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The Jacobi-Davidson Method for Eigenvalue and Generalized Eigenvalue Problems

by
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joint work with

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Program

- The method
- The name
- Related methods
- Deflation
- Generalized eigenproblems
- Examples

The problem

$$\boxed{\mathbf{A}\,\mathbf{x} = \lambda\,\mathbf{x}}$$

Suppose $\mathbf{u} \approx \mathbf{x}$.

$$\theta \equiv \frac{\mathbf{u}^*\mathbf{A}\mathbf{u}}{\mathbf{u}^*\mathbf{u}}, \qquad \mathbf{r} \equiv \mathbf{A}\,\mathbf{u} - \theta\,\mathbf{u}$$

Find $\mathbf{z} \perp \mathbf{u}$ such that $\mathbf{x} = \mathbf{u} + \mathbf{z}$.

$$(\mathbf{A} - \lambda\,\mathbf{I})\,\mathbf{z} = -\mathbf{r} + (\lambda - \theta)\,\mathbf{u}$$

or, equivalently,

$$\begin{cases} \mathbf{u}^*\mathbf{A}\mathbf{x} = \lambda\,\mathbf{u}^*\mathbf{x} \\ \left(\mathbf{I} - \frac{\mathbf{u}\,\mathbf{u}^*}{\mathbf{u}^*\mathbf{u}}\right)(\mathbf{A} - \lambda\,\mathbf{I})\left(\mathbf{I} - \frac{\mathbf{u}\,\mathbf{u}^*}{\mathbf{u}^*\mathbf{u}}\right)\mathbf{z} = -\mathbf{r} \end{cases}$$

The correction equation

Solve $\mathbf{z} \perp \mathbf{u}$ such that

$$\boxed{\left(\mathbf{I} - \frac{\mathbf{u}\,\mathbf{u}^*}{\mathbf{u}^*\mathbf{u}}\right)(\mathbf{A} - \theta\,\mathbf{I})\left(\mathbf{I} - \frac{\mathbf{u}\,\mathbf{u}^*}{\mathbf{u}^*\mathbf{u}}\right)\mathbf{z} = -\mathbf{r}}$$

Iteration by simple updating.

repeat until convergence:

$$\begin{cases} \bullet \ \text{solve the correction equation} \\ \bullet \ \mathbf{u} \leftarrow \mathbf{u} + \mathbf{z}, \qquad \theta \leftarrow \frac{\mathbf{u}^*\mathbf{A}\mathbf{u}}{\mathbf{u}^*\mathbf{u}} \end{cases}$$

Accelerated updating:

find the best approximation of $\mathbf{x}$ in

$$\text{span}(\mathbf{u}_1, \mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{k-1})$$

- Compute an orth.norm. basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$;
- Project the problem: $\boxed{\mathbf{V}^*\mathbf{A}\mathbf{V}\, y = \theta\, y}$

  where $\mathbf{V} \equiv [\mathbf{v}_1 \,|\, \ldots \,|\, \mathbf{v}_k]$;
- Solve the **projected problem**;
- Select a y with associated θ and compute

$$\mathbf{u} \equiv \mathbf{V}\, y.$$

Jacobi-Davidson

1. **Start:** Choose an initial $\mathbf{z}$.
   - Set $\mathbf{V} = [\,]$.
2. **Repeat:**
   - $\mathbf{V} \leftarrow$ ModGS($[\mathbf{V} \,|\, \mathbf{z}]$), $\mathbf{W} \leftarrow \mathbf{A}\mathbf{V}$
   - Compute $H \leftarrow \mathbf{V}^*\mathbf{W}$.
   - Compute the largest eigenpair (θ, y) of H (with $\|\, y \,\| = 1$).
   - Compute $\mathbf{u} \leftarrow \mathbf{V} y$,

     $$\mathbf{r} \leftarrow \mathbf{A}\,\mathbf{u} - \theta\,\mathbf{u}.$$
   - **Stop if** satisfied.
   - Solve (approximately)

     $$(\mathbf{I} - \mathbf{u}\,\mathbf{u}^*)\,(\mathbf{A} - \theta\,\mathbf{I})\,(\mathbf{I} - \mathbf{u}\,\mathbf{u}^*)\,\mathbf{z} = -\mathbf{r}.$$

The name

Jacobi-Davidson [S&V '94]:

- solve approximately $\mathbf{z} \perp \mathbf{u}$

  $(\mathbf{I} - \mathbf{u}\,\mathbf{u}^*)(\mathbf{A} - \theta\,\mathbf{I})(I - \mathbf{u}\,\mathbf{u}^*)\,\mathbf{z} = -\mathbf{r}$
- accelerated update

Davidson 1975:

- solve approximately $\mathbf{z}$

  $(\mathbf{A} - \theta\,\mathbf{I})\,\mathbf{z} = -\mathbf{r}$
- accelerated update

Jacobi 1846:

- solve approximately $\mathbf{z} \perp e_1$

  $(\mathbf{I} - e_1 e_1^*)(\mathbf{A} - \theta\,\mathbf{I})(\mathbf{I} - e_1 e_1^*)\,\mathbf{z} = -\mathbf{r}$
- $\mathbf{u} \leftarrow \mathbf{u} + \mathbf{z}$, $\theta \leftarrow e_1^*\mathbf{A}\mathbf{u}/e_1^*\mathbf{u}$,

Solve approximately

$$\left(\mathbf{I} - \frac{\mathbf{u}\,\mathbf{u}^*}{\mathbf{u}^*\mathbf{u}}\right)(\mathbf{A} - \theta\,\mathbf{I})\left(\mathbf{I} - \frac{\mathbf{u}\,\mathbf{u}^*}{\mathbf{u}^*\mathbf{u}}\right)\mathbf{z} = -\mathbf{r}.$$

Any preconditioned it. method can be used.

Preconditioning: $\mathrm{M} \approx \mathbf{A} - \theta\,\mathbf{I}$

Preconditioning equation:

$$\left(\mathbf{I} - \frac{\mathbf{u}\,\mathbf{u}^*}{\mathbf{u}^*\mathbf{u}}\right)\mathrm{M}\left(\mathbf{I} - \frac{\mathbf{u}\,\mathbf{u}^*}{\mathbf{u}^*\mathbf{u}}\right)\mathbf{z} = -\mathbf{r}$$

Solution:

$$\mathrm{M}\,\mathbf{z} = -\mathbf{r} + \alpha\,\mathbf{u} \quad \text{such that} \quad \mathbf{z} \perp \mathbf{u},$$

or

$$\mathbf{z} = -\mathrm{M}^{-1}\mathbf{r} + \alpha\,\mathrm{M}^{-1}\mathbf{u} \quad \& \quad \alpha = \frac{\mathbf{u}^*\mathrm{M}^{-1}\mathbf{z}}{\mathbf{u}^*\mathrm{M}^{-1}\mathbf{u}}.$$

Solve approximately

$$\left(\mathbf{I} - \frac{\mathbf{u}\,\mathbf{u}^*}{\mathbf{u}^*\mathbf{u}}\right)(\mathbf{A} - \theta\,\mathbf{I})\left(\mathbf{I} - \frac{\mathbf{u}\,\mathbf{u}^*}{\mathbf{u}^*\mathbf{u}}\right)\mathbf{z} = -\mathbf{r}$$

$$\neq$$

Solve approximately

$$\left.\begin{array}{l}(\mathbf{A} - \theta\,\mathbf{I})\,\mathbf{z}_r = \mathbf{r}\\ (\mathbf{A} - \theta\,\mathbf{I})\,\mathbf{z}_u = \mathbf{u}\end{array}\right\} \;\&\; \begin{array}{l}\mathbf{z} = -\mathbf{z}_r + \alpha\,\mathbf{z}_u\\ \text{s.t.}\quad \mathbf{z} \perp \mathbf{u}\end{array}$$

except if

"solve appr." $=$ 1 step with fixed precond. $\mathbf{M}$.

Example. With static polynomial method p:

$$p\left(\left(\mathbf{I} - \frac{\mathbf{u}\,\mathbf{u}^*}{\mathbf{u}^*\mathbf{u}}\right)(\mathbf{A} - \theta\,\mathbf{I})\left(\mathbf{I} - \frac{\mathbf{u}\,\mathbf{u}^*}{\mathbf{u}^*\mathbf{u}}\right)\right) \neq$$

$$\left(\mathbf{I} - \frac{\mathbf{u}\,\mathbf{u}^*}{\mathbf{u}^*\mathbf{u}}\right)p\,(\mathbf{A} - \theta\,\mathbf{I})\left(\mathbf{I} - \frac{\mathbf{u}\,\mathbf{u}^*}{\mathbf{u}^*\mathbf{u}}\right)$$

Related methods

with $\mathbf{M}$ a fixed preconditioner

Expand the search space by the sol. $\mathbf{z}$ of

- $\mathbf{M}\,\mathbf{z} = -\mathbf{r} + \alpha\,\mathbf{u}$ & $\alpha = \mathbf{u}^*\mathbf{M}^{-1}\mathbf{z}/\mathbf{u}^*\mathbf{M}^{-1}\mathbf{u}$

  Olsen's method ['90 for symm. $\mathbf{A}$]

- $\mathbf{M}\,\mathbf{z} = -\mathbf{r}$

  Generalized Davidson [Morgan '92]

- $\mathbf{M}\,\mathbf{z} = \mathbf{u}$

  acc. inexact Shift-and-Invert

$$\mathbf{z} = -\mathbf{M}^{-1}\mathbf{r} + \alpha\,\mathbf{M}^{-1}\mathbf{u}$$

Extreme choices

- $\mathbf{M} = \mathbf{I}$:

  $\mathbf{z} = -\mathbf{r} + \alpha\,\mathbf{u} = -\mathbf{A}\,\mathbf{u} + (\theta + \alpha)\mathbf{u} \sim \mathbf{A}\,\mathbf{u}$.

  J-D $\sim$ Gen. Dav. $\sim$ Arnoldi,

  inexact Shift-and-Invert stagnates.

- $\mathbf{M} = \mathbf{A} - \theta\,\mathbf{I}$:

  $\mathbf{z} = -\mathbf{u} + \alpha\,(\mathbf{A} - \theta\,\mathbf{I})^{-1}\mathbf{u} \sim (\mathbf{A} - \theta\,\mathbf{I})^{-1}\mathbf{u}$.

  J-D $\sim$ acc. Shift-and-Invert with opt. shift,

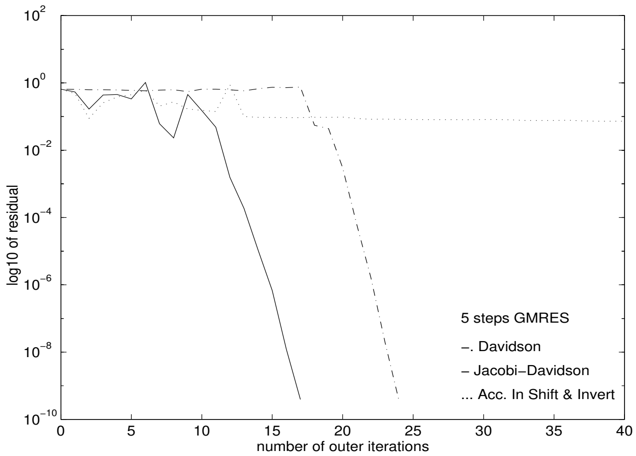
  Generalized Davidson stagnates.

$\lambda \approx 0$, $\mathbf{A} =$ SHERMAN4 – .5$\mathbf{I}$, $n = 1104$

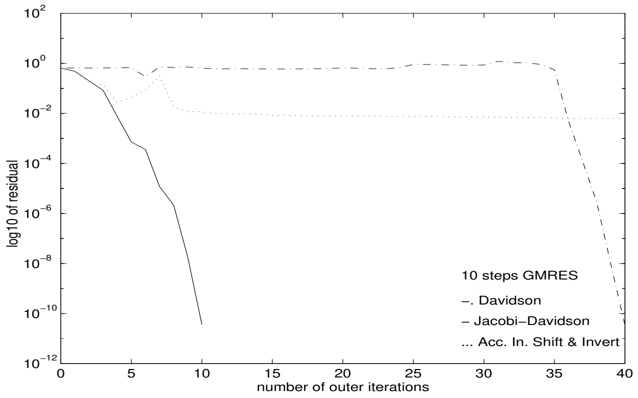
CorEq:

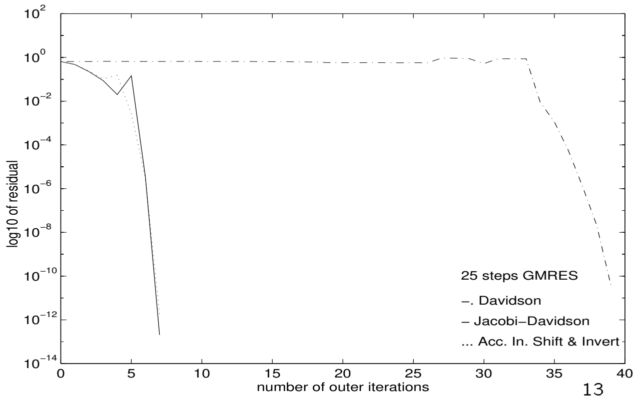
- $(\mathbf{I} - \mathbf{u}\,\mathbf{u}^*)(\mathbf{A} - \theta\,\mathbf{I})(\mathbf{I} - \mathbf{u}\,\mathbf{u}^*)\,\mathbf{z} = -\mathbf{r}$ [J-D]
- $(\mathbf{A} - \theta\,\mathbf{I})\,\mathbf{z} = -\mathbf{r}$ [Davidson]
- $(\mathbf{A} - \theta\,\mathbf{I})\,\mathbf{z} = \mathbf{u}$ [Shift-and-Invert]

m steps precond. GMRES with $\mathbf{M} =$ ILU(2).



$m = 5$





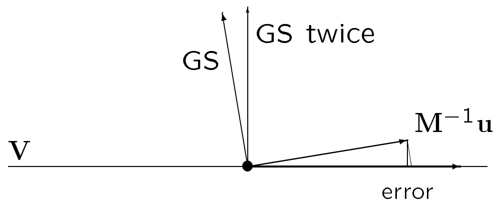
Stability

If $\quad \angle(\mathbf{M}^{-1}\mathbf{u}, \mathbf{V}) \approx 0$

then
- ModGS$_{\mathbf{V}}(\mathbf{M}^{-1}\mathbf{u})$ to be repeated,
- ModGS$_{\mathbf{V}}(-\mathbf{M}^{-1}\mathbf{r} + \alpha\, \mathbf{M}^{-1}\mathbf{u})$ fine.

If $\quad \mathbf{V} \ni \mathbf{y} \approx$ orth. proj$_{\mathbf{V}}(\mathbf{M}^{-1}\mathbf{u}) \quad \Rightarrow$

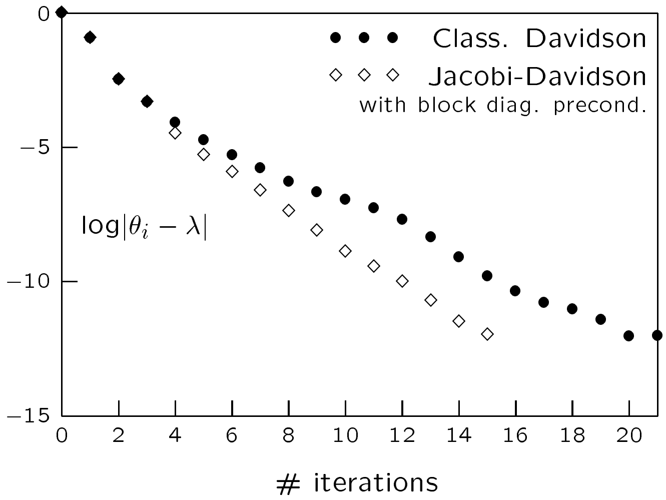
ModGS$_{\mathbf{V}}(\mathbf{M}^{-1}\mathbf{u} - \mathbf{y}) \sim$ twice ModGS$_{\mathbf{V}}(\mathbf{M}^{-1}\mathbf{u})$



$\mathbf{M}^{-1}\mathbf{r} \approx \mathbf{M}^{-1}\mathbf{u} \approx \mathbf{u} \approx \mathbf{y}$

[van Dam, van Lenthe '94]

Example [Quantum Chemistry]



$\mathbf{A}$ symmetric, $n \approx 50{,}000$

block diag. precond. with max. block size 66

J-D as a Newton scheme

$$\mathbf{F}(\mathbf{u},\theta) \equiv \mathbf{A}\,\mathbf{u} - \theta\,\mathbf{u}, \qquad \|\,\mathbf{u}\,\| = 1,\ \theta \in \mathbb{C}$$

$$\mathbf{F} : \{\mathbf{u}, \|\,\mathbf{u}\,\| = 1\} \times \mathbb{C} \to \mathbb{C}^n$$

$$\boxed{\mathbf{A}\,\mathbf{x} = \lambda\,\mathbf{x} \quad \& \quad \|\,\mathbf{x}\,\| = 1} \;\Leftrightarrow\; \boxed{\mathbf{F}(\mathbf{x},\lambda) = 0}$$

Newton:
- solve $\quad \mathbf{z} \perp \mathbf{u}$
  $$\frac{\partial\,\mathbf{F}}{\partial\,(\mathbf{u},\theta)}(\mathbf{z},\epsilon) = -\mathbf{r} \equiv -\mathbf{F}(\mathbf{u},\theta)$$
- $\mathbf{u} \leftarrow \mathbf{u} + \mathbf{z}, \qquad \theta \leftarrow \theta + \epsilon$

The sol. $\mathbf{z}$ of the Newton correction satisfies

$$\mathbf{z} \perp \mathbf{u}, \quad (\mathbf{I} - \mathbf{u}\,\mathbf{u}^*)\,(\mathbf{A} - \theta\,\mathbf{I})\,\mathbf{z} = -\mathbf{r}$$

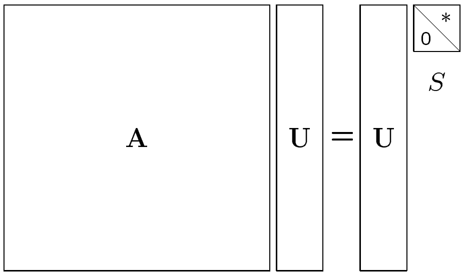
J-D = accelerated (inexact) Newton

$\Rightarrow$ *($\approx$) quadratic convergence*

QR [S. & Fokkema]

Compute m eigenvalues close to some $\tau \in \mathbb{C}$.

- (Explicit) deflation
- Schur vectors
- Assembly.



$(\mathbf{I} - \mathbf{U}\,\mathbf{U}^*)(\mathbf{A} - \theta\,\mathbf{I})(\mathbf{I} - \mathbf{U}\,\mathbf{U}^*)$

$\mathbf{A}\,\mathbf{U} = \mathbf{U}\,S$ with S upper Δ
diag$(S) = (\lambda_1, \ldots, \lambda_m)$
$|\lambda_i - \tau|$ increases

Restart: retain ℓ Ritz vectors with Ritz values closest to τ.

1. **Start:** Choose an initial $\mathbf{z}$.
   - Set $\Lambda = [\,]$, $\mathbf{U} = [\,]$, $\mathbf{V} = [\,]$.
2. **Repeat:**
   a. $\mathbf{V} \leftarrow$ ModGS$([\mathbf{V}\,|\,\mathbf{z}])$, $\mathbf{W} \leftarrow \mathbf{A}\mathbf{V}$

   b. Compute $H \leftarrow \mathbf{V}^*\mathbf{W}$.

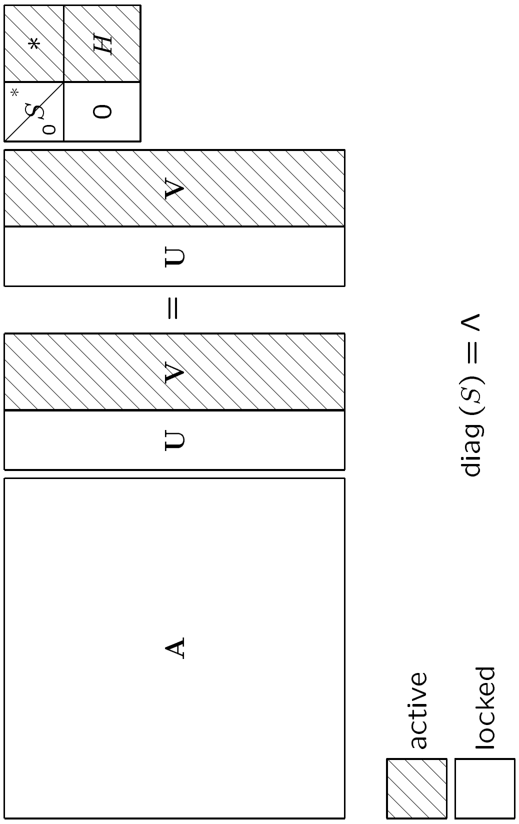
   c. Compute $H = QRQ^*$
   with $Q^*Q = I$, R upper Δ,
   $|R_{11} - \tau| \leq |R_{22} - \tau| \leq \ldots$.

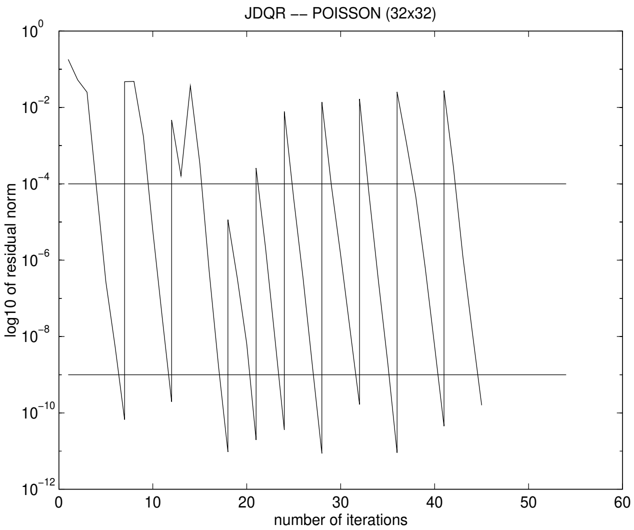
   d. $\theta \leftarrow R_{11}$, $y \leftarrow Qe_1$, $\mathbf{u} \leftarrow \mathbf{V}y$,
   $\mathbf{r} \leftarrow$ ModGS$_{\mathbf{U}}(\mathbf{A}\,\mathbf{u} - \theta\,\mathbf{u})$.

   e. **If** $\|\,\mathbf{r}\,\| \leq$ *tol*
   - $\Lambda \leftarrow [\Lambda\,|\,\theta]$, $\mathbf{U} \leftarrow [\mathbf{U}\,|\,\mathbf{u}]$.
   - $J = [\,e_2\,|\,\ldots\,|\,e_{\dim(\mathbf{V})}\,]$,
     $\mathbf{V} \leftarrow \mathbf{V}(QJ)$, $Q \leftarrow I$, $R \leftarrow J^*RJ$.
   - **Goto** 2.d.

   f. Solve (app.) $\mathbf{z} \perp \widetilde{\mathbf{U}} \equiv [\,\mathbf{U}\,|\,\mathbf{u}\,]$

   $$(\mathbf{I} - \widetilde{\mathbf{U}}\,\widetilde{\mathbf{U}}^*)\,(\mathbf{A} - \theta\,\mathbf{I})\,(\mathbf{I} - \widetilde{\mathbf{U}}\,\widetilde{\mathbf{U}}^*)\,\mathbf{z} = -\mathbf{r}.$$





2D Poisson Problem (32×32)

Target = 0d0

Max Dim = 15, Min Dim = 10

GMRES(10) plus ILU(0)

eps = 1d-9

```
lambda( 1): ( 9.0562E-03, 0.0000E+00 )
lambda( 2): ( 2.2599E-02, 0.0000E+00 )
lambda( 3): ( 2.2599E-02, 0.0000E+00 )
lambda( 4): ( 3.6143E-02, 0.0000E+00 )
lambda( 5): ( 4.5035E-02, 0.0000E+00 )
lambda( 6): ( 4.5035E-02, 0.0000E+00 )
lambda( 8): ( 5.8578E-02, 0.0000E+00 )
lambda( 7): ( 5.8578E-02, 0.0000E+00 )
lambda( 9): ( 7.6160E-02, 0.0000E+00 )
lambda(10): ( 7.6160E-02, 0.0000E+00 )
```

2D Poisson Problem (32×32)

Target = 0d0

Max Dim = 15, Min Dim = 10

GMRES(10) with ILU(0)

eps = 1d-9

Generalized eigenproblems

$$\mathbf{A}\,\mathbf{x} = \lambda\,\mathbf{B}\,\mathbf{x}$$

$\mathbf{V}$ <u>search space</u>, $\mathbf{W}$ <u>test space</u>

- Projected problem:

$$\mathbf{W}^*\,\mathbf{A}\,\mathbf{V}\,y - \theta\,\mathbf{W}^*\,\mathbf{B}\,\mathbf{V}\,y = 0.$$

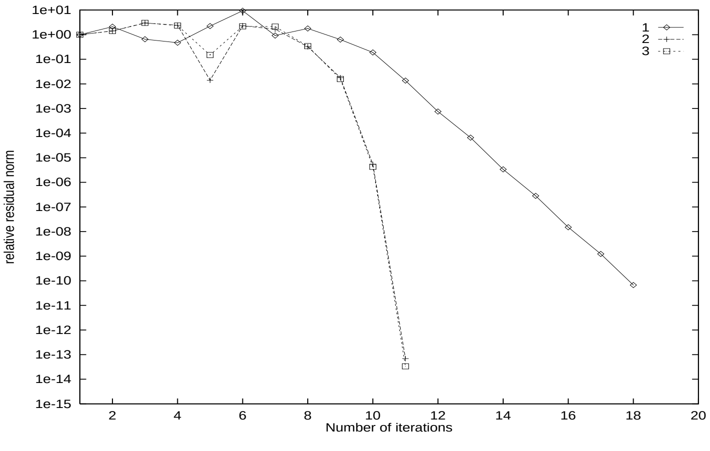
$$\mathbf{u} = \mathbf{V}\,y, \qquad \mathbf{r} \equiv \mathbf{A}\,\mathbf{u} - \theta\,\mathbf{B}\,\mathbf{u}$$

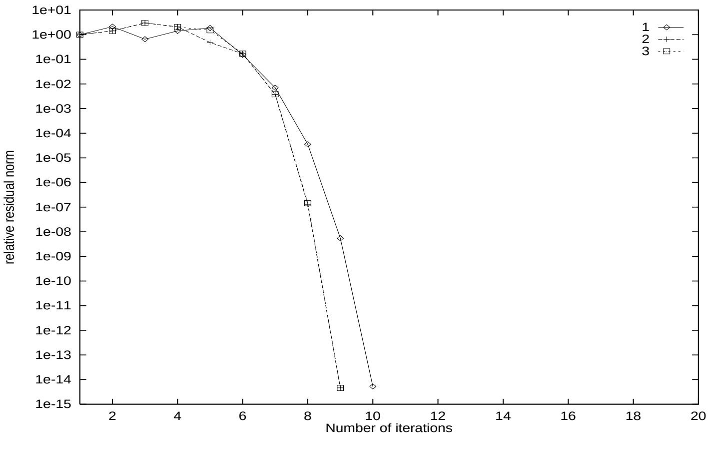
- The correction equation:

$$\left(\mathbf{I} - \frac{\widetilde{\mathbf{w}}\,\mathbf{w}^*}{\mathbf{w}^*\widetilde{\mathbf{w}}}\right)(\mathbf{A} - \theta\,\mathbf{B})\left(\mathbf{I} - \frac{\mathbf{u}\,\widetilde{\mathbf{u}}^*}{\widetilde{\mathbf{u}}^*\mathbf{u}}\right)\mathbf{z} = -\mathbf{r}$$

<u>**Th.**</u> $\mathbf{w} \in \mathbf{W}$ & $\widetilde{\mathbf{w}} = \mathbf{B}\,\mathbf{u}$

& exact solve $\Rightarrow$ *quadratic convergence.*

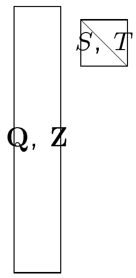




Partial generalized Schur form

$\mathbf{A}\,\mathbf{Q} = \mathbf{Z}\,S$ $\qquad$ $\mathbf{Q}$, $\mathbf{Z}$ orth.,

$\mathbf{B}\,\mathbf{Q} = \mathbf{Z}\,T$ $\qquad$ S, T upper Δ.



$$\mathbf{A}\,\mathbf{Q} = \mathbf{B}\,\mathbf{Q}\,\underbrace{T^{-1}S}_{\text{upper }\Delta} \qquad \text{span}\,(\mathbf{Z}) = \text{span}\,(\mathbf{B}\,\mathbf{Q})$$

$\mathbf{Z}^*\mathbf{A}\,\mathbf{Q} = S$ $\qquad$ $\mathbf{V} \ni \mathbf{u} \to \mathbf{q}$

$\mathbf{Z}^*\mathbf{B}\,\mathbf{Q} = T$ $\qquad$ $\mathbf{W} \ni \mathbf{w} \to \mathbf{z}$

Natural choices in connection with QZ:

span ($\mathbf{W}$) = span ($\mathbf{B}\,\mathbf{V}$), $\quad \mathbf{w} = \mathbf{B}\,\mathbf{u}$, $\quad \widetilde{\mathbf{u}} = \mathbf{u}$.

Choosing spaces & projections

$$\mathbf{u} = \mathbf{V}\,y, \quad \mathbf{w} = \mathbf{B}\,\mathbf{u}, \quad \mathbf{r} = \mathbf{A}\,\mathbf{u} - \theta\,\mathbf{B}\,\mathbf{u}$$

- The correction equation:

$$\left(\mathbf{I} - \frac{\mathbf{w}\,\mathbf{w}^*}{\mathbf{w}^*\mathbf{w}}\right)(\mathbf{A} - \theta\,\mathbf{B})\left(\mathbf{I} - \frac{\mathbf{u}\,\mathbf{u}^*}{\mathbf{u}^*\mathbf{u}}\right)\mathbf{z} = -\mathbf{r}.$$

- The search space and test space:

$\mathbf{V} = \text{modGS}\,([\,\mathbf{V}\,|\,\mathbf{z}\,]), \quad \mathbf{W} = \text{modGS}\,([\,\mathbf{W}\,|\,\mathbf{B}\,\mathbf{z}\,])$

Preconditioning

$$\mathbf{M} \approx \mathbf{A} - \theta\,\mathbf{B}$$

$$\left(\mathbf{I} - \frac{\mathbf{w}\,\mathbf{w}^*}{\mathbf{w}^*\mathbf{w}}\right)\mathbf{M}\left(\mathbf{I} - \frac{\mathbf{u}\,\mathbf{u}^*}{\mathbf{u}^*\mathbf{u}}\right)\mathbf{z} = -\mathbf{r}.$$

Explicit left preconditioning:

- The correction equation:

$$\left(\mathbf{I} - \frac{\mathbf{v}\,\mathbf{u}^*}{\mathbf{u}^*\mathbf{v}}\right)\mathbf{M}^{-1}(\mathbf{A} - \theta\,\mathbf{B})\left(\mathbf{I} - \frac{\mathbf{v}\,\mathbf{u}^*}{\mathbf{u}^*\mathbf{v}}\right)\mathbf{z} = -\mathbf{r}'.$$

where

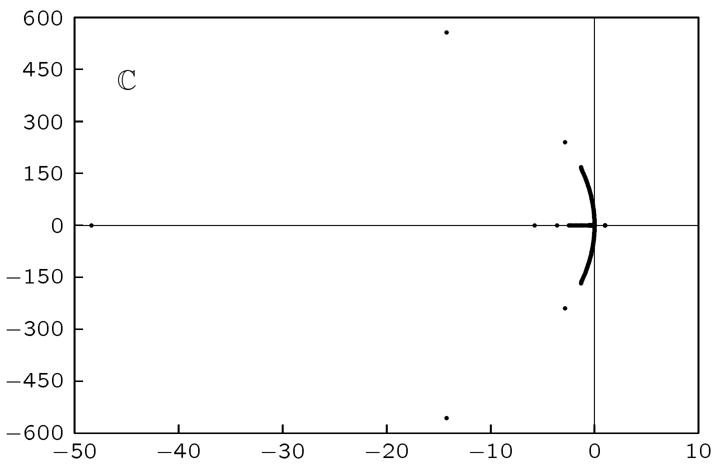
$$\mathbf{M}\,\mathbf{v} = \mathbf{w} = \mathbf{B}\,\mathbf{u} \quad \& \quad \mathbf{r}' = \left(\mathbf{I} - \frac{\mathbf{v}\,\mathbf{u}^*}{\mathbf{u}^*\mathbf{v}}\right)\mathbf{r}.$$

Properties

- JDQZ algorithm exists.
- Finds also multiple eigenvalues.
- Easy selection of wanted eigenvalues.
- No matrix inversion involved.
- Ingredients:

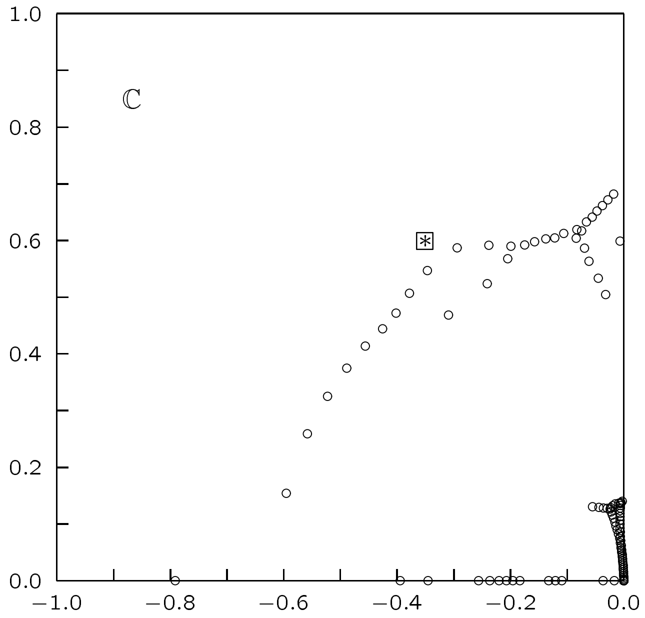
  vector updates, Dot-products, MVs,
  preconditioners
  small problems.

Spectrum MHD test problem

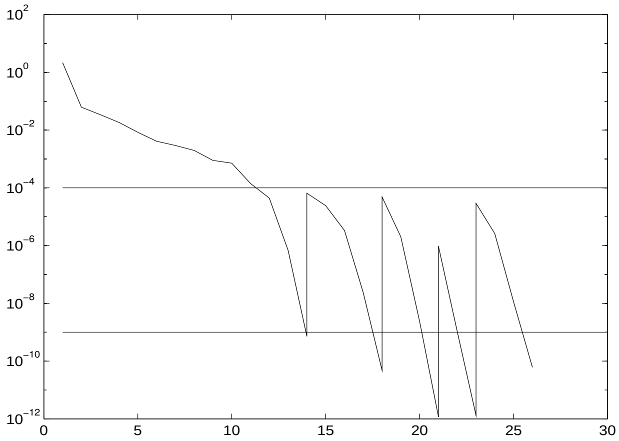


Entire spectrum MHD test problem ($n = 416$).
Correct eigenvalues (.) are shown.

Alfvén spectrum



MHD test problem ($n = 416$).
Correct eigenvalues (∘), target (⊛).



MHD Problem (416)

Target = (-0.3, 0.5)

Max Dim = 15, Min Dim = 10

GMRES(10) with ILU(2)

eps = 1d-9

---

lambda(1): (-3.0929E-01, 4.6875E-01)
lambda(2): (-3.4656E-01, 5.4689E-01)
lambda(3): (-2.4103E-01, 5.2381E-01)
lambda(4): (-3.7810E-01, 5.0716E-01)
lambda(5): (-2.9401E-01, 5.8715E-01)

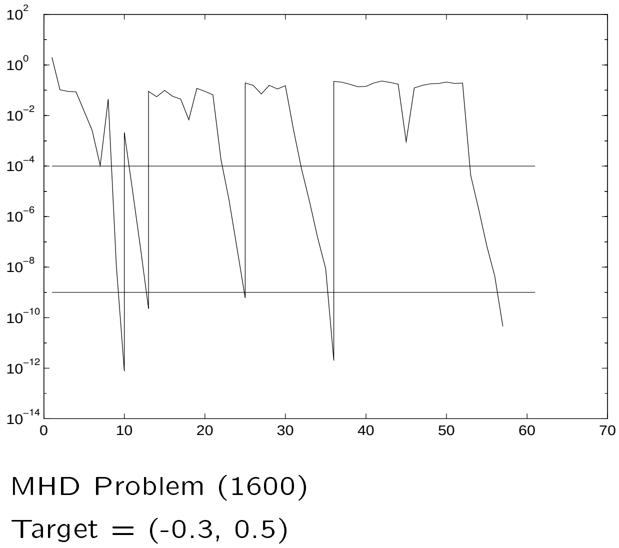
MHD Problem (416)

Target = (-0.3, 0.5)

Max Dim = 15, Min Dim = 10

GMRES(10) with ILU(2)

eps = 1d-9



MHD Problem (1600)

Target = (-0.3, 0.5)

Max Dim = 15, Min Dim = 10

GMRES(10) with ILU(2)

eps = 1d-9

```
lambda( 1): ( -3.3746E-01, 4.8958E-01 )
lambda( 2): ( -3.5776E-01, 4.6050E-01 )
lambda( 3): ( -2.3839E-01, 4.4657E-01 )
lambda( 4): ( -3.8189E-01, 5.0369E-01 )
lambda( 5): ( -2.2157E-01, 5.3364E-01 )
```

MHD Problem (1600)

Target = (-0.3, 0.5)

Max Dim = 15,   Min Dim = 10

GMRES(10) with ILU(2)

eps = 1d-9

Generalized eigenproblems

$$\mathbf{A}\,\mathbf{x} = \lambda\,\mathbf{x}.$$

[van Dam, van Lenthe **Q. Chem.**]

$$\mathbf{A}\,\mathbf{x} = \lambda\,\mathbf{B}\,\mathbf{x}.$$

[Booten, Fokkema **MHD**]

$$\mathbf{P}(\lambda)\mathbf{x} \equiv \mathbf{A}_0\,\mathbf{x} + \lambda\,\mathbf{A}_1\,\mathbf{x} + \ldots + \lambda^\ell\,\mathbf{A}_l\,\mathbf{x} = 0$$

[van Gijzen, Raeven **acoustics, con. mech.**]

$$\mathbf{W}^*\mathbf{P}(\theta)\mathbf{V}y = 0,\ \mathbf{u} = \mathbf{V}y,\ \mathbf{w} \equiv \mathbf{P}'(\theta)\mathbf{u},\ \mathbf{r} \equiv \mathbf{P}(\theta)\mathbf{u}$$

- The correction equation:
$$\left(\mathbf{I} - \frac{\mathbf{w}\,\mathbf{u}^*}{\mathbf{u}^*\mathbf{w}}\right)\mathbf{P}(\theta)\left(\mathbf{I} - \frac{\mathbf{u}\,\mathbf{u}^*}{\mathbf{u}^*\mathbf{u}}\right)\mathbf{z} = -\mathbf{r}$$
- The spaces: $\mathbf{V} = \text{modGS}\,([\,\mathbf{V}\,|\,\mathbf{z}\,]),\ \mathbf{W} = \mathbf{V}$

J-D

- is an accelerated inexact Newton scheme
  — quadratic convergence —
- relies on orthogonal projections
- allows any precond. it. method
  to solve the linear (Jacobian) system
- robust

J-D is <u>not</u>

- a Krylov subspace method,
- Davidson with Jacobi-preconditioning,
- a method that produces eigenvectors for
  a preconditioned eigenproblem

Current research

- Fine tuning inner-outer loop and problem.
- Assembly strategies.
- Stopping criterion.
- Preconditioner.
- Tracking strategies.