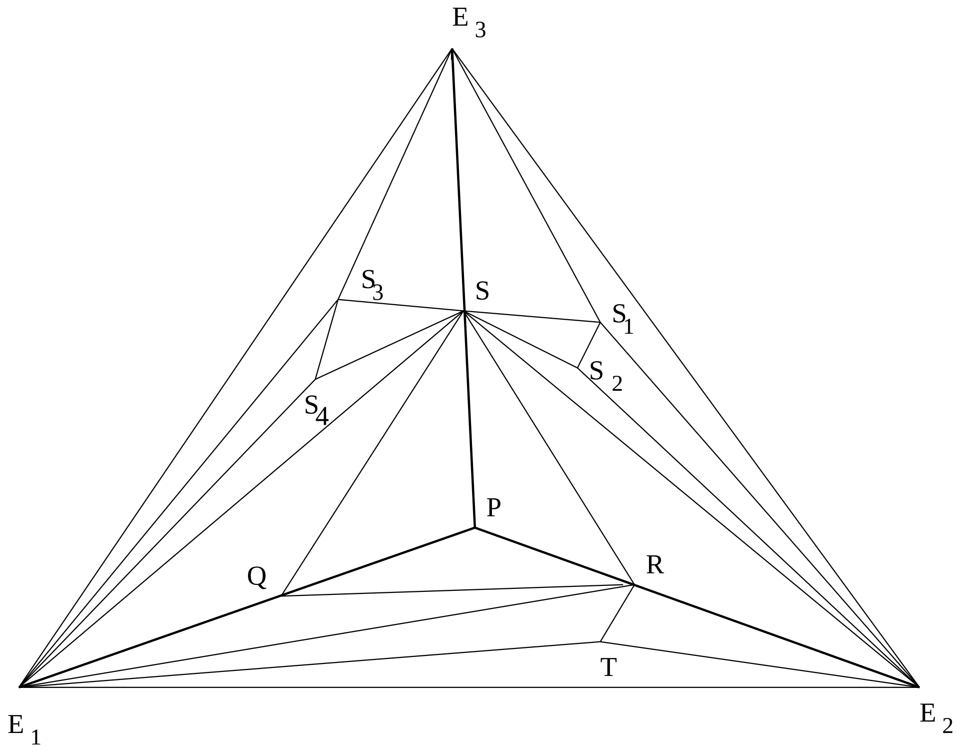

FIGURE 2

$R = {}^{\mathrm{T}}(7,5,2)$, $S = {}^{\mathrm{T}}(3,2,1)$, $S_1 = {}^{\mathrm{T}}(1,1,1)$, $S_2 = {}^{\mathrm{T}}(2,2,1)$, $S_3 = {}^{\mathrm{T}}(2,1,2)$, $S_4 = {}^{\mathrm{T}}(2,1,1)$, $T = {}^{\mathrm{T}}(4,3,1)$. Only the exceptional divisors corresponding to P, Q, R, S have non-empty intersections with the strict transform of the surface $G(2,3,7)$. On the chart corresponding to the triangle $\sigma = \Delta QPS$, we have the modification map represented by

$$x = u^{11}v^{21}w^3, y = u^7v^{14}w^2, z = u^3v^6w,$$

where (u,v,w) is the coordinates on the chart corresponding to σ. The strict transform of $G(2,3,7)$ is defined by $u+w+1=0$. The exceptional divisor corresponding to S is defined by $u+1=0, w=0$. The blowing down of the line $v = c \neq 0, u+w+1=0$ is the smooth curve in $(\mathbb{C}^3, 0)$:

$$x = -c^{21}(1+t)^{11}t^3, y = -c^{14}(1+t)^7t^2, z = -c^6(1+t)^3t,$$

where $t \in \mathbb{C}$ is the parameter. This curve is on $G(2,3,7)$ with torsion number 3.

Remark 14. If one applies the formula in lemma 11 to this example, one gets a different parameterization. This shows that the parameterizations of the lines in the same family might be different. This is because the toric modifications of the surface and the chart used might be different. However, the torsion numbers are invariant.

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