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THE DUAL OF THE SPACE OF MEASURES WITH CONTINUOUS TRANSLATIONS

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1. INTRODUCTION

Let S be a locally compact semigroup. Consider the space L(S) of all bounded Radon measures μ on S of which the translations r_μ and ℓ_μ [$r_\mu(x) := \mu*\bar{x}$, $\ell_\mu(x) := \bar{x}*\mu$ $(x \in S)$, where $\bar{x}$ denotes the point-mass at x] are maps on S that are continuous with respect to the total variation norm. Assume that S is a foundation semigroup; i.e. clo $\cup\{\text{supp}(\mu) \mid \mu \in L(S)\} = S$ [cf. [1],§4]. Certainly, if in addition S has an identity element [S is said to be a foundation stip; cf. [6], (2.2)], in view of the results in e.g. [6], one may state that L(S) is the analogue of the group algebra $L^1(G)$ of a locally compact group G. However, unlike the group case, in general, there is no Radon measure m on S for which L(S) can be identified with $L^1(S,m)$. Consequently, it is not obvious whether $L(S)^*$, the topological dual of L(S), can be identified with the space of the bounded complex-valued L(S)-measurable [i.e. μ-measurable for all $\mu \in L(S)$] functions on S.

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For instance, any non-discrete locally compact Hausdorff space X provided with a trivial multiplication [i.e. fix a $0 \in X$ and define $xy := 0$ for all $x,y \in X$] is a foundation semigroup on which every bounded Radon measure belongs to L(X) and clearly $L(X)^*$ cannot be viewed as a space of functions on X. If S is a foundation stip, then in this case, as well, no such "dominant" measure m need exist [cf. (2.9)]. However, in this note, for such a semigroup S, we identify $L(S)^*$ with the above-mentioned space of L(S)-measurable functions.

We shall shed some light upon the structure of a foundation stip [(2.1)-(3.2)]. The results shall be used in order to obtain a generalization of the localization property in ch.IV, §5, no.9 of [3]. As a corollary, we find the description of $L(S)^*$ as mentioned above.

This localization property is a basic requirement if one whishes to apply the main results in [2]; the constructions, considered by J.-P. Bertrandias in this paper, may turn out to be important in order to obtain analogues on foundation stips of the $L^p(G)$ spaces on a group G ($p \in (1,\infty]$).

If, the foundation stip is commutative, the results in this note follow easily from the ones in ch.XIV of [5]. However, in the non-commutative case, we have to do some work. Some of the lemmas (2.4)-(2.8), in one or another form, can also be found in [5]; for the convenience of the reader we include a [simplified] proof here.

2. THE COUNTABLE CLOSURE OF THE SMALLEST DENSE IDEAL

The conventions, notations and definitions that are not explained in the text are the same as the ones in [6]. Definition (2.1) and proposition (2.2) are basic in the theory of stips; their proofs can be found in [6].

(2.1) DEFINITION. [cf. [6], (2.1)-(2.4)]. Let S be a locally compact semigroup with identity element 1. S is said to be a stip if for each neighbourhood U of 1 we have that

(i) $x \in \mathrm{int}[U^{-1}(Ux) \cap (xU)U^{-1}]$ for all $x \in S$

(ii) $1 \in \mathrm{int}[U^{-1}v \cap wU^{-1}]$ for some $v,w \in U$.

If, in addition, (iii) $\mathrm{clo}(\bigcup\{\mathrm{supp}(\mu)\,|\,\mu \in L(S)\}) = S$, we say that S is a foundation stip. Put $\dot{S} := \bigcap\{J\,|\,J \subseteq S,\ \mathrm{clo}\ J = S,\ JS \cap SJ \subseteq J\}$.

Throughout the sequel, S is a stip.

(2.2) PROPOSITION. [cf. [6], (2.4)-(2.7), (3.13)]. $\mathrm{clo}(\dot{S}) = S$, $S\dot{S}S = \dot{S}\dot{S} = \dot{S}$; $\dot{S}$ is the smallest dense ideal of S. Let U,V be open sets, $x \in S$, $v \in V \cap \dot{S}$. Then $U^{-1}(Vx)$, $(xV)U^{-1}$, $(U \cap \dot{S})^{-1}x$, $x(U \cap \dot{S})^{-1}$ are open and $x \in \mathrm{int}[(xV)v^{-1} \cap v^{-1}(Vx)]$. L(S) is an L-ideal in the space M(S) of all bounded Radon measures on S. □

$\mathcal{K}$ denotes the collection of all compact subsets F of S, while $\mathcal{N}$ is the subcollection of all $F \in \mathcal{K}$ that are L(S)-negligible [i.e. $\mu(F) = 0$ for all $\mu \in L(S)$]. $L(S)_1$ is the unit-ball $\{\mu \in L(S)\,|\,\|\mu\| \le 1\}$ of L(S).

(2.3) NOTATION. $S_\omega := \bigcup\{\mathrm{clo}(A)\,|\,A \text{ is a countable subset of } \dot{S}\}$.
Note that S_ω is a two-sided dense ideal in S.
E is the collection of all idempotents e in S [$e^2 = e$] and
$E_\delta := S_\omega \cap E$.

(2.4) LEMMA. Let $(V_n)_{n\in\mathbb{N}}$ be a sequence of neighbourhoods of 1. Then

$$E_\delta \cap \bigcap\{V_n\,|\,n \in \mathbb{N}\} \neq \emptyset.$$

Proof. There exists a sequence $(W_n)_{n\in\mathbb{N}}$ of open relatively compact neighbourhoods of 1 such that

$$\bar{W}_n^2 \subseteq W_{n-1} \cap \mathrm{int}(V_{n-1}), \quad n = 2,3,\ldots$$

Put $H := \bigcap\{W_n | n \in \mathbb{N}\}$. Then H is a compact subsemigroup of S. Let e be an idempotent in the kernel of H. For each $n \in \mathbb{N}$, take $w_n \in W_n \cap \dot{S}$. Let $x \in \bigcap_n \mathrm{clo}\{w_m | m \geq n\}$. Then $x \in H$ and there is a $y \in S$ such that $e = exye$. Then $e \in \mathrm{clo}\{ew_m ye | m \in \mathbb{N}\}$, while $ew_m ye \in \dot{S}$. □

(2.5) LEMMA. For each $e \in E$, eSe is closed. $S_\omega = \bigcup\{eSe | e \in E_\delta\}$.

Proof. The first claim is obvious. Let A be a countable subset of $\dot{S}$. There is an $e \in E_\delta \cap \bigcap\{S^{-1}a | a \in A\} \cap \bigcap\{aS^{-1} | a \in A\}$. Then $A \subseteq eSe$, whence $\mathrm{clo}(A) \subseteq eSe$. □

(2.6) LEMMA. Let $A \subseteq S_\omega$ be countable, let F be a σ-compact subset of S. There is an $f \in E_\delta$ such that $AF \cap FA \subseteq fSf$.

Proof. Let B be a countable subset of $\dot{S}$ such that $A \subseteq \mathrm{clo}(B)$. For each $x \in S$ both xSa^{-1} and $a^{-1}Sx$ are neighbourhoods of x $(a \in B)$. Since, F is σ-compact, for each $a \in B$ there are countable subsets P_a, P'_a of F such that $Fa \subseteq P_a S$ and $aF \subseteq SP'_a$. Let $f \in E_\delta \cap \bigcap\{pS^{-1} | p \in P_a, a \in B\} \cap \bigcap\{S^{-1}p | p \in P'_a, a \in B\} \cap \bigcap\{S^{-1}a | a \in B\} \cap \bigcap\{aS^{-1} | a \in B\}$. If $x \in F$ then $xa = pt$ for some $p \in P_a$, $t \in S$. Since $fp = p$, we have that $fxa = fpt = pt = xa$. Therefore

$$FB \cup BF \subseteq fSf \text{ and } FA \cup AF \subseteq \mathrm{clo}(FB \cup BF) \subseteq fSf. \qquad □$$

(2.7) LEMMA. Let S be a foundation stip. Then $S_\omega = \bigcup\{\mathrm{supp}(\mu) | \mu \in L(S)\}$.

Proof. Put $T := \bigcup\{\mathrm{supp}(\mu) | \mu \in L(S)\}$. Let $x \in S$. Since $\dot{S}^{-1}x$ is open and non-empty, there is a $\mu \in L(S)^+$ such that $\mu(\dot{S}^{-1}x) \neq 0$. If $t \in \mathrm{supp}(\mu) \cap \dot{S}^{-1}x$ then $x \in St \subseteq \bigcup\{\mathrm{supp}(\bar{y}*\mu) | y \in S\}$. Hence $\dot{S} \subseteq T$.

If $(\nu_n)_{n\in\mathbb{N}}$ is a sequence in $L(S)_1^+$, then $\nu := \sum_{n=1}^{\infty} \nu_n 2^{-n} \in L(S)$ and $\mathrm{supp}(\nu) = \mathrm{clo}\, \bigcup\{\mathrm{supp}(\nu_n) \mid n \in \mathbb{N}\}$. Therefore $S_\omega \subseteq T$.

Let $\mu \in L(S)^+$. Put $d := \sup\{\mu(aK) \mid a \in \dot{S},\ K \in \mathcal{K}\}$. Then $\mu(AF) = d$ for some countable subset A of $\dot{S}$ and some σ-compact subset F of S. Consider $\nu := \mu|_{S\setminus AF}$. If $K \in \mathcal{K}$ then $\nu(aK) = 0$ for all $a \in \dot{S}$. Since $1 \in \mathrm{clo}(\dot{S})$ and $\nu \in L(S)$ we have that $\nu(K) = 0$ for all $K \in \mathcal{K}$. Therefore $\mu(S\setminus AF) = 0$. By (2.6) there is an $f \in E_\delta$ with $AF \subseteq fSf$. Hence $\mathrm{supp}(\mu) \subseteq \mathrm{clo}(AF) \subseteq \mathrm{clo}\, fSf = fSf$. □

(2.8) <u>LEMMA</u>. <u>Let</u> S <u>be a foundation stip</u>. <u>Let</u> $F \in \mathcal{K}$ <u>and</u> $e \in E_\delta$ <u>with</u> $F \subseteq Se$.

<u>There is an</u> $m \in L(S)^+$ <u>such that</u> $\mu|_F \ll m$ <u>for all</u> $\mu \in L(S)$.

<u>Proof</u>. Let $\nu \in L(S)_1^+$ such that $e \in \mathrm{supp}(\nu)$. There is a sequence $(x_n)_{n\in\mathbb{N}}$ in F for which $\{\bar{x}_n * \nu \mid n \in \mathbb{N}\}$ is dense in $\{\bar{x} * \nu \mid x \in F\}$. Put $m := \Sigma 2^{-n} \bar{x}_n * \nu$. If $K \in \mathcal{K}$ such that $m(K) = 0$ then $\bar{x} * \nu(K) = 0$ for all $x \in F$, whence $\mu|_F * \nu(K) = 0$ $(\mu \in L(S))$. Since $e \in \mathrm{supp}(\nu)$, $\mu|_F * \bar{e}(K) = 0$ $(\mu \in L(S))$. Finally, the observation that $\mu|_F * \bar{e} = \mu$ $(\mu \in L(S))$ completes the proof. □

(2.9) <u>EXAMPLE</u>. There exists a foundation stip S for which $S_\omega \neq S$. In particular, for this stip S, for each $m \in M(S)^+$ we have $L(S) \not\subseteq L^1(S,m)$. [Take an uncountable index set I, and let S be the product space $\{0,1\}^I$ endowed with the product topology and coordinate wise multiplication [per coordinate we have the usual multiplication of the integers]. Then $\dot{S}$ consists of the elements x in S for which only finitely many coördinates are equal to 1 [$\{i \in I \mid x(i) = 1\}$ is finite]. Furthermore, $L(S)$ is the closed subspace of $M(S)$ generated by $\{\bar{x} \in M(S) \mid x \in \dot{S}\}$. Therefore, S is a foundation stip. However, S_ω consists of the elements in S for which at most countably many coordinates are equal to 1 and

$S_\omega \neq S$. Now, suppose there is an $m \in M(S)^+$ such that $L(S) \subseteq L^1(S,m)$. Then there is an $m_1 \in L(S)^+$ such that $L(S) = L^1(S,m)$. Therefore, by (2.7), $S_\omega = \mathrm{supp}(m_1)$. However, this violates the facts that clo $S_\omega = S$ and $S \neq S_\omega$.]

3. THE LOCALIZABILITY OF S WITH RESPECT TO L(S)

In order to show that any foundation stip S is "localizable" with respect to L(S) [cf. (3.4)] we need another two lemmas. In these lemmas we give a partition of $\dot{S}$ into L(S)-measurable non-L(S)-negligible parts.

(3.1) LEMMA. *Let H be a compact subsemigroup of S with identity element* 1. *Put*

$$E(H) := E \cap H,$$

$$I(e) := \bigcup\{Sf \mid f \in E(H),\ HfH = HeH\} \qquad (e \in E(H)),$$

$$H(e) := I(e) \setminus \bigcup\{I(f) \mid f \in E(H),\ e \notin HfH\} \qquad (e \in E(H)).$$

Then (1) $I(e)$ *is a closed left ideal in* S $\qquad (e \in E(H))$,

(2) $\{H(e) \mid e \in E(H)\}$ *is a partition of* S,

(3) *if* $e \in E(H)$ *then* $H(e) \cap \dot{S} \neq \emptyset$ *if and only if* $H(e)$ *is open in* $I(e)$,

(4) *The collection* $\{H(e) \mid e \in E(H),\ H(e) \cap \dot{S} \neq \emptyset\}$ *covers* $\dot{S}$.

Proof. Before we prove (1)-(4), we make some observations.

(i) Note that $H(e) = H(f)$ for all $e,f \in E(H)$ for which $HeH = HfH$. Take an $x \in S$. Put $F(x) := \{y \in H \mid xy = x\}$. Since $F(x)$ is a compact semigroup, $F(x)$ contains some minimal idempotent e_1 [i.e. $e_1 \in F(x) \cap E$ and $e_1 \in F(x)yF(x) \subseteq HyH$ for all $y \in F(x)$].

(ii) If $f \in E(H)$ such that $x \in I(f)$ then $x \in Sf_1$ for some $f_1 \in E(H)$ for which $Hf_1H = HfH$. Clearly, $f_1 \in F(x)$ and therefore,

$e_1 \in Hf_1H = HfH$. Obviously, $x \in I(e_1)$ and, consequently, $x \in H(e_1)$.

(iii) Furthermore, if $e \in E(H)$ such that $x \in H(e)$, then, by definition of $H(e)$ and the fact that $x \in I(e_1)$ we have that $e \in He_1H$. Since e_1 is minimal, we see that $e \in He_1H \subseteq HyH$ for all $y \in F(x)$.

From (ii), it follows that $\{H(e) | e \in E(H)\}$ covers S and also that $\{H(e) | e \in E(H), H(e) \cap \dot{S} \neq \emptyset\}$ covers $\dot{S}$. In view of (i), it is not hard to see that, for any $f,e \in E(H)$, either $H(f) = H(e)$ or $H(f) \cap H(e) = \emptyset$.

In order to prove (1) and (3), let $e_0 \in E(H)$.

The closedness of $I(e_0)$ follows easily from the compactness of $\{f \in E(H) | HfH = He_0H\}$.

Now, assume that $H(e_0) \cap \dot{S} \neq \emptyset$.

First we shall prove that

(5) $\dot{S}f \cap H(e_0) \cap \dot{S} \neq \emptyset$ for all $f \in E(H)$ for which $HfH = He_0H$.

Take an $x \in H(f) \cap \dot{S}$. Let $e \in E(H)$ such that $HeH = HfH$ and $x \in Se$. Let $v,w \in H$ such that $e = vfw$. Consider xvf. Obviously $xvf \in \dot{S}f \cap \dot{S} \cap I(f)$. If $xvf \in Se_1$ for some $e_1 \in E(H)$ then $xvfe_1 = xvf$ and, therefore $xvfe_1w = xe = x$, which shows that $vfe_1w \in F(x)$ [where $F(x)$ is as above]. Hence, by (iii), $f \in Hvfe_1wH \subseteq He_1H$. Apparently, $xvf \in H(f)$ and $\dot{S}f \cap H(f) \cap \dot{S} \neq \emptyset$. Since $H(f) = H(e_0)$ for all $f \in E(H)$ with $HfH = He_0H$, this proves (5).

A combination of (5) and (2.2) shows that

(6) $\dot{S}^{-1}H(e_0)$ is an open set containing $\{f \in E(H) | HfH = He_0H\}$.

Let $f \in E(H)$ such that $e_0 \notin HfH$. Suppose that $f \in \dot{S}^{-1}H(e_0)$. Then $tf \in H(e_0)$ for some $t \in \dot{S}$. Clearly, $tf \in I(f)$. Since $tf \in H(e_0)$, we must have that $e_0 \in HfH$, which is impossible. Apparently

(7) if $f \in E(H)$ such that $e_0 \notin HfH$ then $f \notin \dot{S}^{-1}H(e_0)$.

Finally, to prove that $H(e_0)$ is open in $I(e_0)$, let $x \in \mathrm{clo}(I(e_0)\backslash H(e_0))$.

Note that, if $y \in I(e_0)\backslash H(e_0)$ and f is a minimal idempotent in $F(y)$ then $f \in He_0H$ and $e_0 \notin HfH$. Therefore, there are nets $(x_\lambda)_{\lambda\in\Lambda}$ in $I(e_0)\backslash H(e_0)$ converging to x and $(f_\lambda)_{\lambda\in\Lambda}$ in $E(H)$ such that

$$x_\lambda f_\lambda = x_\lambda,\ f_\lambda \in He_0H,\ e_0 \notin Hf_\lambda H \qquad (\lambda \in \Lambda).$$

Since $E(H) \cap He_0H$ is compact, we may assume that $(f_\lambda)_{\lambda\in\Lambda}$ converges to an $f \in E(H) \cap He_0H$. (6), (7) and "$f \in He_0H$" tells us that $e_0 \notin HfH$. Also we have that $xf = x$, which shows that $x \notin H(e_0)$.

The other property in the lemma has a simple proof; this is omitted. □

(3.2) LEMMA. *Let H be a compact G_δ-subsemigroup of S such that $1 \in H$. Using the same notation as in the preceding lemma, we have that*

(1) *each $e \in E(H)$ belongs to E_δ as soon as $H(e) \cap \dot{S} \neq \emptyset$;*

(2) *for each $\mu \in L(S)$ there exists a countable subcollection $\mathcal{P}$ of $\{H(e) | e \in E(H),\ H(e) \cap \dot{S} \neq \emptyset\}$ such that $|\mu|(S\backslash \cup\mathcal{P}) = 0$.*

Proof. Let $e \in E(H)$ such that $H(e) \cap \dot{S} \neq \emptyset$. since H is a G_δ-set and $H(e)$ is open in $I(e)$ [cf. (3.1.3)], we can find a compact subsemigroup H' of H that is a G_δ-set in $I(e)$ and with $e \in H' \subseteq H(e)$. Note that for each $f \in E(H) \cap H'$ we have that $HfH = HeH$. Since H' is a G_δ-set in $I(e)$, there exists a countable subset D of $\dot{S}$ [use (2.2)] such that $H' \cap \mathrm{clo}(D) \neq \emptyset$. Therefore, since each compact semigroup contains minimal idempotents, in view of our note, we have that $e \in \mathrm{clo}(hDg)$ for some $h,g \in H$. Now (1) follows from the fact that $hDg \subseteq \dot{S}$.

Put $\dot{E} := \{e \in E(H) | H(e) \cap \dot{S} \neq \emptyset\}$.

To prove (2), let $\mu \in L(S)^+$.

Since $\dot{S} \subseteq \cup\{H(e) | e \in \dot{E}\}$ by an adaptation of the argument in the proof of (2.7), we can find a sequence $(F_n)_{n\in\mathbb{N}}$ of compact subsets of $\dot{S}$ and a countable subset C of $\dot{E}$ such that

(3) $\mu(S\backslash CA) = 0$,

where $A := \cup\{F_n | n \in \mathbb{N}\}$.

Now, let $F \in \{F_n | n \in \mathbb{N}\}$ and $e \in C$. Put $\mu' := \mu|_{eF}$. We shall show that there exists a countable subset $\mathcal{P}'$ of $\{H(e) | e \in E\}$ such that $\mu'(S\backslash \cup\mathcal{P}') = 0$; then, in view of (3), we may conclude that (2) holds. Let $\varepsilon > 0$. Since e belongs to E_δ, $\bar{e}*\mu' = \mu'$ and $\mu' \in L(S)$, there is a sequence $(a_n)_{n\in\mathbb{N}}$ in $\dot{S}$ such that $e \in \mathrm{clo}\{a_n | n \in \mathbb{N}\}$ and

$$||\mu'-\bar{a}_n*\mu'|| < \varepsilon.2^{-n} \qquad (n \in \mathbb{N}).$$

Now, we shall prove that

(4) $$eS \cap H(f) \subseteq \cup\{\mathrm{int}(a_n^{-1}H(f)) | n \in \mathbb{N}\} \qquad (f \in \dot{E}).$$

Let $f \in \dot{E}$ and take an $x \in eS \cap H(f)$. The facts that $ex = x$ and $e \in \mathrm{clo}\{a_n | n \in \mathbb{N}\}$ imply that $x \in \mathrm{clo}\{a_n x | n \in \mathbb{N}\}$. I(f) is a left ideal and hence, $a_n x \in I(f)$ $(n \in \mathbb{N})$. Now, the openness of H(f) in I(f) shows that $a_k x \in H(f)$ for some $k \in \mathbb{N}$. Let O be an open set in S such that $H(f) = O \cap I(f)$. Then we have that

$$x \in a_k^{-1}O \cap I(f) \subseteq a_k^{-1}O \cap (\dot{S}^{-1}a_k)^{-1}I(f) \subseteq a_k^{-1}O$$
$$\cap\ \mathrm{int}(a_k^{-1}I(f)) \subseteq \mathrm{int}(a_k^{-1}O \cap a_k^{-1}I(f)) \subseteq \mathrm{int}(a_k^{-1}H(f)).$$

This shows (4). Since eF is compact and $eF \subseteq \dot{S} \subseteq \cup\{H(f) | f \in \dot{E}\}$, (4) leads to the existence of a finite subcollection $\mathcal{P}_\varepsilon$ of $\{H(f) | f \in \dot{E}\}$ such that

$$eF \subseteq D^{-1}P_\varepsilon, \text{ where } P_\varepsilon := \cup\mathcal{P}_\varepsilon \text{ and } D := \{a_n | n \in \mathbb{N}\}.$$

Now note that

$$eF\backslash P_\varepsilon \subseteq D^{-1}P_\varepsilon\backslash P_\varepsilon, \text{ while}$$
$$\mu'(D^{-1}P_\varepsilon\backslash P_\varepsilon) \leq \sum_{n=1}^{\infty} \mu'(a_n^{-1}P_\varepsilon\backslash P_\varepsilon) \leq \varepsilon.$$

Put $P := \cup\{P_{1/n} | n \in \mathbb{N}\}$. Then P is the union of countably many elements of $\{H(f) | f \in \dot{E}\}$ and, furthermore, $eF\backslash P \subseteq D^{-1}P_{1/n}\backslash P_{1/n}$ $(n \in \mathbb{N})$, which implies that $\mu'(eF\backslash P) = 0$. □

The proof of the following lemma is an adaptation of the one of proposition in ch.IV, §5, no.9 in [3]. The subsequent theorem can be viewed as a generalization of this result in [3].

(3.3) LEMMA. Let S be a foundation stip. Let H be a compact G_δ-subsemigroup of S with $1 \in H$. Let $e \in E(H)$ for which $H(e) \cap \dot{S} \neq \emptyset$. There exists a family $\mathcal{A}$ of compact subsets of H(e) such that

(i) $A \cap B = \emptyset$ for all $A,B \in \mathcal{A}$, $A \neq B$,

(ii) $A \notin \mathcal{N}$ for all $A \in \mathcal{A}$ [where $\mathcal{N}$ as in §2],

(iii) for each $C \in \mathcal{K}$ with $C \subseteq H(e)$, the collection $\{A | A \in \mathcal{A}, A \cap C \neq \emptyset\}$ is countable and $C\backslash\cup\mathcal{A} \in \mathcal{N}$,

(iv) for each $A \in \mathcal{A}$ there is an $m \in L(S)^+$ such that $\mu|_A << m$ $(\mu \in L(S))$.

Proof. For each $F \in \mathcal{K}$, put $d(F) := \{x \in F | X \cap F \notin \mathcal{N}$ for all $X \in \mathcal{K}$ with $x \in int(X)\}$. Note that $d(F) \subseteq F$ and $F\backslash d(F) \in \mathcal{N}$.

By Zorn's lemma there exists a family of compact subsets of S that is maximal with respect to property (i), (ii) and (v): $A = d(A)$ for all $A \in \mathcal{A}$.

Let $C \in \mathcal{K}$ with $C \subseteq H(e)$. Take an open relatively compact neighbourhood U of C and put $\tilde{C} := \bar{U} \cap I(e)$. There is an idempotent $f \in E_\delta$ such that $eH \subseteq fSf$, whence $I(e) \subseteq SeH \subseteq Sf$ [cf.(2.6)]. By (2.8) there is an $m \in L(S)^+$ such that

$$\mu|_{\tilde{C}} << m \qquad \text{for all } \mu \in L(S).$$

If $F \in \mathcal{A}$ such that $F \cap C \neq \emptyset$ then $\emptyset \neq F \cap U \subseteq I(e) \cap \bar{U} = \tilde{C}$. Since $d(F) = F$ there is a $\mu \in L(S)$ with $\mu(F \cap U) \neq 0$. Consequently, $m(F) \neq 0$. Apparently $\{F \in \mathcal{A} | F \cap C \neq \emptyset\} \subseteq \{F \in \mathcal{A} | m(F) \neq 0\}$. The countability property follows easily now. In particular $C\backslash\cup\mathcal{A}$ is measurable. The negligibility follows from the maximality of $\mathcal{A}$. □

(3.4) THEOREM. *Let S be a foundation stip. There exists a family $\mathcal{H}$ of compact subsets of S such that*

(i) $A \cap B = \emptyset$ *for all* $A, B \in \mathcal{H}$, $A \neq B$;

(ii) $A \notin \mathcal{N}$ *for all* $A \in \mathcal{H}$;

(iii) *for each* $\mu \in L(S)^+$, $\mu(S \setminus \bigcup\{A \in \mathcal{H} \mid \mu(A) \neq 0\}) = 0$;

(iv) *for each* $A \in \mathcal{H}$ *there is an* $m \in L(S)^+$ *for which*

$$\mu|_A << m \qquad (\mu \in L(S)). \qquad \square$$

The theorem follows by an obvious combination of (3.2) and (3.3).

Viewing the above result, the reader will not find it too hard to prove the following theorem and proposition [cf. [2]].

(3.5) THEOREM. *Let S be a foundation stip. Let B be a subcollection of* $L(S)_{loc}$, *the space of Radon measures* μ *on S that are locally contained in* $L(S)$ [i.e. $\mu|_F \in L(S)$ for all compact subsets F of S]. *If* $(f_\rho)_{\rho\in B}$ *is a family of complex-valued functions on S for which for each* $\rho \in B$, f_ρ *is* ρ*-measurable, and* $f_\rho = f_\sigma$ *locally* ρ*-a.e. whenever,* $\rho, \sigma \in B$, $\rho << |\sigma|$, *then there is a complex-valued function f on S that is* ρ*-measurable for each* $\rho \in B$ *and* $f_\rho = f$ *locally* ρ*-a.e. for all* $\rho \in B$. $\square$

With the above B, let $L^\infty(S,B)$ be the quotient Banach lattice of $\mathcal{L}^\infty(S,B)$ with respect to $\mathcal{N}(S,B)$, where $\mathcal{L}^\infty(S,B)$ is the space of all bounded complex-valued B-measurable functions on S provided with the sup-norm and $\mathcal{N}(S,B) := \{f \in \mathcal{L}^\infty(S,B) \mid f = 0 \ \mu\text{-a.e. for all } \mu \in B\}$.

In [4], de Jonge shows that the property of B as described in (3.5) is equivalent to the Dedekind completeness of $L^\infty(S,B)$. One also encounters these properties in the field of statistics; the interested reader is referred to [2] and [4] for detailed information.

(3.6) PROPOSITION. Let S be a foundation stip. Then $L(S)^*$ is isometrically isomorphic to $L^\infty(S,L(S))$ [more precise: for each $h \in L(S)^*$ there is an $f \in \mathcal{L}^\infty(S,L(S))$ such that $h(\mu) = \mu(f)$ $(\mu \in L(S))$ and $||h|| = ||f||_\infty$]. □

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