# On minimal round functions

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**Abstract**

We describe the structure of minimal round functions on closed surfaces and three-folds. The minimal possible number of critical loops is determined and typical non-equisingular round function germs are interpreted in the spirit of isolated line singularities. We also discuss a version of Lusternik-Schnirelmann theory suitable for round functions.

**Key words:** round function, equisingular critical loop, almost Morse round function, Lusternik-Schnirelmann category

## 1 Introduction

We will deal with round functions on low-dimensional manifolds, that is with such smooth functions that their critical sets are smooth one-dimensional submanifolds (which are not assumed to be non-degenerate, in the sense of R.Bott [3]). This is an evident extension of the notion of a *round Morse function* introduced by W.Thurston [21].

Our main concern in this paper are round functions with the minimal possible number of critical loops (so-called minimal round functions), especially changes in topology of their Lebesque sets and typical local models of their singular behaviour. The set-up and approach accepted in this note are much in the spirit of F.Takens' paper [20] containing a comprehensive treatment of similar questions for functions with isolated critical points.

It should be noted that, unlike to the round Morse functions which gained a lot of attention [1], [21], [17], [7], round functions with degenerate critical loops are rather poorly understood. For example, it is still unclear how to describe the class of compact closed manifolds which possess round functions. Some results about general (not necessarily Morse) round functions may be found in several papers [1], [2], [17], [18], but to the best of our knowledge there exists no systematic exposition of this topic in the literature. The present paper may be considered as an attempt to fill this gap and create certain framework for further investigations.

As we were able to conclude from [2], [17], [18] and discussions with colleagues, many natural questions about round functions remain unanswered even in low dimensions. Thus we decided to begin with discussing round functions on low-dimensional manifolds. Specifically, we consider round functions on surfaces and three-folds (smooth three-dimensional manifolds) and their local behaviour near critical loops. Some of these results are rather simple and we do not exclude that they may be known for the experts or even belong to "mathematical folklore", but we have good evidence to hope that, in any case, our presentation contains certain novelties arising from the treatment of round functions from the singularity theory viewpoint.

One of the basic ideas we want to formulate and illustrate here, is that round functions with degenerate critical loops appear quite naturally in certain simple context and their transversal singular behaviour along critical loops resembles some patterns exhibited by so-called isolated line singularities [19]. The context we have in mind, is related to certain homotopy invariants similar to the classical Lusternik-Schnirelmann category [12]. We describe this setting in some detail, as well as typical examples of degenerate round functions in low dimensions.

In order to endow the whole topic with a proper background, we first address the general existence problem for round functions on closed manifolds. We recall main results in this direction from [1], [16] and complement them by some observations about the Euler characteristic of Lebesque sets. Results of this section imply, in particular, that degenerate round functions, generally speaking, cannot be approximated by round Morse functions. This shows that degenerate round functions are in some sense inevitable and should be studied by themselves.

We proceed by considering examples of minimal round functions on compact closed surfaces. It turns out that, in the orientable case, critical loops of such functions are transversally non-degenerate except a finite number of points of Whitney umbrella type ($D_\infty$-points in notation of [19]). For functions with such critical loops, we describe possible changes in topology of Lebesque sets under passing of a critical level. This enables us, in particular, to determine the minimal possible number of $D_\infty$-points on a given orientable two-surface. We also establish that round functions exist on all closed three-folds, which is in a contrast with the fact that not all of three-folds possess *round Morse functions* [17].

Thus minimal round functions on closed surfaces admit a rather detailed description. Actually, these two-dimensional results serve as the main paradigm for our research and suggest an approach to higher-dimensional cases, although it should be noted that in higher dimensions the situation is much more complicated and leaves small chances for such a complete description. In particular, it is already not so simple to compute the round complexity by visual geometric considerations so one has to develop some general topological machinery suitable for this purpose.

With this in mind, we describe some tools sufficient to obtain general lower estimates for the number of critical loops. As is well known, the classical Lusternik-Schnirelmann category gives a lower estimate for the minimal number of isolated critical points of a smooth function on a given manifold [12]. We follow the same pattern in the context of round functions by using an appropriate version of generalized category-like invariants introduced by M.Clapp and D.Puppe [4] (cf. also [2]). In such a way we come to reasonably effective lower estimates for the minimal possible number of critical loops in terms of these invariants and conclude the section by computing them in some simple cases.

In the last section we discuss minimal round functions on three-folds. In particular, we present complete lists of (homeomorphy types of) three-folds possessing round functions with two or three critical loops and obtain some corollaries concerned with the computation of round categories and round complexities. These results require rather involved geometric arguments and may be considered as our main new contribution to the topic.

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## 2 Definitions and setting

For brevity, throughout the whole text the word "smooth" means "infinitely differentiable". Manifolds are always assumed to be smooth compact and closed (without boundary), those with boundary will be referred to as $\partial$-manifolds.

Let $f : M \to \mathbf{R}^1$ be a smooth function on a (smooth compact) manifold $M$. As usually, the critical set $C(f)$ of function $f$ is defined as the set of all points where differential $d_p f$ vanishes. In our situation $C(f)$ is evidently non-empty and carries a lot of topological information about manifold $M$. The latter circumstance is well known and has spectacular manifestations in the case of functions with isolated critical points, especially Morse functions [15]. It should be noted that often it is also necessary (or useful) to consider functions with non-isolated critical points [3].

In many problems of differential topology and Hamiltonian mechanics an important role is played by so-called round Morse functions, with critical sets consisting of several smooth loops which are non-degenerate, in the sense of R.Bott [1], [21]. In some situations the condition of non-degeneracy does not seem natural, so we will work with a little bit more general class of functions.

**Definition 1** *Function $f$ is called a round function if its critical set $C(f)$ is a disjoint union of several smooth simple loops (images of smooth embeddings of the circle $S^1$). Components of $C(f)$ will be called critical loops of $f$.*

*If all critical loops of $f$ are non-degenerate, in the sense of Bott, then following [21] we will say that $f$ is a round Morse function. If all critical loops are non-degenerate except a finite number of points on them, we will say that $f$ is an almost Morse round function.*

Examples of round functions are immediate, but unlike to Morse functions, they do not exist on all manifolds.

**Proposition 1** *No round functions exist on two-sphere $S^2$ and real projective plane $\mathbf{RP}^2$.*

Indeed, on $S^2$ any critical loop bounds an embedded disc in interior of which the function in question evidently should have further critical points (maxima or minima, at least). On the other hand, by an evident compactness argument one always finds a critical loop containing no other critical loops in its interior. So there should exist some isolated minima or maxima in its interior, which contradicts the definition of round function. In the case of $\mathbf{RP}^2$ one arrives at the desired conclusion by considering lifts of functions to the universal covering space $S^2$.

At the same time there are many evident examples of round functions in all dimensions. For example, on any manifold of the form $M \times S^1$ round functions arise from arbitrary functions with isolated critical points on $M$. The same holds for circle bundles over manifolds, which provides, in particular, round functions on odd-dimensional spheres from Hopf fibrations $S^{2n+1} \to \mathbf{CP}^n$, the most visual example being a well-known function ("height-Hopf") on $S^3$ having two critical loops which are linked (as fibres of the Hopf fibration $S^3 \to S^2$).

Thus the existence issue for round functions is not completely trivial and we will comment upon it in the sequel. Note that in three-dimensional case mutual position of critical loops is also a non-trivial issue.

**Proposition 2** *On $S^3$ there are no functions with two critical loops which are unlinked.*

Indeed, any function with two critical loops gives a decomposition of $S^3$ in a union of two solid tori $S^1 \times D^2$ (which are suitable tubular neighborhouds of critical loops) glued by certain diffeomorphism of their boundaries. Clearly, only homotopy classes of gluing diffeomorphisms are important, and those are known to be described by an element of $SL(2, \mathbf{Z})$ expressing the effect of gluing on $H_1(T, \mathbf{Z})$ [11]. Discussion in [11] shows that the result of such gluing is diffeomorphic to $S^3$ if and only if the core circles of those solid tori are linked with linking number $\pm 1$ (in other cases one obtains $S^1 \times S^2$ or such lens spaces which are not even homeomorphic to $S^3$).

On the other hand, it is clear that round functions with unlinked critical circles exist on direct products of the form $M^2 \times S^1$. These observations suggest some natural problems which seem rather promising.

**Problem 1** *Characterize closed three-manifolds which possess round functions with unlinked critical circles.*

**Problem 2** *For a given compact connected three-fold, characterize links which may be represented as critical sets of round functions.*

Using Bott's theory of non-degenerate critical manifolds it is quite simple to indicate a topological invariant responsible for the existence of round Morse functions. Indeed, it is clear that if a non-degenerate critical manifold is homeomorphic to the circle, then under passage of this critical level the Euler characteristic of Lebesque sets remains unchanged [3]. Thus round Morse functions may only exist on manifolds with vanishing Euler characteristic.

D.Asimov proved that the converse is also true, except in the three-dimensional case [1]. As follows from [13], one can generalize this observation by looking at possible changes of the Euler characteristic in so-called transversally equisingular case. It turns out that the transversally equisingular behaviour is often exhibited by minimal round functions, so we give a precise definition.

**Definition 2** *A function $f$ is called transversally equisingular at a critical manifold $C$ if, for every point $p \in C$, germs at $p$ of restrictions of $f$ to small discs transversal to $C$ at $p$ belong to the same right-left equivalence class [8].*

**Proposition 3** *([13]) If $K$ is a transversally equisingular critical submanifold of function $f$, then under the passage of level $f(K)$ the Euler characteristic of Lebesgue sets $\chi(\{f \leq a\})$ is changed by an integer multiple of $\chi(K)$.*

**Corrolary 1** *Equisingular round functions exist on manifold $M$ if and only if $\chi(M) = 0$.*

In order to prove this proposition, one uses the multiplicativity property of the Euler characteristic and existence of a locally trivial fibration structure in a neighbourhood of an equisingular critical submanifold (cf. similar statements in [5]), to show that the total change of the Euler characteristic of Lebesgue sets $\triangle\chi\{f \leq a\}$ is equal to the product $\triangle\chi(slice) \cdot \triangle\chi(K)$. Actually, it is not difficult to show that in this case $\triangle\chi(slice)$ is equal to the gradient index $ind_p\, grad\,(f \,|\, D_p)$ of restrictions of $f$ to small transversal discs $D_p$ at $p$.

**Remark 1** *According to [13], the proof may be also obtained by a deformation argument which enables one to substitute an equisingular degenerate submanifold $K$ by a number of Bott submanifolds diffeomorphic to $K$, and then refer to classical results of Bott [3]. This is a sort of "morsification" procedure in the class of functions with critical submanifolds.*

This proposition remains valid under a weaker assumption that function $f$ is only topologically equisingular but, as we will see below, imposing certain condition of equisingularity is essential and a similar statement is not true for arbitrary round functions. In other words, the existence issue for round Morse functions and general round functions have essentially different features. Actually, up to now there is no complete description of the class of manifolds which possess round functions. In the next section we will clarify this issue for surfaces and three-folds.

**Remark 2** *In connection with said above it is natural to ask if every round function can be approximated by round Morse functions. In the next section we will see that this is not always possible, so degenerate round functions are really inevitable, even from the topological point of view.*

For further reference, it is also convenient to introduce another natural class of degenerate round functions.

**Definition 3** *A point $p \in K$ on a critical loop of a round function $f$ is called a point of $D_\infty$-type (for $f$) if, in some system of local coordinates $(x_1, ..., x_n)$ around $p$, function $f$ takes the form $x_1 x_2^2 \pm x_3^2 \pm ... \pm x_n^2$.*

*A critical loop $K$ is called a Morse-Whitney critical loop if $f$ is transversally non-degenerate on $K$ with only possible exception of a finite number of points which are all of $D_\infty$-type. Finally, we will say that a round function is a Morse-Whitney function if all of its critical loops are Morse-Whitney critical loops.*

**Remark 3** *The term "Morse-Whitney" is chosen for the reason that level surfaces near a $D_\infty$-point exhibit behaviour similar to that of the Whitney umbrella. In principle, it would be even more logical to say "Bott-Whitney" but we want to emphasize analogy with the term "round Morse functions".*

In this paper we will be basically concerned with estimating the minimal possible number of critical loops of a round function on a given manifold. Recall that F.Takens in [20] introduced an interesting topological invariant $F.(M)$ of a smooth manifold $M$, defined as the minimal number of critical points of smooth functions on $M$. In some cases he was able to show that $F.(M)$ coincides with the Lusternik-Schnirelmann category $cat\, M$, and explicitly constructed so-called exact functions which have precisely $catM$ critical points on $M$. Our research strategy is to mimic his approach in the context of round functions.

**Definition 4** *Round complexity $roc\, M$ of a manifold $M$ is defined as the minimal possible number of critical loops of round functions on $M$. If round functions on $M$ do not exist we put $roc\, M = \infty$. A round function $f$ is called a minimal round function if the number of components of $C(f)$ is equal to $roc\, M$.*

Round complexities are usually hard to compute. Below we will introduce an appropriate homotopy invariant of $M$, round category $Tcat\, M$, which gives a lower estimate for $roc\, M$. We will be interested in finding cases in which $roc\, M = Tcat\, M$. If this is the case, we will say that a round function is exact if it has precisely $Tcat\, M$ critical loops.

In the sequel we will be mainly concerned with minimal and exact round functions on surfaces and three-folds.

## 3 Round functions in low dimensions

Here we describe some constructions of round functions on surfaces and three-folds and discuss the structure of minimal round functions.

**Theorem 1** *Round functions exist on all closed surfaces, except* $S^2$ *and* $\mathbf{RP}^2$*. Transversally equisingular round functions, as well as round Morse functions, exist only on* $T^2$ *and Klein bottle* $K^2$*. The round complexity is equal to two for* $T^2$ *and* $K^2$*, and it is equal to three in all remaining cases.*

*On all closed surfaces there exist almost Morse round functions. On surfaces with the even Euler characteristic there always exist minimal round functions which are Morse-Whitney functions. The minimal number of* $D_\infty$*-points of a minimal Morse-Whitney function on an orientable surface of genus* $g$ *is equal to* $2g-2$*.*

We prove this by using a sort of surgery suitable for round functions on surfaces the best description of which could be probably given just by drawing some pictures. We prefer nevertheless to make concise comments which should make clear the main point of construction. For simplicity, we only consider the orientable case and start with a standard model of a (round) torus $T^2$ with the evident minimal round function on it ("height on a lying tyre").

We take two such tori and arrange that the circle of maxima on the first copy lies on the same level with the circle of minima on the second copy, say on the zero-level. Then we take a point $P$ on one circle and a point $Q$ on another, delete small discs around these points and perform our surgery. To this end we glue a cylinder to boundaries of deleted discs and try to extend our function to that cylinder.

A simple visual examination of arising picture shows that this is really possible and the simplest way of doing so is to join "free ends" of original critical circles by two segments on the cylinder with one Whitney umbrella ($D_\infty$-point) on each of those segments. As one immediately sees, the result is a round function with three critical loops on a connected sum of two tori and it becomes clear that this procedure may be iterated, which yields round functions on all orientable closed surfaces.

A slight modification of the same surgery enables one to fuse any pair of critical loops containing points of different types (max vs. min). This shows that the number of critical loops may be reduced to three, for any surface with the genus higher than one, and the arising round functions are

Morse-Whitney functions. Using standard Morse theory it is easy to check that the number of Whitney umbrellas on every critical loop is even and the Euler characteristic of Lebesgue sets changes by $-2k$ under passage of a Morse-Whitney loop with $2k$ points of $D_\infty$-type on it, which gives the last statement of the theorem referring to the orientable case.

In the non-orientable case one can develop similar surgery which uses gluing of Möbius bands. It should be noted that critical loops provided by this procedure have the transversal $A_2$-type [19] so in this way one cannot derive existence of almost Morse functions on non-orientable closed surfaces. To prove the latter statement one may apply another natural procedure which uses blow-ups of isolated extrema and produces almost Morse critical loops with points of $J_{2,\infty}$-type, in the notation of [19].

**Remark 4** *Despite its simplicity, this theorem is instructive since it gives an example of a fairly complete description of minimal round functions, both in local and global aspects. The description of their local behaviour follows from the explicit geometric constructions which we apply and fits nicely into the framework of isolated line singularities [19]. It is also remarkable that constructions of round functions and types of local singularities are different in the orientable and non-orientable case. In higher dimensions one cannot hope for such a complete description but it is clearly helpful to keep in mind a sample result.*

We consider now round functions on three-folds. Here situation is substantially more complicated and our results are less complete. For the sake of clarity we first formulate the existence result. Recall that there exists an especially well understood class of three-folds, so-called Waldhausen class, which consists of unions of Seifert fibrations patched along parts of their boundaries [11].

**Theorem 2** *Round functions exist on all closed three-folds. Round complexity of a closed three-fold does not exceed four. Transversally equisingular round functions, as well as round Morse functions, exist only on three-folds of Waldhausen class.*

The simplest way to prove the first statement is to refer to the well known fact that any three-fold is a union of solid tori glued along parts of their boundaries [11]. This is indeed sufficient because we are also able to show that round functions can be constructed on any such union by applying a "parameterized version" of Takens' trick ([20], Theorem 2.7), in such a way that critical loops are precisely the core circles of those solid tori.

In order to prove that the round complexity does not actually exceed four, one may refer to a more elaborated procedure of "surgery on links" [11], which in particular yields that every three-fold may be obtained from $S^3$ by a surgery on a two-component link. This basically reduces to deleting two solid tori from $S^3$ and gluing them back with possible twists defined by certain diffemorphisms of their boundaries. Granted this, we may start by taking a standard round function on $S^3$ ("height·Hopf") and then properly modify it on the interiors of deleted solid tori using the same "parameterized version" of the Takens' construction.

The statement about transversally equisingular round functions is reduced to the particular case of round Morse functions by a procedure of "round morsification" mentioned in Remark 1. The statement about round Morse functions follows from the results of [17]. Indeed, in [17] it was shown that a three-fold belongs to Waldhausen class if and only if it admits so-called round handle decompositions introduced in [1]. The proof is completed by noticing that, as was established in [16], existence of a round handle decomposition is equivalent to existence of a round Morse function.

**Remark 5** *Another application of a "parameterized version of Takens's trick" may be found in [5]. We emphasize that the setting there is substantially different from ours and results of [5] cannot be automatically applied in our case because they were obtained under the assumption that critical submanifolds are simply connected. Similar but less precise statements about round functions on surfaces and three-folds may be also found in [2]. It should be noted that [2] contains only announcements of results without any comments on proofs.*

Two important general conclusions which one may derive from these results, are, firstly, that degenerate round functions are inevitable outside Waldhausen class and, secondly, that arbitrary round functions not always can be approximated by round Morse functions. Of course this well may be caused by the low-dimensionality of manifolds in question so it would be interesting to know if such approximation is possible in higher dimensions, but this issue remains uninvestigated.

## 4 Round category and cup-length

In higher dimensions, there is already little hope to succeed in studying round functions by purely geometric means and to this end one has to develop some suitable topological tools. Here we use two relevant notions, round category and round cup-length, and present analogies of the two basic inequalities of Lusternik-Schnirelmann theory (Propositions 4 and 5). Similar notions and results are found in [2] and [14] but actually they go back to [4]. Then we compute these invariants in some cases of interest, including closed surfaces and certain three-folds. Combined with results of Section 3, this proves existence of exact round functions on closed surfaces and certain connected sums of Seifert fibrations.

Recall that a subset $A \subset X$ of a topological space $X$ is called $T$-categorical if the inclusion map $i : A \to X$ may be factored through the circle $T$ up to homotopy [4], that is, there exist continuous maps $\phi : A \to T$ and $\psi : T \to X$ such that $\psi\phi$ is homotopic to $i$.

**Definition 5** *(cf. [4], [2], [14]) Round category $Tcat\, X$ of a connected paracompact space $X$ is defined as the minimal possible cardinality of coverings of $X$ by $T$-categorical closed subsets. A round function on manifold $M$ is called an exact round function if the number of its critical loops is equal to $Tcat\, M$.*

This is just a special case of the general definition from [4] so we may use results from [4]. In particular, from the discussion in [4] it follows that, for any closed manifold $M$ which is not homeomorphic to $S^1$, one has inequalities: $2 \leq Tcat\, M \leq cat\, M \leq dim\, M + 1$. In particular, it becomes clear that, for any $n$, $Tcat\, S^n = 2$.

**Proposition 4** *For a closed manifold $M$, one has: $Tcat\, M \leq roc\, M$.*

This follows from a more general statement found in [4] (Theorem 2.3) so we omit the proof. It is now easy to verify that Theorem 1 enables one to compute round category for closed surfaces and yields examples of exact round functions.

**Corrolary 2** *Exact round functions exist on all surfaces except two-torus and Klein bottle.*

For evident reasons, it is tempting to compare round category of a circle bundle with the usual Lusternik-Schnirelmann category of its base. It may be proved that for a circle bundle $E$ over a closed surface $M$, one always has $Tcat\, E = cat\, M$. There is little hope that the same holds for arbitrary CW-complexes but there is good evidence that this is true for direct products with the circle.

**Problem 3** *Prove that $Tcat\,(X \times S^1) = cat\, X$ for any two-dimensional CW-complex $X$.*

We are nearly sure that in higher dimensions this equality cannot hold for all smooth manifolds, so looking for a corresponding counter-example might be a reasonable enterprise. At the same time, the equality holds for many manifolds with sufficiently simple cellular decompositions.

**Problem 4** *Find topological conditions on manifold $M$ which guarantee that $Tcat\,(M \times S^1) = cat\, M$.*

Computation of round category in higher-dimensional cases usually cannot be done by purely geometric considerations like those in Theorems 1 and 2. Some tools from algebraic topology are helpful here and we borrow one of them from [4].

**Definition 6** *([4]) Round cup-length $Tcl(X)$ of a topological space $X$ is defined as the nilpotency index [12] of the subring $H^*_T(X, \mathbf{Z})$ equal to the intersection of all kernels of induced mappings $F^* : H^*(X, \mathbf{Z}) \to \mathbf{Z}$, where $F : T \to X$ runs over all continuous mappings of the circle $T$ into $X$.*

**Proposition 5** *For any CW-complex $X$, one has: $Tcl(X) + 1 \leq Tcat\, X$.*

As in the case of Proposition 4, the proof is obtained by a simple modification of the proof of a similar statement in [4] (Proposition 3.1) and is therefore omitted. For a closed manifold $M$, according to Poincaré duality, one can of course reformulate this estimate in a more geometric form by looking at suitable intersections of cycles on $M$.

This result enables one to compute the round category in some higher-dimensional cases. We present but two results of this kind which illustrate some general phenomena exhibited by these invariants.

**Proposition 6** $roc\, S^5 = 3$.

We prove this by analyzing possible topological types of unions of two copies of $S^1 \times D^4$ glued along some diffeomorphism of their boundaries. Using van Kampen's theorem it is not difficult to show that the result of such gluing never can have the homotopy type of $S^5$.

**Proposition 7** *$roc\, T^n = Tcat\, T^n = n$.*

This follows by first showing that $Tcat\, T^n \geq n$ and then constructing a round function with $n$ critical loops. An elementary examination of cup-products in $H^*(T^n)$ shows that $Tcl(T^n) = n-1$ (cf. [4]), and it is also easy to obtain a desired function on $T^n$. Indeed, it is well known that $cat\, T^{n-1} = n$ [12] and one easily shows by induction that on $T^{n-1}$ exists a function with exactly $n$ isolated critical points.

Thus exact round functions exist on tori, while $S^5$ admits no exact round functions. Hence inequality in Proposition 4 cannot be substituted by equality and of course the same refers to inequality in Proposition 5. Nevertheless, we know many cases when it is possible to prove that one or both of these equalities take place, so this issue deserves further investigation.

**Remark 6** *It is also possible to prove that, for all $n \geq 3$, $roc\, S^{2n-1} \geq 3$ by a similar topological analysis of unions of two copies of $S^1 \times D^{2n-2}$. Actually, one can even compute the round complexity in question and show that, for all $n \geq 2$, $roc\, S^{2n-1} = n$. We do not make here any attempts to describe the proof, since our argument requires some results of Conley index theory which did not seem appropriate to discuss in this note.*

We would like to conclude this section by formulating another interesting problem suggested by the inequalities obtained above.

**Problem 5** *For any $n \geq 4$, construct an $n$-dimensional manifold $M$ with $Tcat\, M = n + 1$.*

# 5 Minimal round functions on three-folds

In order to formulate the main result, we need some notations and conventions. The symbol # will always denote connected sum of two closed 3-folds. By $B_j$ we will denote a copy of the product $S^1 \times S^2$ and by $B = \#B_j$ an arbitrary finite connected sum of such products. By $L_j$ or simply $L$ we will denote any lens space with a non-trivial finite fundamental group. Finally, let $Sf(3)$ denote any Seifert fibration [11] with no more than three exceptional fibres.

**Theorem 3** *Let $M$ be a compact closed three-fold. If $roc\, M$ is equal to two, then $M$ is homeomorphic to $S^3, \mathbf{RP}^3, S^1 \times S^2$, or $L$. If $roc\, M$ is equal to three, then $M$ is homeomorphic to one of the manifolds of the following type:*

$$L\#B, L_1\#L_2, L_1\#L_2\#B, L_1\#L_2\#L_3, L_1\#L_2\#L_3\#B, Sf(3), Sf(3)\#B.$$

**Corrolary 3** *There exist manifolds $M$ from Waldhausen class with $roc\, M = 4$, in other words the upper bound four established in Theorem 2 is sharp.*

**Remark 7** *Theorem 3 enables one to compute the round complexity for many three-folds and may be considered as an analogy of Theorem 3.3 from [20]. Despite apparent similarity of formulations of these two results, their proofs use essentially different techniques. In particular, we make no use of so-called fillings [20] playing the crucial role in the Takens' approach.*

Proof of Theorem 3 makes an essential use of the existing comprehensive structural theory of three-folds [11]. The crucial ingredient is an analysis of possible homeomorphy types of unions of several solid tori in the spirit of [10]. First, applying standard Morse theory we show that a manifold with the round complexity not exceeding three is representable as a union of two or three solid tori appearing as suitable tubular neighbourhoods of critical loops. Results of [10] actually contain the topological classification for certain types of unions of two solid tori, and with some additional effort we are able to show that they are applicable in our situation. It remains to extend the classification to unions of three solid tori, which is done in an analogous way by making proper use of results of [9].

**Remark 8** *A straightforward attempt to prove that in situation of Theorem 3 manifold $M$ is actually diffeomorphic to one of the manifolds in these lists, meets some serious difficulties typical for low-dimensional differential topology. Situation here is analogous with [20] since availability of classification of three-folds $M$ with $F.(M) = 2$ depends on validity of Poincaré conjecture.*

Thus it turns out that the round category is more hard to compute than the round complexity. Nevertheless, Theorem 3 apparently computes the round category of any three-fold $M$ with $roc\, M \leq 3$. It is also possible to develop explicit constructions of minimal round functions on many three-folds from the above lists and obtain some information on the transversal behaviour of resulting functions along their critical loops.

Note that one should additionally check existence of exact round functions on three-folds from the above lists, since they are only *homeomorphic* to three-folds on which the existence of an exact round function is granted, while we do not have a proof of the fact that $roc\, M$ is a topological invariant. We approach the construction of desired functions by using an extension of surgery applied in the proof of Theorem 1.

Such surgery enables one, in particular, to fuse two extremal loops of different types (a max-min pair) lying on the same critical level, in such a way that the result is again a round function with only two exceptional points on the corresponding critical loop. This leads to an inductive construction of round functions on three-folds from our lists and yields exact round functions on some of them.

**Proposition 8** *All three-folds from the above lists possess almost Morse round functions.*

We prove this by checking that this surgery produces an almost Morse round function on a connected sum of two three-folds from a pair of almost Morse round functions on summands.

**Remark 9** *We want to emphasize that existence of exact round functions on these three-folds cannot be discussed since we have not yet computed their round categories. Moreover, even their round complexities should be computed by a separate argument since we do not know if round complexity is a topological invariant (from the definition it only follows that round complexity is an invariant of diffeomorphy type).*

Thus it remains unclear if exact round functions exist on all three-folds. The latter fact would be established, if the following problem has positive solution, which seems to us very plausible.

**Problem 6** *Prove that for a compact closed three-fold $M$, $Tcat\, M = 3$ if and only if $roc\, M = 3$.*

As a natural first step, one should of course try to compute round categories for three-folds in our lists. It is quite simple to see that three-folds with round complexity equal to two, have the round category also equal to two. We have checked that the round category is equal to three for some manifolds from our second list, but we do not see any "a priori" reason why the same should hold in remaining cases so the whole issue is open even for our "models".

In general, round category is difficult to compute and situation here is much more complicated than with the classical Lusternik-Schnirelmann category. For example, it is well-known that a manifold $M$ with $cat\, M = 2$ is homotopy equivalent to a sphere [12], but we are not aware of any reasonable description of manifolds with the round category equal to two.

**Remark 10** *In order to keep this text within a reasonable length, we consciously do not discuss round functions in higher dimensions, despite this is related to some interesting observations and open problems. For example, the vanishing of Euler characteristic is not indeed an obstruction for existence of round functions in higher-dimensions, as may be seen by considering products of Riemann surfaces of high genera. At the same time, we are not aware of any description of manifolds which admit round functions, so finding such a description is apparently a meaningful and urgent problem.*

In the conclusion, we would like to mention that similar results are available for low-dimensional $\partial$-manifolds and some other types of stratified spaces. These developments, as well as relevant versions of Lusternik-Schnirelmann theory, will be published elsewhere.

Another promising line of development is related with functions on manifolds endowed with codimension-one foliations [6]. Those results are technically more involved and require additional preliminary explanations so they are also left for future publications.

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