Singularities at infinity and their vanishing cycles

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Abstract

We study the topology of the fibres of polynomials f with "singularities at infinity". We first define $\mathcal{W}$-singularities at infinity, where $\mathcal{W}$ refers to a certain Whitney stratification on the space $\mathbf{X}$ which is the union of compactified fibres of f. We show that the general fibre of a polynomial with isolated $\mathcal{W}$-singularities at infinity has the homotopy type of a bouquet of spheres and that the number of these spheres is $\mu_f + \lambda_f$, where μ_f is the Milnor number and λ_f is the sum of some polar numbers at infinity. As a byproduct, we improve the connectivity estimation of the general fibre of any polynomial. In the last part we define t-regularity at infinity of a polynomial and show that it implies topological triviality at infinity. We relate isolated $\mathcal{W}$-singularities at infinity to t-nonregularity via polar curves at infinity.

1 Introduction

It has been remarked by René Thom [T] that a polynomial $f : \mathbf{C}^n \to \mathbf{C}$ induces a locally trivial fibration $f : \mathbf{C}^n \setminus f^{-1}(\Lambda) \to \mathbf{C} \setminus \Lambda$ above the complement of a finite set Λ, see e.g. [Mi] for a proof. The points in Λ are either critical values of f or *atypical* values coming from "the singularities at infinity of f".

There is up to now no definition for "singularity at infinity" of f. What one can only see is the effect of such a hidden thing, namely the change in the topology of the fibre (non-fiberability).

We propose a natural and rather large class of polynomials where "singularity at infinity" has a precise meaning, as follows. First we extend the function f to a proper function $t : \mathbf{X} \to \mathbf{C}$. Then endow $\mathbf{X}$ with a certain Whitney stratification: t becomes a stratified mapping, thus having stratified singularities. We shall call *$\mathcal{W}$-singularity at infinity* the germ at $\mathbf{X} \setminus \mathbf{C}^n$ of the singular locus of t.

It would be possible that some of these $\mathcal{W}$-singularities do not count as singularities at infinity of f (for instance if they do not produce atypical values). On the other hand, it appears that, like in the local case, these singularities can be nonisolated, so the hope to find more precise results than general connectivity statements would be rather small.

It was therefore reasonable for us to focus on polynomials with *isolated $\mathcal{W}$-singularities at infinity*. In this case we prove that "$\mathcal{W}$-singularities at infinity" is an incarnation of "singularities at infinity of f". This class includes the reduced plane curves and includes strictly the class of polynomials with "isolated singularities at infinity" in the sense used by Parusinski [Pa], see our Examples 2.6.

To give an account on the difficulties one may encounter in the study of isolated $\mathcal{W}$-singularities, let us refer to the local situation. If $g : (Y, y) \to (\mathbf{C}, 0)$ is a function germ with isolated singularity with respect to a Whitney stratification of Y at y (see [L4] for the definition) then one may hope that y being a singularity of g is equivalent to existence of vanishing cycles in the Milnor fibre of g. This is well-known to be true for smooth Y, but not anymore if Y is singular (see e.g. [Ti] for examples).

Our main result is that the general fibre of a polynomial with isolated $\mathcal{W}$-singularity at infinity is homotopy equivalent to a wedge of spheres of dimension $n-1$. We then relate the existence of an isolated $\mathcal{W}$-singularity at infinity to the existence of a certain polar curve as germ at that point. We prove that, if the germ of such a polar curve is nonvoid, then the point is a singularity where a certain positive number of cycles of a general fibre vanish.

We give an estimation (Corollary 3.6) of the connectivity of a general fibre of any polynomial. This estimation is better (usually by one) than the one of Dimca [Di].

In the second part, we get topological triviality results by assuming "t-regularity at infinity", a natural condition that we introduce for any polynomial. This condition fits well into the context of polynomials with isolated $\mathcal{W}$-singularities at infinity. For instance, we prove that, if the polar curve at some isolated $\mathcal{W}$-singularity at infinity is void, then this point, loosely speaking, does not change the topology of the fibre.

We prove that atypical values of polynomials with isolated $\mathcal{W}$-singularities are characterised by the variation of the Euler number of the fibre (Theorem 5.6). This extends the results of Hà and Lê [HL] and Parusinski [Pa]. We also extend some results in two variables which are more or less known to specialists (Corollaries 5.8, 3.5).

2 Stratification at infinity

Let $f : \mathbf{C}^n \to \mathbf{C}$ be a polynomial of degree d and let $\tilde{f}(x, x_0)$ be the homogenized of f by the new variable x_0. One replaces $f : \mathbf{C}^n \to \mathbf{C}$ by a proper mapping $t : \mathbf{X} \to \mathbf{C}$ which depends on the chosen system of coordinates on $\mathbf{C}^n$, as follows (see [Br-2]). Consider the closure in $\mathbf{P}^n \times \mathbf{C}$ of the graph of f, that is the hypersurface

$$\mathbf{X} := \{((x; x_0), t) \in \mathbf{P}^n \times \mathbf{C} | F := \tilde{f}(x, x_0) - t x_0^d = 0\},$$

which fits into the commuting diagram

$$\begin{tikzcd} \mathbf{C}^n \arrow[rr, "i"] \arrow[dr, "f"'] & & \mathbf{X} \arrow[dl, "t"] \\ & \mathbf{C} & \end{tikzcd},$$

where i denotes the inclusion $x \mapsto (x, f(x))$ and t is the projection on the second factor.

2.1 Let H_∞ denote the hyperplane at infinity $\{x_0 = 0\} \subset \mathbf{P}^n$. The singularities of $\mathbf{X}$ are contained in the part "at infinity" $\mathbf{X}_\infty := \mathbf{X} \cap (H_\infty \times \mathbf{C})$, namely:

$$\mathbf{X}_{\text{sing}} := A \times \mathbf{C}, \quad \text{where} \quad A := \{\frac{\partial f_d}{\partial x_1} = \cdots = \frac{\partial f_d}{\partial x_n} = 0, \; f_{d-1} = 0\} \subset H_\infty.$$

The singularities of f, the affine set $\mathrm{Sing}\, f := Z(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$, can be identified, by the above diagram, with the singularities of t on $\mathbf{X} \setminus \mathbf{X}_\infty$. One can prove, by an easy computation, that $\overline{\mathrm{Sing}\, f} \cap H_\infty \subset A$, where $\overline{\mathrm{Sing}\, f}$ denotes the closure of $\mathrm{Sing}\, f$ in $\mathbf{P}^n$. In particular $\dim \mathrm{Sing}\, f \le 1 + \dim A$.

Let us recall that, for an analytic function ψ on a complex space Y endowed with a complex Whitney stratification $\mathcal{C}$, one has the well defined notion of *stratified singularity of* ψ (alternatively: singularity of ψ with respect to $\mathcal{C}$). Namely, the stratified singularities of ψ, denoted by $\mathrm{Sing}\, \psi$, are the union $\cup_{\mathcal{C}_i \in \mathcal{C}} \overline{\mathrm{Sing}\, \psi_{|\mathcal{C}_i}}$ (see e.g. [GM], [L3]). We introduce the following definitions:

2.2 Definition (Canonical stratification at infinity)
Let $\mathcal{W}$ be the least fine Whitney stratification of $\mathbf{X}$ that contains the strata $\mathbf{X} \setminus \mathbf{X}_\infty$ and $\mathbf{X}_\infty \setminus \mathbf{X}_{\mathrm{sing}}$. This is a canonical Whitney stratification (see [Ma, §4]) with two imposed strata instead of the smooth open $\mathbf{X} \setminus \mathbf{X}_{\mathrm{sing}}$. We may call $\mathcal{W}$ the *canonical Whitney stratification at infinity of* $\mathbf{X}$.

2.3 Let $\mathrm{Sing}\, t$ be the singularities of $t : \mathbf{X} \to \mathbf{C}$ with respect to the canonical Whitney stratification at infinity and denote $\mathrm{Sing}^\infty f := \mathrm{Sing}\, t \cap \mathbf{X}_\infty$. Using 2.1, one then gets the following equality:

$$\mathrm{Sing}\, t = \mathrm{Sing}\, f \cup \mathrm{Sing}^\infty f.$$

Let us also remark that $\mathrm{Sing}\, t \cap (\mathbf{X}_\infty \setminus \mathbf{X}_{\mathrm{sing}}) = \emptyset$ and that $\dim \mathrm{Sing}^\infty f \le \dim A$.

The class of polynomials we want to focus at is defined, by making use of Lê's definition of *isolated singularities* [L3, Definition 1.1], as follows.

2.4 Definition (Isolated $\mathcal{W}$-singularities at infinity)
We say that the polynomial $f : \mathbf{C}^n \to \mathbf{C}$ has *isolated $\mathcal{W}$-singularities at infinity* if the projection $t : \mathbf{X} \to \mathbf{C}$ has isolated singularities with respect to the stratification $\mathcal{W}$ (equivalently: $\dim \mathrm{Sing}\, t \le 0$).

2.5 Remarks (a) "Isolated $\mathcal{W}$-singularities at infinity" implies that the polynomial f has isolated singularities in $\mathbf{C}^n$ (in the usual sense), which fact does not depend any more on coordinates.

(b) If t already has isolated singularities with respect to some Whitney stratification which is finer than $\mathcal{W}$ then obviously t has isolated singularities with respect to $\mathcal{W}$. We therefore introduce the following finer (sometimes more handy in examples) stratification $\mathcal{W}'$. There is a canonical (i.e. the least fine) Whitney stratification $\mathcal{S}$ of the projective hypersurface $\{f_d = 0\} \subset H_\infty \simeq \mathbf{P}^{n-1}$ such that A is a union of strata. Then $\mathcal{S} \times \mathbf{C}$ is a Whitney stratification of $\mathbf{X}_\infty$. It is foliated by lines $\{a\} \times \mathbf{C}$, for $a \in \{f_d = 0\}$. The stratification of $\mathbf{X}$ which consists of $\mathbf{X} \setminus \mathbf{X}_\infty$ and the strata of $\mathcal{S} \times \mathbf{C}$ may be not a Whitney stratification of $\mathbf{X}$, but it can be refined to such a one by (eventually) introducing new strata on $\mathbf{X}_{\mathrm{sing}}$ only. The least fine Whitney stratification of $\mathbf{X}$ obtained in this way will be denoted by $\mathcal{W}'$.

2.6 Examples (a) If the polynomial f has isolated singularities in $\mathbf{C}^n$ and $\dim A \le 0$ then f has isolated $\mathcal{W}$-singularities at infinity. This is the case for all reduced plane curves.

(b) $h = x^3y + x + z^2 : \mathbf{C}^3 \to \mathbf{C}$ has isolated $\mathcal{W}$-singularities at infinity (with respect to $\mathcal{W}'$) but $\dim A = 1$.

(c) The polynomial $g := x^2y + x : \mathbf{C}^3 \to \mathbf{C}$ has non-isolated $\mathcal{W}$-singularities at infinity (namely in the fibre $g^{-1}(0)$). It turns out that it has nonisolated $\mathcal{W}$-singularities at infinity in any coordinates, see also our next Remark.

3 A bouquet theorem

We prove here a global version of the local bouquet theorems of J. Milnor [Mi, Theorem 6.5] and Lê D.T. [L3, Theorem 5.1].

3.1 Theorem *Let $f : \mathbf{C}^n \to \mathbf{C}$ be a polynomial with isolated $\mathcal{W}$-singularities at infinity. Then the general fibre of f is homotopy equivalent to a bouquet of spheres of real dimension $n-1$.*

Remark As in the local case, if f has nonisolated $\mathcal{W}$-singularities at infinity, then one cannot expect to get a bouquet of the type in the Theorem above. The Example 2.6(c) shows that even if the polynomial has smooth fibres (but some nonisolated $\mathcal{W}$-singularity contained in $\mathbf{X}_\infty$), the result above is no more true: the general fibre of g is a circle (whereas it should have been a bouquet of spheres of dimension 2).

3.2 Proof of Theorem 3.1
Step 1. We prove first that the reduced homology of a general fibre is concentrated in dimension $n-1$. Relatively to the stratification $\mathcal{W}$, the projection $t : \mathbf{X} \to \mathbf{C}$ has only isolated singularities, namely a finite number of points situated on $\mathbf{X}_\infty$ and another finite set on $\mathbf{X} \setminus \mathbf{X}_\infty$ which coresponds to the set $\operatorname{Sing} f$. Let R be the set of critical values of t.

For each $b \in R$, let δ_b be a small enough disc centred at b. Then $t : \mathbf{X} \cap t^{-1}(\mathbf{C} \setminus \cup_{b\in R}\delta_b) \to \mathbf{C} \setminus \cup_{b\in R}\delta_b$ is a stratified topological fibration (with respect to $\mathcal{W}$), hence its restriction to $f^{-1}(\mathbf{C} \setminus \cup_{b\in R}\delta_b)$ is a locally trivial topological fibration, by Thom first isotopy lemma.

Let $\mathbf{X}_S := t^{-1}(S)$, $F_S := f^{-1}(S)$, for some $S \subset \mathbf{C}$. Let us fix $c \in \mathbf{C} \setminus \cup_{b\in R}\delta_b$ and $c_b \in \partial\delta_b$. We get, as usually by deformation retraction and excision, the following splitting:

$$\tilde{H}_i(F_c) = H_{i+1}(\mathbf{C}^n, F_c) = \oplus_{b\in R} H_{i+1}(F_{\delta_b}, F_{c_b}).$$

We stick to such a term: for simplicity, let D be one of the discs δ_b and fix some $d \in \partial D$. We have, according to [Br-2, Proposition 5.2]:

$$H_\bullet(F_D, F_d) \cong H^{2N-\bullet}(\mathbf{X}_D, \mathbf{X}_d).$$

It remains to prove that $H^\bullet(\mathbf{X}_D, \mathbf{X}_d)$ is concentrated. Let b be the centre of D. The singularity of $t_{|\mathbf{X}_D}$ are on $\mathbf{X}_b$, let those be denoted by $a_1, \ldots, a_k$. We may choose a good neighbourhood of a_i, say of the form $B_i \cap \mathbf{X}_D$, where B_i is a small enough closed ball in some local chart and also suppose D small enough such that the restriction $t : B_i \cap \mathbf{X}_D \to D$ is a Milnor representative of the germ $t : (\mathbf{X}_D, a_i) \to (\mathbf{C}, b)$. Since this germ is an isolated singularity with respect to the induced stratification, it follows that the fibres $t^{-1}(u)$, $\forall u \in D$, are transversal to a certain semi-algebraic Whitney stratification of $\partial B_i \cap \mathbf{X}_D$, constructed as in the proof of [L2, Theorem 1.1] or [L3, Theorem 1.3]. Thus there is a trivial topological fibration:

$$t : \mathbf{X}_D \setminus \cup_{i=1,k} B_i \to D$$

and, by an excision, we get the isomorphism:

$$H^\bullet(\mathbf{X}_D, \mathbf{X}_d) = \oplus_{i=1,k} H^\bullet(B_i \cap \mathbf{X}_D, B_i \cap \mathbf{X}_d) = \oplus_{i=1,k} \tilde{H}^{\bullet-1}(B_i \cap \mathbf{X}_d),$$

where $B_i \cap \mathbf{X}_d$ is the local Milnor fibre of the germ of t at a_i.

We may conclude our proof by applying a theorem due to Lê D.T. (see e.g. [L3, Theorem 5.1] for a more general result) which says that the Milnor fibre of an isolated singularity function germ on a hypersurface of pure dimension n has the homotopy type of a bouquet of spheres of dimension $n-1$.

Step 2. To get the homotopy result, we would like to replace in the above proof the homology excision by the homotopy excision. By inspecting the proof, it easily appears that one can do this until the local situation. For instance, since the trivial topological fibration $t : \mathbf{X}_D \setminus \cup_{i=1,k} B_i \to D$ is a stratified one, it restricts to the trivial fibration $t : (\mathbf{X}_D \setminus \mathbf{X}_\infty) \setminus \cup_{i=1,k} B_i \to D$.

It remains to manage the local situation. Let first $a_i \in \mathbf{X}_\infty$ be a singular point of t. This fits into a statement due to Hamm and Lê [HL1, Theorem 4.2.1, Corollary 4.2.2]: the conditions are obviously fulfilled, namely t has isolated singularities with respect to $\mathcal{W}$ and $\mathrm{rHd}(\mathbf{X} \setminus \mathbf{X}_\infty) \geq n$ (since $\mathbf{X} \setminus \mathbf{X}_\infty$ is smooth). The ingredients in the proof are homotopy excision (Blakers-Massey theorem) and stratified Morse theory.

Now by a very slight modification of the above cited result of Hamm and Lê (i.e. by using cylindrical neighbourhoods, which are conical by [GM, p. 165]), we get that the pair

$$J(B_i \cap \mathbf{X}_D \setminus \mathbf{X}_\infty, B_i \cap \mathbf{X}_d \setminus \mathbf{X}_\infty)$$

is $(n-1)$-connected.

We may apply [Sw, Prop. 6.13] to conclude that $B_i \cap \mathbf{X}_D \setminus \mathbf{X}_\infty$ is obtained from $B_i \cap \mathbf{X}_d \setminus \mathbf{X}_\infty$ by adding cells of dimensions $\geq n$.

A similar (even better) situation is encountered on the affine piece: if $a_j \in \mathbf{X}_b \setminus \mathbf{X}_\infty$ is a singularity of $t_{|\mathbf{X}_D}$ then it is well known that $B_j \cap F_D$ is obtained from $B_j \cap F_d$ by attaching n-cells.

For global fibres, it follows that F_D is obtained (up to homotopy) by adding a finite number of cells of dimension $\geq n$ to F_d.

Finally, the whole space $\mathbf{C}^n = F_{\mathbf{C}}$ is obtained, up to homotopy, by attaching a finite number of cells of dimensions $\geq n$ to a general fibre F_c. Since F_c has the homotopy type of a n-dimensional CW-complex, we get $F_c \overset{\mathrm{ht}}{\simeq} \vee_\gamma S^{n-1}$, by Whitehead theorem.

□

3.3 Note Our homotopy statement extends the one for polynomials with "good behaviour at infinity": tame [Br-2], quasi-tame [N], M-tame [NZ]. Recall that "tame" implies "quasi-tame", which implies in its turn "M-tame" [N], [NZ]. It will turn out from Corollary 5.8, Proposition 5.3 and Corollary 3.5(a) that if such a polynomial has isolated $\mathcal{W}$-singularities at infinity then actually it has no $\mathcal{W}$-singularities at infinity.

Under the additional hypothesis $\dim A = 0$, the homology statement was proved (with a different proof) by Broughton [Br-2, Theorem 5.2]. In this case he also gets Corollary 3.5(a) below.

3.4 Definition We denote by λ_a the number of spheres in the Milnor fibre of the germ $t : (\mathbf{X}, a) \to (\mathbf{C}, b)$ and call it *the Milnor number at infinity, at a.*

3.5 Corollary *Let f be a polynomial with isolated $\mathcal{W}$-singularities at infinity. Then:*

(a) *The number γ of spheres in a general fibre is equal to the sum $\mu_f + \lambda_f$, where μ_f is the total Milnor number of f and λ_f is the sum of the Milnor numbers at infinity. In particular λ_f is invariant under diffeomorphisms of $\mathbf{C}^n$.*

(b) *Let μ_{F_b} denote the sum of the Milnor numbers of all the singularities of the fibre F_b and let λ_{F_b} denote the sum of all Milnor numbers at infinity at $\mathbf{X}_b \cap \mathbf{X}_\infty$. Then*

$$\chi(F_u) - \chi(F_b) = (-1)^{n-1}(\lambda_{F_b} + \mu_{F_b}),$$

where F_u is a general fibre of f. □

3.6 Corollary (General connectivity estimation) *Let $f : \mathbf{C}^n \to \mathbf{C}$ be any polynomial. Then its general fibre F_u is at least $(n - 2 - \dim(A \cup \operatorname{Sing} f))$-connected.*

Note This result improves, in case $\dim \operatorname{Sing} f \leq \dim A$ the connectivity estimation of Dimca [Di, Theorem 1], which improved in its turn the one of Kato [Ka]. Comparing to [Di], one should remark that our statement is only true for general fibres: the connectivity of atypical fibres can be 1 less (see for instance the example $f = x^2y + x : \mathbf{C}^2 \to \mathbf{C}$).

We actually can prove the following more natural and possibly sharper (see 2.3) connectivity upper bound: $(n - 2 - \dim \operatorname{Sing} t)$, but the proof is different from the one below and we defer it to some future paper.

Proof We give a brief account of the proof and refer the reader to [Di] for details. Using the Lefschetz type theorem of Goresky and MacPherson [GM, p. 153], one gets that the pair $(F_u, F_u \cap H)$ is $(n-2)$-connected, where H is a generic hyperplane in $\mathbf{C}^n$. Let $k := \dim(A \cup \operatorname{Sing} f)$ and notice that $\dim \operatorname{Sing} f \leq 1 + \dim A$. As slicing, the dimensions of the sets A and $\operatorname{Sing} f$ (see 2.1) drop by one and within k steps we arive to a general fibre of a polynomial (in k less coordinates) with isolated singularities and the corresponding set A of dimension 0 (or just void). We know in this case, by our Theorem, that the general fibre is homotopy equivalent to a bouquet of spheres of middle dimension. In case $\dim A \geq \dim \operatorname{Sing} f$, the difference to Dimca's proof is only in this last point: he uses a bouquet result valid only if $A = \emptyset$, therefore has to slice one more time. □

4 Polar curves and vanishing cycles at infinity

The proof of the above Theorem shows that the general fibre of f has cycles that vanish at the $\mathcal{W}$-singularities at infinity. We give here another interpretation of the number of vanishing cycles at infinity, in terms of polar numbers. In section 5, this will allow us to answer affirmatively the following natural question that rises up. Loosely speaking: if there are no cycles that vanish at points at infinity of some fibre of f, then is there topological triviality at infinity, at this fibre?

4.1 The space $\mathbf{X}$ is covered by the affine charts $U_i := \{((x;x_0),t) \in \mathbf{P}^n \times \mathbf{C} \mid x_i \neq 0\}$, for $i \in \{0,1,\ldots,n\}$. Let $p \in \mathbf{X}_\infty \cap U_i$, for some $i \in \{1,\ldots,n\}$ and consider in this chart the function

$$x_0 : (\mathbf{X}, p) \to (\mathbf{C}, p_0).$$

This induces a local Thom (A_{x_0})-stratification at p. A priori, this stratification depends on the choice of the chart U_i at p (since the function x_0 itself does), but one can prove the following:

4.2 Lemma *The canonical Whitney stratification at infinity of $\mathbf{X}$ verifies the (A_{x_0})-condition at any $p \in \mathbf{X}_\infty$, in any chart U_i.*

Proof For any $i \in \{1,\ldots,n\}$, the morphism $x_0 : \mathbf{X} \cap U_i \to \mathbf{C}$ is a stratified morphism with respect to the induced Whitney stratification on $\mathbf{X} \cap U_i$, respectively the stratification $\{\mathbf{C} \setminus \{0\}, \{0\}\}$ on $\mathbf{C}$. Indeed, a simple computation shows that the singular locus of the above morphism is included in $\operatorname{Sing} \mathbf{X}_\infty$, hence included in $\{x_0 = 0\}$.

We conclude, by using the main result of Briançon, Maisonobe and Merle in [BMM, Theorem 4.2.1], that the strata of the globally well-defined Whitney stratification $\mathcal{W}$ verify the (A_{x_0})-condition. □

4.3 Let $a \in \mathbf{X}_\infty \cap U_i$ and $b := t(a)$. We consider the germ at a of the mapping

$$\Phi := (t, x_0) : \mathbf{X} \to \mathbf{C}^2.$$

Since t can have at most an isolated singularity at a, with respect to $\mathcal{W}$ and since the strata of $\mathcal{W}$ verify the (A_{x_0})-condition at points of $\mathbf{X}_\infty$ (Lemma 4.2), it follows that the following variety, as germ at a:

$$\Gamma_a(t, x_0) := \operatorname{closure}(\operatorname{Sing} \Phi \setminus \mathbf{X}_\infty)$$

is of dimension ≤ 1 [L2, Lemma 2.2]. We call it *(nongeneric) polar curve.*

Following [L1], there is a fundamental system of privileged open polydiscs in U_i, centred at a, of the form $(D_\alpha \times P_\alpha)_{\alpha \in K}$ and a corresponding fundamental system $(D_\alpha \times D'_\alpha)_{\alpha \in K}$ of 2-discs at $(b, 0)$ in $\mathbf{C}^2$, such that Φ induces, for any $\alpha \in K$, a mapping

$$\Phi_\alpha : \mathbf{X} \cap (D_\alpha \times P_\alpha) \to D_\alpha \times D'_\alpha$$

which is a topological fibration over $D_\alpha \times D'_\alpha \setminus \Phi_\alpha(\operatorname{Sing} \Phi)$.

One may identify the Milnor fibre of the germ $t : (\mathbf{X}, a) \to (\mathbf{C}, b)$ by ${\Phi_\alpha}^{-1}(\{b+\varepsilon\} \times D'_\alpha)$, for some $\varepsilon \in \mathbf{C}$ close enough to 0. We know, from Lê's [L3, Theorem 5.1], that this Milnor fibre is homotopy equivalent to a bouquet of λ_a spheres of real dimension $n-1$.

4.4 Definition-Proposition *Let $\operatorname{int}(\Gamma_a(t,x_0), \mathbf{X}_b)_a$ be the intersection number of the polar curve with the hypersurface $\mathbf{X}_b$ in $\mathbf{X}$, at a. We call it the polar number at infinity of f, at the point a. It is equal to the Milnor number λ_a.*

It follows that the polar number at $a \in \mathbf{X}_\infty$ does not depend on the chart U_i, since the local Milnor number of t at a depends only on the point.

Proof We have a finite number of isolated singularities of the function $x_0 : P_\alpha \cap \mathbf{X}_{b+\varepsilon} \cap x_0^{-1}(D'_\alpha \setminus \{0\}) \to D'_\alpha \setminus \{0\}$ which are precisely the intersections of the polar curve with $\mathbf{X}_{b+\varepsilon} \subset U_i$. All of them project to $D'_\alpha \setminus D''$, where $D'' \subset D'_\alpha$ is a small enough disc centred at 0. Thus the Milnor fibre is made up by attaching cells of dimension $n-1$ to $P_\alpha \cap \mathbf{X}_{b+\varepsilon} \cap x_0^{-1}(D'')$. But this last space is contractible, by the following reason. By construction, the fibres $x_0^{-1}(s)$, for $s \in D''$, are transversal to the induced stratification of the space $P_\alpha \cap \mathbf{X}_{b+\varepsilon}$. Since all the singularities of $x_{0|\mathbf{X}_{b+\varepsilon}}$ are on $\{x_0 = 0\}$, we may apply a result of A. Durfee [Du] to conclude that $P_\alpha \cap \mathbf{X}_{b+\varepsilon} \cap x_0^{-1}(D'')$ is homotopy equivalent to $P_\alpha \cap \mathbf{X}_{b+\varepsilon} \cap x_0^{-1}(0)$. In its turn, the latter space is contractible, since it can be identified with the complex link of the space $\mathbf{X}_\infty$ at the point a, which is contractible, by the product structure of the Whitney stratification $\mathcal{S} \times \mathbf{C}$ of $\mathbf{X}_\infty$. $\square$

5 t-regularity and topological triviality

We prove that a $\mathcal{W}$-singularity at infinity with $\lambda_a = 0$ is in fact a "false singularity". Let us first introduce the following:

5.1 Definition (t-regularity at infinity) We say that the fibre $f^{-1}(t_0)$ *is t-regular at* $s \in \{f_d = 0\} \subset H_\infty$ if there is an affine chart U_i such that, for any sequence of points $z \in \mathbf{X} \setminus \mathbf{X}_\infty$, $z \to p$, where $p := (s, t_0)$, the limit of the tangent hyperplanes $\lim_{z\to p} T_z(\mathbf{X} \cap \{x_0 = z_0\})$, whenever it exists, is transverse within $\mathbf{P}^n \times \mathbf{C}$ to the hyperplane $\{t = t_0\} \subset \mathbf{P}^n \times \mathbf{C}$, at p.

If $f^{-1}(t_0)$ is t-regular at s, $\forall s \in \{f_d = 0\}$, then we say that this fibre is *t-regular at infinity.*

5.2 Remark It is obvious from the definition that the t-regularity of $f^{-1}(t_0)$ at s is implied by the transversality of $\{t = t_0\}$ to the strata of some (A_{x_0})-stratification, at p. In particular, by use of Lemma 4.2, t-regularity is implied by the transversality of $\{t = t_0\}$ to the strata of $\mathcal{W}$ which are in $\mathbf{X}_\infty$. One can moreover prove the following, by using the same lemma.

For any polynomial f, the fibres which are not t-regular at infinity are finitely many. If f has isolated $\mathcal{W}$-singularities at infinity, then these singularities are the only points where the corresponding fibres of f may be not t-regular.

5.3 Proposition *Let t have an isolated singularity at $p = (s, t_0) \in \mathbf{X}_\infty$. Then $f^{-1}(t_0)$ is t-regular at s if and only if $\Gamma_p(t, x_0) = \emptyset$.*

Proof Let $C_{x_{0|\mathbf{X}\cap B}}$ be the conormal space relative to the function $x_0 : \mathbf{X} \cap B \to \mathbf{C}$, for some small enough open ball B centred at p within a fixed chart U_i, that is:

$$C_{x_{0|\mathbf{X}\cap B}} := \text{closure}\{(x, H) \in (\mathbf{X} \setminus \mathbf{X}_\infty) \cap B \times \check{\mathbf{P}}^n \mid T_x(\mathbf{X} \cap \{x_0 = \text{const.}\}) \subset H\},$$

where $\check{\mathbf{P}}^n$ is the dual of $\mathbf{P}^n$, i.e. the set of hyperplanes in $\mathbf{C}^{n+1}$, at 0. Let π_1 be the projection on the first factor and π_2 be the projection on $\check{\mathbf{P}}^n$. Since p is an isolated singularity, by Remark 5.2, p is an isolated point where t-regularity is eventually not fulfilled, hence we get $\pi_1(\pi_2^{-1}(\{t = t_0\})) \cap \mathbf{X}_\infty \cap B \subset \{p\}$. We already know that $\Gamma_p(t, x_0)$ is a curve. It is precisely the germ at p of $\pi_1(\pi_2^{-1}(\{t = t_0\}))$. Now $f^{-1}(t_0)$ is not t-regular at s if and only if $(p, \{t = t_0\}) \in C_{x_{0|\mathbf{X}\cap B}}$ and this, if and only if $\pi_1(\pi_2^{-1}(\{t = t_0\})) \cap \mathbf{X}_\infty \cap B = \{p\}$. This last equality implies that $\pi_2^{-1}(\{t = t_0\})$ has dimension at least 1, since nonvoid and since $\dim C_{x_0|\mathbf{X}\cap B} = n+1$. We conclude our proof by noticing that $\Gamma_p(t, x_0) \neq \emptyset$ is equivalent to $\lambda_p \neq 0$ (by 4.4) and this does not depend on the chart U_i. $\square$

We are now in the position to prove the following topological triviality statements.

5.4 Proposition (local triviality at infinity)
If $\lambda_p = 0$ (equivalently, $\Gamma_p(t, x_0) = \emptyset$) then the restriction of t

$$t : (\mathbf{X} \setminus \mathbf{X}_\infty) \cap (D_\alpha \times P_\alpha) \to D_\alpha$$

is a trivial fibration.

Proof Since $\operatorname{Sing} \Phi_\alpha \cap \mathbf{X} \setminus \mathbf{X}_\infty = \emptyset$, we can lift the complex vector field $\partial/\partial t$ into the tangent space $T_z(\mathbf{X} \cap \{x_0 = \text{constant}\})$, at any point $z \in (\mathbf{X} \setminus \mathbf{X}_\infty) \cap (D_\alpha \times P_\alpha)$. This gives a nowhere zero complex vector field on $(\mathbf{X} \setminus \mathbf{X}_\infty) \cap (D_\alpha \times P_\alpha)$ which is moreover tangent to the levels $\mathbf{X} \cap \{x_0 = \text{constant}\}$). $\square$

5.5 Proposition (global triviality) *Let f be any polynomial and let t_0 be a regular value of f. If $f^{-1}(t_0)$ is t-regular at infinity, then the fibre $f^{-1}(t_0)$ is not atypical.*

Proof We have $\mathbf{X} \cap U_n = \{F_n = 0\} \subset \mathbf{C}^n \times \mathbf{C}$, where $F_n := F(x_1, \ldots, x_{n-1}, 1, x_0, t)$. By definition, t-regular at s, say with respect to U_n, is equivalent to the following: there is $\delta > 0$ such that

$$\delta \left| \frac{\partial(F_n)}{\partial t} \right| < \left\| \frac{\partial(F_n)}{\partial x_1}, \ldots, \frac{\partial(F_n)}{\partial x_{n-1}} \right\|,$$

in a neighbourhood of $p = (s, t_0)$. We divide by $|x_0^{d-1}|$, then change the chart to U_0: we have to replace x_0 by x_n^{-1}. We then get

$$\delta < |x_n| \cdot \left\| \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{n-1}} \right\|, \text{ for } x \to p.$$

This clearly implies the *Malgrange condition*:

(*)
$$||x|| \cdot ||\mathbf{grad} f|| > \delta.$$

If this happens for all s in the compact $\mathbf{X}_\infty \cap t^{-1}(t_0)$, then the condition (*) is fulfilled, with a certain $\delta > 0$, for all $x \in \mathbf{C}^n$ such that $||x|| \to \infty$ and $|f(x)| \to t_0$. Now it is standard (see [Mi] and also [Br-2, p. 229], [NZ, p. 684]) that this alows one to construct a complex vector field which trivializes the map $f : f^{-1}(D) \to D$, where D is a small enough disc centred at t_0. Namely, one first projects the vector $\mathbf{grad} f(x)$ into the complex tangent hyperplane at x to the sphere S^{2n-1}, centred at $0 \in \mathbf{C}^n$. This gives a nonzero vector $\mathbf{v}(x) := \mathbf{grad} f(x) - \frac{\langle x, \mathbf{grad} f(x)\rangle}{||x||} \cdot x$ which one renormalizes into $\mathbf{w}(x) := \frac{\mathbf{v}(x)}{\langle \mathbf{v}(x), \mathbf{grad} f(x)\rangle}$. For $\|x\|$ sufficiently large, we thus get a nowhere zero vector field $\mathbf{w}$ with $\langle \mathbf{w}, \mathbf{grad} f\rangle = 1$. We then glue it (by a partition of unity) to the vector field $\frac{\mathbf{grad} f}{||\mathbf{grad} f||^2}$ which is nowhere zero within a big ball $B \subset \mathbf{C}^n$. The resulting vector field on $f^{-1}(D) \cap B$ is the one we are looking for. Details of the explicit elementary computation can be found in [Pa, Lemma 1.2]. $\square$

5.6 Theorem *Let f be a polynomial with isolated $\mathcal{W}$-singularities at infinity. Then a fibre F_b is atypical or not smooth if and only if $\chi(F_b) \neq \chi(F_u)$, where F_u is a general fibre.*

Proof If F_b is not smooth, then $\mu_{F_b} > 0$ and we quickly conclude by Corollary 3.5(b). Let thus F_b be smooth. By Proposition 5.5 on the one hand and by Propositions 5.3, 4.4 and Corollary 3.5(b) on the other hand, F_b is not atypical if and only if it is t-regular at infinity. Thus F_b atypical implies that $\Gamma_a(t, x_0) \neq \emptyset$, for at least a point $a \in \mathbf{X}_b \cap \mathbf{X}_\infty$. By Corollary 3.5(b), this introduces a difference of λ_a in the Euler characteristics. The "if" part of the statement is trivial. □

Note For reduced plane curves, Theorem 5.6 was proved by Hà and Lê [HL]. For polynomials with isolated (affine) singularities and $\dim A = 0$ it was proved by Parusinski [Pa]. The family of hypersurfaces $\{\mathbf{X}_t\}_{t \in \mathbf{C}}$ can be viewed as a family of local isolated hypersurface singularities at $\{a\} \times \mathbf{C}$, for some $a \in A$. Then the above Theorem can be rephrased by saying that μ-constancy of this family at some t_0, for all $a \in A$, is equivalent to topological triviality at t_0 of the polynomial fibration $f : \mathbf{C}^n \to \mathbf{C}$. (This is in turn equivalent to the constancy or the Euler number of the global fibre F_t, for t in a neighbourhood of t_0.) Moreover, by adapting the result of Lê-Saito [LS] to our situation, the μ-constancy is equivalent to the fact that, locally at t_0, the stratum $\{a\} \times \mathbf{C}$ verifies the (A_{x_0})-condition.

5.7 Remark Let $f : \mathbf{C}^2 \to \mathbf{C}$ be a reduced polynomial. Then t-regularity at infinity is equivalent to *equisingularity at infinity*. By Abyankhar-Moh Theorem, a t-regular at infinity polynomial f with $\mu_f = 0$ is in the $\mathrm{Aut}(\mathbf{C}^2)$-orbit of a coordinate of $\mathbf{C}^2$.

Then the Jacobian Conjecture is equivalent to the following: For any $f, g : \mathbf{C}^2 \to \mathbf{C}$ such that $\mathrm{Jac}(f, g) = \mathbf{C}[x, y]$, one has $\mu_f = \lambda_f = 0$.

Question Let $f : \mathbf{C}^n \to \mathbf{C}$ have isolated $\mathcal{W}$-singularities at infinity. If $\mu_f = \lambda_f = 0$, then is there $\phi \in \mathrm{Aut}(\mathbf{C}^n)$ such that $f \circ \phi$ is a coordinate? This is Abyankhar-Moh Theorem in case $n = 2$.

The t-regularity at infinity is related to M-tameness as follows. Recall the definition from [NZ]: a polynomial is called M-tame if there is no sequence of points $x_k \in \mathbf{C}^n$ such that, as $k \to \infty$, $\lim ||x_k|| = \infty$, $\lim f(x_k) = t_0$ and $\mathbf{grad} f(x_k) = \lambda_k x_k$ for some $\lambda_k \in \mathbf{C}^n$.

5.8 Corollary *Let f have isolated $\mathcal{W}$-singularities at infinity.*
Then f is M-tame if and only if f is t-regular at infinity (equivalently: $\lambda_f = 0$). In particular, $f : \mathbf{C}^n \to \mathbf{C}$ is a trivial fibration if and only if f is M-tame and $\mu_f = 0$.

Proof If f is M-tame, then the total number of spheres γ is equal to μ_f [NZ], hence $\lambda_f = 0$. But this implies that f is t-regular at infinity (Proposition 5.3). Conversely, if f is t-regular, then the M-tameness is satisfied, as showed in the last part of the proof of Proposition 5.5 above. We were informed by Pierrette Cassou-Noguès that a result of this type was proved in [Ha], in case of plane curves ($n = 2$). □

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