On the formal completion of the Chow group $CH^2(X)$ for a smooth projective surface in characteristic 0

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ABSTRACT

Bloch [1] defined the formal completion of the group of 0-cycles modulo rational equivalence on a surface X and studied it in case X is defined over an algebraic number field. In this paper we investigate in detail the situation for ground fields which are extensions of $\mathbb{Q}$ of finite transcendence degree. We look in particular at the kernel of the formal analogue of the Abel-Jacobi mapping from Chow group to Albanese variety. It turns out that the influence of the derivations of the ground field k can be described completely in terms of the Gauss-Manin connection on $H^2_{DR}(X/k)$.

INTRODUCTION

Let X be a smooth projective surface over a field k of characteristic 0 and of finite transcendence degree over $\mathbb{Q}$. Given an imbedding of k in $\mathbb{C}$ one can form the complex surface $X_{\mathbb{C}} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \mathbb{C}$. The Abel-Jacobi map gives a surjective homomorphism

(0.1) $\quad CH^2(X_{\mathbb{C}})_0 \twoheadrightarrow \operatorname{Alb}(X_{\mathbb{C}})$

from the group of 0-cycles of degree 0 on $X_{\mathbb{C}}$ modulo rational equivalence onto the Albanese variety of $X_{\mathbb{C}}$. Mumford showed in [10] that, in contrast to what happens for 0-cycles on curves, this map cannot be an isomorphism, if the geometric genus p_g of X is not zero. Bloch's conjecture is that conversely, the Abel-Jacobi map (0.1) is an isomorphism if $p_g = 0$ [1]. This has been verified in some cases, but the general conjecture is still open. For surfaces with $p_g > 0$ the kernel of (0.1) is a true mystery, without even a guess as to what its structure may be.

In the hope to get more insight into the structure of the group $CH^2(X)$ of 0-cycles modulo rational equivalence on X Bloch proposed to study its "formal completion at the origin" [1]. The definition of this completion is motivated by Bloch's formula

$$CH^2(X) = H^2(X, \mathscr{K}_{2,X}),$$

where for any scheme Y $\mathscr{K}_{2,Y}$ is the sheaf for the Zariski topology on Y associated to the pre-sheaf whose group of sections over an open U is the group $K_2(\Gamma(U, \mathscr{O}_Y))$, which Milnor's functor K_2 assigns to the ring $\Gamma(U, \mathscr{O}_Y)$ [11]. Let $\mathscr{C}$ be the category of artinian local k-algebras with residue field k. The objects of $\mathscr{C}$ will be denoted as $(A, \mathfrak{m})$ where A is the local ring and $\mathfrak{m}$ its maximal ideal. For $(A, \mathfrak{m}) \in \text{obj } \mathscr{C}$ write $X_A = X \times_{\text{Spec } k} \text{Spec } A$. Now we are ready to define the "formal completion at the origin of $CH^2(X)$". It is the covariant functor

$$\widehat{CH}^2_X : \mathscr{C} \to \text{abelian groups}$$

given by

$$\widehat{CH}^2_X(A, \mathfrak{m}) = \ker\, [H^2(X_A, \mathscr{K}_{2,X_A}) \xrightarrow{\mathfrak{m}\to 0} H^2(X, \mathscr{K}_{2,X})].$$

In [1] Bloch studied this functor (denoting it as F^2_0) in case k is algebraic over $\mathbb{Q}$ (see also [2]). The purpose of the present paper is to show what one finds without this assumption.

We look first for an analogue of the Abel-Jacobi map (0.1). The tangent space at the origin of the Albanese variety is $H^2(X, \Omega^1_{X/k})$. This means that the formal completion at the origin of the Albanese variety, viewed as a covariant functor

$$\widehat{\text{Alb}}_X : \mathscr{C} \to \text{abelian groups},$$

is given by

$$\widehat{\text{Alb}}_X(A, \mathfrak{m}) = H^2(X, \Omega^1_{X/k}) \otimes_k \mathfrak{m}$$

We shall prove in § 2:

THEOREM 1. Let X and k be as above. Then

(i) There is a surjective natural transformation

(0.2) $\quad \widehat{CH}^2_X \to \widehat{\text{Alb}}_X.$

(ii) Every homomorphism from $\widehat{CH}^2_X$ into a smooth commutative formal group (which we view as a covariant functor from $\mathscr{C}$ to 'abelian groups') factors via (0.2) through $\widehat{\text{Alb}}_X$.

(iii) The map (0.2) is an isomorphism if and only if $p_g = 0$. □

Parts (i) and (iii) are in the number field case also proved in [1]. It was actually the discovery of (iii) which lead Bloch to his conjecture.

Next let us concentrate on ker $(\widehat{CH}^2_X \to \widehat{\text{Alb}}_X)$. This is of course also a covariant functor $\mathscr{C} \to$ (abelian groups). On the last line of Bloch's paper [1] one can read that in case k is algebraic over $\mathbb{Q}$ this functor is naturally isomorphic to the one which assigns to an object (A, m) of $\mathscr{C}$ the group

$$H^2(X, \mathcal{O}_X) \otimes_k \Omega^1_{A/k}/dA.$$

If k is not algebraic over $\mathbb{Q}$, the description of ker $(\widehat{CH}^2_X \to \widehat{\text{Alb}}_X)$ and the hypotheses which one has to assume, are influenced by the derivations of k over $\mathbb{Q}$. The Gauss-Manin connection determines an action of Der$(k/\mathbb{Q})$ on $H^2_{DR}(X/k)$ (see § 3 for a summary of the construction and main properties of the Gauss-Manin connection). It turns out that the description of ker $(\widehat{CH}^2_X \to \widehat{\text{Alb}}_X)$ and the necessary hypotheses can be formulated in terms of the Gauss-Manin connection. Generalities are given in § 3. As a special case we mention here

THEOREM 2. Let X be a smooth projective surface over a field k which has finite transcendence degree over $\mathbb{Q}$. Then the following statements are equivalent:

(i) The map $H^1(X, \Omega^1_{X/k}) \to H^2(X, \mathcal{O}_X) \otimes_k \Omega^1_{k/\mathbb{Q}}$ which is induced by the Gauss-Manin connection (and which is equal to cup-product with the Kodaira-Spencer mapping) is surjective.

(ii) The functor ker $(\widehat{CH}^2_X \to \widehat{\text{Alb}}_X)$ is naturally isomorphic to the one which assigns to an object (A, m) of $\mathscr{C}$ the group

$$H^2(X, \mathcal{O}_X) \otimes_k \Omega^1_{A/k}/dA. \quad \square$$

This theorem is proved in § 3 in the form of theorem 2bis. Giving the map $H^1(X, \Omega^1_{X/k}) \to H^2(X, \mathcal{O}_X) \otimes_k \Omega^1_{k/\mathbb{Q}}$ is equivalent to giving a k-linear map

$$\text{(0.3)} \qquad \text{Der}\,(k/\mathbb{Q}) \to \text{Hom}_k(H^0(X, \Omega^2_{X/k}), H^1(X, \Omega^1_{X/k}))$$

(cf. (3.9)). In case X has genus 1, surjectivity of the former map is equivalent to injectivity of the latter. Thus we get the following corollary.

COROLLARY. Let X and k be as in theorem 2. Assume in addition that X has genus $p_g = 1$. Then the following statements are equivalent

(i) The map Der $(k/\mathbb{Q}) \to \text{Hom}_k(H^0(X, \Omega^2_{X/k}), H^1(X, \Omega^1_{X/k}))$ which is induced by the Gauss-Manin connection, is injective.

(ii) The functor ker $(\widehat{CH}^2_X \to \widehat{\text{Alb}}_X)$ is naturally isomorphic to the one which assigns to (A, m) the group $\Omega^1_{A/k}/dA$. $\square$

The map (0.3) which appears here, is well-known in deformation theory. It has the following interpretation.

Choose a regular $\mathbb{Q}$-algebra of finite type, R, and a smooth projective map $\pi : X' \to \text{Spec}\, R = S$ whose generic fibre is $X \to \text{Spec}\, k$ (i.e. the field of fractions of R is k and $X = X' \times_S \text{Spec}\, k$). By base change relative to the inclusion $\mathbb{Q} \subset \mathbb{C}$ we obtain from $X' \to S \to \text{Spec}\, \mathbb{Q}$ a smooth family of complex surfaces $X'_{\mathbb{C}} \xrightarrow[\pi_{\mathbb{C}}]{} S_{\mathbb{C}} \to \text{Spec}\, \mathbb{C}$. An imbedding $\sigma : k \hookrightarrow \mathbb{C}$ determines a point of $S_{\mathbb{C}}$, which we also denote by σ. The fibre of $\pi_{\mathbb{C}}$ over this point is precisely the surface $X_\sigma = X \times_{\operatorname{Spec} k} \operatorname{Spec} \mathbb{C}$ which is obtained from X by base change relative to $\sigma : k \hookrightarrow \mathbb{C}$. The Hodge and De Rham cohomology of $X'_{\mathbb{C}}/S_{\mathbb{C}}$ are sheaves of $\mathscr{O}_{S_{\mathbb{C}}}$-modules on $S_{\mathbb{C}}$. The ones relevant for our purpose are $\mathscr{H}^2_{DR}(X'_{\mathbb{C}}/S_{\mathbb{C}})$, $\mathscr{H}^{2,0}_{\text{Hodge}}$, $\mathscr{H}^{1,1}_{\text{Hodge}}$. The stalks of these sheaves at the point σ are, respectively,

$$(\mathscr{H}^2_{DR}(X'_{\mathbb{C}}/S_{\mathbb{C}}))_\sigma = H^2_{DR}(X_\sigma/\mathbb{C}) \simeq H^2_{DR}(X/k) \otimes_k \mathbb{C}$$

$$(\mathscr{H}^{2,0}_{\text{Hodge}})_\sigma = H^0(X_\sigma, \Omega^2_{X_\sigma/\mathbb{C}}) \simeq H^0(X, \Omega^2_{X/k}) \otimes_k \mathbb{C}$$

$$(\mathscr{H}^{1,1}_{\text{Hodge}})_\sigma = H^1(X_\sigma, \Omega^1_{X_\sigma/\mathbb{C}}) \simeq H^1(X, \Omega^1_{X/k}) \otimes_k \mathbb{C}$$

where in the tensor products $\mathbb{C}$ is considered as a k-algebra via σ. The isomorphisms are canonical isomorphisms (see Deligne's paper [4] for a discussion of the compatibilities between various cohomology theories). The Gauss-Manin connection for $X'_{\mathbb{C}} \to S_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ induces an $\mathscr{O}_{S_{\mathbb{C}}}$-linear map between sheaves of $\mathscr{O}_{S_{\mathbb{C}}}$-modules

$$\Omega^{1\,\vee}_{S_{\mathbb{C}}/\mathbb{C}} = \operatorname{Der}\,(S_{\mathbb{C}}/\mathbb{C}) \to \mathscr{H}om_{\mathscr{O}_{S_{\mathbb{C}}}}(\mathscr{H}^{2,0}_{\text{Hodge}}, \mathscr{H}^{1,1}_{\text{Hodge}})$$

In the stalks at the point σ this map is

(0.4) $\quad T_{S_{\mathbb{C}},\sigma} \to \operatorname{Hom}_{\mathbb{C}}(H^0(X_\sigma, \Omega^2_{X_\sigma/\mathbb{C}}), H^1(X_\sigma, \Omega^1_{X_\sigma/\mathbb{C}}))$

where $T_{S_{\mathbb{C}},\sigma}$ is the tangent space to $S_{\mathbb{C}}$ at σ. It follows from the construction that

$(0.4) = (0.3) \otimes_k \mathbb{C}$
(0.3) is injective $\Leftrightarrow (0.4)$ is injective.

The map (0.4) has the following interpretation (cf. [8] p. 168). As said, $X'_{\mathbb{C}} \to S_{\mathbb{C}}$ is a smooth family of smooth complex surfaces. We can pass to the analytic context (without adding new notations). Choose a marking of the family, i.e. an isomorphism from $R^2\pi_{\mathbb{C}*}\mathbb{Z}$ onto a fixed lattice L. Associated with such a marked family is a

period mapping : $S_{\mathbb{C}} \to$ (period space).

The period space is a piece of some flag manifold and the period mapping assigns to a point s of $S_{\mathbb{C}}$ the Hodge filtration on $H^2_{DR}(X_s/\mathbb{C}) = L \otimes_{\mathbb{Z}} \mathbb{C}$. The interpretation of (0.4) is:

(0.5) $\quad$ 'The map (0.4) is the differential of the period mapping'.

As the Hodge filtration has the property $F^1 = F^{2\perp}$, the image of s in the period space is completely determined by the position of the line $H^0(X_s, \Omega^2_{X_s/\mathbb{C}})$ in $L \otimes_{\mathbb{Z}} \mathbb{C}$. This is position is classically expressed by the periods of a holomorphic 2-form on X. That much to the statement (i) in the corollary.

As for (ii) in the corollary, one can remark that the simplicity of the result allows a simple description of a natural transformation from the functor of (infinitesimal) points on X to $\ker\,(\widehat{CH}^2_X \to \widehat{\text{Alb}}_X)$. For this we fix a non-zero

2-form $\omega \in H^0(X, \Omega^2_{X/k})$. Let (A, m) be an object of $\mathscr{C}$. An A-valued point of X is just a morphism f : Spec $A \to X$. Let x be the k-rational point underlying f, i.e. the composite Spec $k \to$ Spec $A \to X$. Let $B = \mathscr{O}^\wedge_{X,x}$ be the completion of the local ring at x. Then f is actually just a surjective ring homomorphism $f' : B \to A$. The form ω "is" an element of $\Omega^2_{B/k}$. It is a closed form and is therefore exact by the formal Poincaré lemma. This means that ω lies actually in the subspace $\Omega^1_{B/k}/dB$ of $\Omega^2_{B/k}$. To the A-valued point f of X we assign the image of ω under the map $\Omega^1_{B/k}/dB \to \Omega^1_{A/k}/dA$ which is induced by f'. This gives the natural transformation we wanted.

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The technical arguments needed to prove theorems 1 and 2 use the assumption dim $X = 2$ only to assure that the functor $H^2(X, -)$ on the category of sheaves of abelian groups on X is right-exact. They work equally well for analyzing the functor on $\mathscr{C}$ which assigns to an object (A, m) the group

$$\ker\, [H^n(X_A, \mathscr{K}_{2,X_A}) \xrightarrow{m \to 0} H^n(X, \mathscr{K}_{2,X})],$$

when X is a smooth projective variety of dimension n over k. This functor is the "formal completion at the origin" of $H^n(X, \mathscr{K}_{2,X})$. Of course, interpretations in terms of 0-cycles and Albanese are not available. Yet, for a curve X $H^1(X, \mathscr{K}_{2,X})$ seems to be of interest. For this reason the hypothesis for the remaining sections of this paper is $-X$ is a smooth projective n-dimensional variety over a field k of characteristic 0 and finite transcendence degree over $\mathbb{Q}$.

§ 1. THE GENERAL DECOMPOSITION OF $H^n(X, \hat{\mathscr{K}}_{2,X})$

Let (A, m) be an object of $\mathscr{C}$. The schemes X and X_A have the same underlying topological space. The map $\mathscr{K}_{2,X_A} \to \mathscr{K}_{2,X}$ between sheaves on this space splits. Let us denote its kernel as $\hat{\mathscr{K}}_{2,X}(A, m)$. Alternately, one can define $\hat{\mathscr{K}}_{2,X}(A, m)$ as the sheaf on X associated to the pre-sheaf

$$(\text{open } U) \to \ker\, [K_2(\Gamma(U, \mathscr{O}_X) \otimes_k A) \to K_2(\Gamma(U, \mathscr{O}_X))].$$

This sheaf varies in a functorial way with (A, m), i.e. we have a covariant functor

$$\hat{\mathscr{K}}_{2,X} : \mathscr{C} \to (\text{sheaves of abelian groups on } X).$$

One has obviously for every (A, m)

$$H^n(X, \hat{\mathscr{K}}_{2,X}(A, m)) = \ker\, [H^n(X_A, \mathscr{K}_{2,X_A}) \to H^n(X, \mathscr{K}_{2,X})].$$

So the functor which we should investigate is $H^n(X, \hat{\mathscr{K}}_{2,X})$, which assigns to (A, m) the group $H^n(X, \hat{\mathscr{K}}_{2,X}(A, m))$.

Being in characteristic 0 one can use logarithms to translate questions about K_2 (multiplicative) to questions about Ω^1 (additive). Concretely, one has according to [1] or [9] an isomorphism

$$\hat{\mathscr{K}}_{2,X}(A, m) \simeq \Omega^1_{X\otimes A, X\otimes m}/d(\mathcal{O}_X \otimes_k m),$$

where by definition

$$\Omega^1_{X\otimes A, X\otimes m} = \ker\ [\Omega^1_{X_A/\mathbb{Q}} \to \Omega^1_{X/\mathbb{Q}}].$$

These isomorphisms, for varying (A, m), constitute actually an isomorphism of functors on $\mathscr{C}$. We may forget about K-theory. Our problem has become analyzing

$$H^n(X, \Omega^1_{X\otimes A, X\otimes m}/d(\mathcal{O}_X \otimes_k m)),$$

as a functor of (A, m).
Define

$$\Omega^1_{A, m} = \ker\ [\Omega^1_{A/\mathbb{Q}} \to \Omega^1_{k/\mathbb{Q}}].$$

For every k-algebra S there is a surjective homomorphism

$$(1.1) \qquad S\otimes_k \Omega^1_{A, m} \oplus \Omega^1_{S/\mathbb{Q}} \otimes_k m \to \Omega^1_{S\otimes_k A, S\otimes_k m}/d(S\otimes_k m),$$

which sends

$$a_1 \otimes (b_1 db_2) \text{ to } (a_1 \otimes b_1) d(1\otimes b_2) \text{ and } (a_1 da_2) \otimes b_1 \text{ to } (a_1 \otimes b_1) d(a_2 \otimes 1),$$

modulo $d(S\otimes_k m)$. It is obvious that elements of the form $a\otimes db + da\otimes b$ are in the kernel of this map. Passing to sheaves and taking cohomology (H^n is right exact because $\dim X = n$) we find a surjection from

$$(1.2) \qquad \frac{H^n(X, \mathcal{O}_X)\otimes_k \Omega^1_{A, m} \oplus H^n(X, \Omega^1_{X/\mathbb{Q}})\otimes_k m}{I(A, m)}$$

onto

$$H^n(X, \Omega^1_{X\otimes A, X\otimes m}/d(\mathcal{O}_X \otimes_k m)),$$

where $I(A, m)$ is the group generated by the elements $\omega\otimes db + d\omega\otimes b$ with $\omega \in H^n(X, \mathcal{O}_X)$ and $b \in m$.

THEOREM 3. The homomorphism (1.2) is an isomorphism for every (A, m) in $\mathscr{C}$. Thus one finds an isomorphism between $H^n(X, \hat{\mathscr{K}}_{2,X})$ and the functor which assigns to an object (A, m) of $\mathscr{C}$ the group

$$\frac{H^n(X, \mathcal{O}_X)\otimes_k \Omega^1_{A, m} \oplus H^n(X, \Omega^1_{X/\mathbb{Q}})\otimes_k m}{I(A, m)}.$$

PROOF. We first reduce the question to the case of certain special algebras (A, m).

Consider a surjective homomorphism $(A, m) \twoheadrightarrow (A', m')$ in $\mathscr{C}$. Let us denote the map and groups in (1.2) schematically as $L \twoheadrightarrow M$, and its analogue for $(A', \mathfrak{m}')$ as $L' \twoheadrightarrow M'$. The homomorphism $(A, \mathfrak{m}) \twoheadrightarrow (A', \mathfrak{m}')$ induces surjections $L \twoheadrightarrow L'$ and $M \twoheadrightarrow M'$. This yields a commutative diagram

$$\begin{array}{ccc} L & \twoheadrightarrow & M \\ \downarrow & & \downarrow \\ L' & \twoheadrightarrow & M' \end{array}$$

One can easily check, starting with simple calculations at ring level and using the right-exactness of the functor $H^n(X, -)$, that the induced map from $\ker (L \twoheadrightarrow L')$ into $\ker (M \twoheadrightarrow M')$ is surjective. This implies by the snake lemma that the induced map from $\ker (L \twoheadrightarrow M)$ into $\ker (L' \twoheadrightarrow M')$ is also surjective. Consequently, if $L \to M$ is injective, then $L' \to M'$ is injective too. Since every artinian local $\mathfrak{k}$-algebra with residue field $\mathfrak{k}$ is the homomorphic image of an algebra of the form

$$(*) \qquad \begin{aligned} A &= \mathfrak{k}[t_1, \dots, t_q]/(t_1, \dots, t_q)^r \\ \mathfrak{m} &= \ (t_1, \dots, t_q) \end{aligned}$$

our problem is thus reduced to proving the injectivity of (1.2) for $(A, \mathfrak{m})$ as in (*).

We need the following lemma:

LEMMA (1.3). Let $(A, \mathfrak{m})$ be as in (*). Let S be any $\mathfrak{k}$-algebra. Then (1.2) induces an isomorphism

$$\frac{S \otimes_{\mathfrak{k}} \Omega^1_{A, \mathfrak{m}} \oplus \Omega^1_{S/\mathbb{Q}} \otimes_{\mathfrak{k}} \mathfrak{m}}{J(A, \mathfrak{m})} \xrightarrow{\sim} \Omega^1_{S \otimes_{\mathfrak{k}} A, S \otimes_{\mathfrak{k}} \mathfrak{m}} / d(S \otimes_{\mathfrak{k}} \mathfrak{m})$$

where $J(A, \mathfrak{m})$ is the group generated by the elements $a \otimes db + da \otimes b$ with $a \in S$ and $b \in \mathfrak{m}$.

PROOF. Put

$$H = \frac{S \otimes_{\mathfrak{k}} \Omega^1_{A, \mathfrak{m}} \oplus \Omega^1_{S/\mathbb{Q}} \otimes_{\mathfrak{k}} \mathfrak{m}}{J(A, \mathfrak{m})}, \quad K = \Omega^1_{S \otimes_{\mathfrak{k}} A, S \otimes_{\mathfrak{k}} \mathfrak{m}} / d(S \otimes_{\mathfrak{k}} \mathfrak{m}).$$

We want to give explicitly a map $K \to H$ which is inverse to the map $H \to K$ coming from (1.1). Note that K has a presentation by generators and relations. The generators are expressions $\langle f, g \rangle$ with $f, g \in S \otimes_{\mathfrak{k}} A$ and f or $g \in S \otimes_{\mathfrak{k}} \mathfrak{m}$. The defining relations are $\langle f, g+h \rangle = \langle f, g \rangle + \langle f, h \rangle$; $\langle f, gh \rangle = \langle fg, h \rangle + \langle fh, g \rangle$; $\langle 1, g \rangle = 0$. Using the convention $\mathbf{t}^\alpha = t_1^{\alpha_1} t_2^{\alpha_2} \dots t_q^{\alpha_q}$ we may write elements of $S \otimes_{\mathfrak{k}} A$ uniquely as $\sum_\alpha a_\alpha \mathbf{t}^\alpha$ with $a_\alpha \in S$.

To $\langle \sum_\alpha a_\alpha \mathbf{t}^\alpha, \sum_\gamma c_\gamma \mathbf{t}^\gamma \rangle$ we assign the element

$$\sum_{\alpha, \gamma} a_\alpha c_\gamma \otimes \mathbf{t}^\alpha d\mathbf{t}^\gamma + \sum_{\alpha, \gamma} a_\alpha dc_\gamma \otimes \mathbf{t}^{\alpha+\gamma} \text{ modulo } J(A, \mathfrak{m}) \text{ in } H.$$

This assignment clearly respects the above relations. Hence it defines a homomorphism $K \to H$. This homomorphism is right-inverse to the map induced by (1.1), and it is surjective. Thus we see that it is an isomorphism. □

We continue the proof of theorem 3. As a consequence of lemma (1.3) one finds an exact sequence of sheaves of abelian groups on X:

$$\mathcal{O}_X \otimes_{\mathbb{Q}} \mathfrak{m} \xrightarrow{1\otimes d + d\otimes 1} \mathcal{O}_X \otimes_{k} \Omega^1_{A,\mathfrak{m}} \oplus \Omega^1_{X/\mathbb{Q}} \otimes_{k} \mathfrak{m} \to \Omega^1_{X\otimes A, X\otimes \mathfrak{m}}/d(\mathcal{O}_X \otimes_{k} \mathfrak{m}) \to 0.$$

To this sequence we apply the right-exact functor $H^n(X, -)$ and obtain thus the exact sequence

$$H^n(X, \mathcal{O}_X) \otimes_{\mathbb{Q}} \mathfrak{m} \xrightarrow{1\otimes d + d\otimes 1} H^n(X, \mathcal{O}_X) \otimes_{k} \Omega^1_{A,\mathfrak{m}} \oplus H^n(X, \Omega^1_{X/\mathbb{Q}}) \otimes_{k} \mathfrak{m} \to$$

$$\to H^n(X, \Omega^1_{X\otimes A, X\otimes \mathfrak{m}}/d(\mathcal{O}_X \otimes_{k} \mathfrak{m})) \to 0.$$

The remark that the group $I(A, \mathfrak{m})$ is precisely the image of $H^n(X, \mathcal{O}_X) \otimes_{\mathbb{Q}} \mathfrak{m}$ under the map $1\otimes d + d\otimes 1$, concludes the proof of theorem 3. □

REMARK (1.4) I like the following reformulation of theorem 3. Let $\mathcal{R}$ be the non-commutative polynomial ring in one variable, $k[d]$, modulo the two-sided ideal generated by the monomials of degree 2. So $\mathcal{R}$ is a graded non-commutative ring with unit concentrated in degrees 0 and 1, with $\mathcal{R}^0 = k$ and $\mathcal{R}^1 = k d k$. We view $\mathfrak{m} \oplus \Omega^1_{A,\mathfrak{m}}$ as a graded left $\mathcal{R}$-module with $\mathfrak{m}$ in degree 0 and $\Omega^1_{A,\mathfrak{m}}$ in degree 1, upon which d acts as the usual derivation $d: \mathfrak{m} \to \Omega^1_{A,\mathfrak{m}}$ and k acts by multiplication on the left on $\mathfrak{m}$ and $\Omega^1_{A,\mathfrak{m}}$. We view $H^n(X, \mathcal{O}_X) \oplus \oplus H^n(X, \Omega^1_{X/\mathbb{Q}})$ as a graded left $\mathcal{R}$-module in the same way.

Let $(\mathfrak{m} \oplus \Omega^1_{A,\mathfrak{m}})(-1)$ be the graded left $\mathcal{R}$-module with $\mathfrak{m}$ in degree $+1$ and $\Omega^1_{A,\mathfrak{m}}$ in degree $+2$, and with d operating as -1 times the usual $d: \mathfrak{m} \to \Omega^1_{A,\mathfrak{m}}$. So (-1) is the standard shift of complexes.

Every graded left $\mathcal{R}$-module $L^\cdot$ can be considered as a right $\mathcal{R}$-module if one defines:

$$\ell \cdot a = a \cdot \ell, \; \ell d = (-1)^i d\ell \text{ for all } \ell \in L^i, \text{ all } i, \text{ all } a \in k.$$

With these conventions we can view

$$\frac{H^n(X, \mathcal{O}_X) \otimes_{k} \Omega^1_{A,\mathfrak{m}} \oplus H^n(X, \Omega^1_{X/\mathbb{Q}}) \otimes_{k} \mathfrak{m}}{I(A, \mathfrak{m})}$$

also as

$$\{(H^n(X, \mathcal{O}_X) \oplus H^n(X, \Omega^1_{X/\mathbb{Q}})) \otimes_{\mathcal{R}} (\mathfrak{m} \oplus \Omega^1_{A,\mathfrak{m}})(-1)\}^{\deg 2}$$

where deg 2 refers to the homogeneous part of total degree 2. One may interpret this formula as saying that the $\mathcal{R}$-module $H^n(X, \mathcal{O}_X) \oplus H^n(X, \Omega^1_{X/\mathbb{Q}})$ is for the functor $H^n(X, \hat{\mathcal{K}}_{2,X})$ what the Lie algebra is for a formal group or Lie group. In characteristic $p > 0$ there is a similar formula for $H^n(X, \hat{\mathcal{K}}_{2,X})$ in which the crucial module is the part $H^n(X, \mathcal{W}\mathcal{O}_X) \oplus H^n(X, \mathcal{W}\Omega^1_X)$ of the slope spectral sequence for crystalline cohomology, viewed as a module over the Cartier-Dieudonné-Raynaud algebra (see [12]).

§ 2. THE PROOF OF THEOREM 1

One obtains theorem 1 by specifying $n=2$ in the propositions (2.2), (2.4) and (2.6) which are proved in this section.

Let $\mathscr{G}$ be the formal group over k whose tangent space (= Lie algebra) is $H^n(X, \Omega^1_{X/k})$ i.e. the covariant functor

$$\mathscr{G} : \mathscr{C} \to \text{(abelian groups)}$$

defined by

$$\text{(2.1)} \qquad \mathscr{G}(A, m) = H^n(X, \Omega^1_{X/k}) \otimes_k m.$$

PROPOSITION (2.2). There exists a natural surjective homomorphism

$$j : H^n(X, \hat{\mathscr{K}}_{2,X}) \to \mathscr{G}.$$

PROOF. According theorem 3 $H^n(X, \hat{\mathscr{K}}_{2,X})$ is naturally isomorphic to the functor which assigns to an object (A, m) the group

$$\frac{H^n(X, \mathscr{O}_X) \otimes_k \Omega^1_{A,m} \oplus H^n(X, \Omega^1_{X/\mathbb{Q}}) \otimes_k m}{I(A, m)}.$$

There is a canonical surjection $p : H^n(X, \Omega^1_{X/\mathbb{Q}}) \twoheadrightarrow H^n(X, \Omega^1_{X/k})$. Now consider the surjective homomorphism

$$\tilde{j} : H^n(X, \mathscr{O}_X) \otimes_k \Omega^1_{A,m} \oplus H^n(X, \Omega^1_{X/\mathbb{Q}}) \otimes_k m \twoheadrightarrow H^n(X, \Omega^1_{X/k}) \otimes_k m$$

which is the zero map on the first summand and $p \otimes 1$ on the second one. Hodge theory shows that the differential $d : H^n(X, \mathscr{O}_X) \to H^n(X, \Omega^1_{X/k})$ vanishes. As an immediate consequence one finds that $\tilde{j}$ vanishes on $I(A, m)$. Hence the proposition follows. □

REMARK (2.3). Continuing the line of thought started in remark (1.4) one can describe the map $j : H^n(X, \hat{\mathscr{K}}_{2,X}) \to \mathscr{G}$ as being induced by the obvious map of $\mathscr{R}$-modules

$$(H^n(X, \mathscr{O}_X) \oplus H^n(X, \Omega^1_{X/\mathbb{Q}})) \to H^n(X, \Omega^1_{X/k}).$$

Here $H^n(X, \Omega^1_{X/k})$ is considered as $\mathscr{R}$-module concentrated in degree 1. Note that

$$\mathscr{G}(A, m) = H^n(X, \Omega^1_{X/k}) \otimes_k m = \{H^n(X, \Omega^1_{X/k}) \otimes_{\mathscr{R}} (m \oplus \Omega^1_{A,m})(-1)\}^{\deg 2}.$$

PROPOSITION (2.4). The map j in (2.2) is a natural isomorphism if and only if the geometric genus p_g of X is zero. (Recall that $p_g = \dim_k H^0(X, \Omega^n_{X/k}) = \dim_k H^n(X, \mathscr{O}_X)$).

PROOF. The kernel of the map $\tilde{j}$ in the proof of (2.2) contains obviously $H^n(X, \mathscr{O}_X) \otimes_k \Omega^1_{A,m}$. From this fact one deduces immediately that there is a surjection from $\ker j$ (evaluated at (A, m)) onto $H^n(X, \mathscr{O}_X) \otimes_k \Omega^1_{A/k}/dA$. Taking

$A = k[\varepsilon, \delta]/(\varepsilon^2, \delta^2, \varepsilon\delta)$ one has an example in which $\Omega^1_{A/k}/dA$ is not zero. Therefore, if j is an isomorphism, then $H^n(X, \mathcal{O}_X)$ must vanish.

For the converse implication we first recall the exact sequence of sheaves on X

$$0 \to \mathcal{O}_X \otimes_k \Omega^1_{k/\mathbb{Q}} \to \Omega^1_{X/\mathbb{Q}} \to \Omega^1_{X/k} \to 0.$$

The corresponding cohomology sequence shows that the map

$$p : H^n(X, \Omega^1_{X/\mathbb{Q}}) \to H^n(X, \Omega^1_{X/k})$$

is an isomorphism if $H^n(X, \mathcal{O}_X) = 0$. This implies that in case $H^n(X, \mathcal{O}_X) = 0$ the map $\tilde{j}$ in the proof of (2.2) is an isomorphism. The same conclusion follows for j. □

REMARK (2.5) One can arrive at the conclusion of (2.4) without first proving theorem 1 by using instead Bloch's approach via the bi-tangent space (see [3]).

Let me recall some facts about formal groups over k. In this paper "formal group" will always mean "smooth commutative formal group". These are covariant functors

$$F : \mathscr{C} \to \text{(abelian groups)}$$

for which the underlying set-valued functor is naturally isomorphic to the functor $\mathbb{A}^r$ (for some r) which is defined by $\mathbb{A}^r(A, m) = m \times \ldots \times m$ (r factors). The number r is called the dimension of F. It is well-known that over a field of characteristic 0 every r-dimensional formal group is naturally isomorphic to the direct sum of r copies of the additive formal group, i.e.

$$F \simeq \mathbb{G}_a^r, \text{ as group valued functors,}$$

where $\mathbb{G}_a$ is the functor which assigns to (A, m) the additive group m.

PROPOSITION (2.6) Every natural homomorphism from $H^n(X, \hat{\mathscr{K}}_{2,X})$ into a formal group over k factors through the formal group $\mathscr{G}$ defined in (2.1).

PROOF. Since every formal group over k is the direct sum of a number of copies of the additive group $\mathbb{G}_a$, it suffices to prove that every natural homomorphism $H^n(X, \hat{\mathscr{K}}_{2,X}) \to \mathbb{G}_a$ factors through $\mathscr{G}$; or rather that it vanishes on the kernel of $j : H^n(X, \hat{\mathscr{K}}_{2,X}) \to \mathscr{G}$.

Denote by Ω^1_- the covariant functor from $\mathscr{C}$ into the category of k-vector spaces which sends (A, m) to $\Omega^1_{A,m}$. It is obvious from the construction of j in proposition (2.2) that there is a natural surjection from the (group valued) functor $H^n(X, \mathcal{O}_X) \otimes_k \Omega^1_-$ onto $\ker j$. It suffices therefore to show that there are no non-trivial natural homomorphisms

$$\varphi : H^n(X, \mathcal{O}_X) \otimes_k \Omega^1_- \to \mathbb{G}_a.$$

So let us take such a homomorphism φ. For every $(A, m) \in \mathscr{C}$ the group $H^n(X, \mathcal{O}_X) \otimes_k \Omega^1_{A,m}$ is generated by elements $\omega \otimes dt$ and $\omega \otimes x dy$ with

$\omega \in H^n(X, \mathcal{O}_X)$, $t, x, y \in m$. It suffices therefore to prove for all $N \geq 2$ and $\omega \in H^n(X, \mathcal{O}_X)$

$$\varphi(\omega \otimes dt) = 0 \quad \text{in case } A = k[t]/(t^N)$$

$$\varphi(\omega \otimes xdy) = 0 \text{ in case } A = k[x, y]/(x, y)^N.$$

Consider the first case. By definition $\varphi(\omega \otimes dt)$ is some polynomial $f(t)$ in $k[t]/(t^N)$. Now look at $\varphi(\omega \otimes d(2t))$. On the one hand φ is a natural transformation, whence $\varphi(\omega \otimes d(2t)) = f(2t)$, and on the other hand it is a homomorphism, whence $\varphi(\omega \otimes d(2t)) = 2f(t)$. This limits the possibilities for $f(t)$ to $f(t) = ct$ for some $c \in k$.

A similar argument shows that in the second case the element $\varphi(\omega \otimes xdy)$ of $k[x, y]/(x, y)^N$ has to be of the form bxy for some constant $b \in k$. The relation between b and c is found by looking at the computation

$$cxy = \varphi(\omega \otimes d(xy)) = \varphi(\omega \otimes xdy) + \varphi(\omega \otimes ydx) = 2bxy.$$

So $c = 2b$.

Now consider the elements $\varphi(\omega \otimes xd(yz))$, $\varphi(\omega \otimes xydz)$ and $\varphi(\omega \otimes xzdy)$ of $k[x, y, z]/(x, y, z)^N$. By functoriality all three are equal to $bxyz$. But on the other hand

$$\varphi(\omega \otimes xd(yz)) = \varphi(\omega \otimes xydz) + \varphi(\omega \otimes xzdy).$$

Hence $b = 0$. This completes the proof of proposition (2.6). □

§ 3. INVESTIGATION OF ker $[j : H^n(X, \hat{\mathcal{K}}_{2,X}) \to \mathcal{G}]$

$\mathcal{G}$ is the formal group defined in (2.1) and j is the homomorphism constructed in (2.2). For $(A, m) \in \mathcal{C}$ we let $P(A, m)$ be the subgroup of $H^n(X, \mathcal{O}_X) \otimes_k \Omega^1_{A,m}$ which is generated by the elements $\sum_i \omega_i \otimes d(c_i t)$ with $\omega_i \in H^n(X, \mathcal{O}_X)$, $c_i \in k$, $t \in m$ and $\sum_i c_i d\omega_i = 0$ in $H^n(X, \Omega^1_{X/\mathbb{Q}})$.

We define

$$(3.1) \qquad \mathcal{E}(A, m) = \frac{H^n(X, \mathcal{O}_X) \otimes_k \Omega^1_{A,m}}{P(A, m)}.$$

This defines a covariant functor

$$\mathcal{E} : \mathcal{C} \to \text{(abelian groups)}.$$

It is obvious from the definitions that $P(A, m)$ is a subgroup of the group $I(A, m)$, which occurs in theorem 3. Combining this fact with theorem 3 we get a natural homomorphism

$$(3.2) \qquad i : \mathcal{E} \to H^n(X, \hat{\mathcal{K}}_{2,X}).$$

Taking a look at the construction of j in (2.2) one easily sees that the sequence

$$\mathcal{E} \xrightarrow{i} H^n(X, \hat{\mathcal{K}}_{2,X}) \xrightarrow{j} \mathcal{G} \to 0$$

is exact.

THEOREM 4. The homomorphism $\iota : \mathscr{E} \to H^n(X, \hat{\mathscr{K}}_{2,X})$ is injective. In other words, the functor ker $(\jmath : H^n(X, \hat{\mathscr{K}}_{2,X}) \to \mathscr{G})$ is naturally isomorphic to the one which assigns to $(A, \mathfrak{m})$ the group

$$\frac{H^n(X, \mathscr{O}_X) \otimes_k \Omega^1_{A,\mathfrak{m}}}{P(A, \mathfrak{m})}.$$

PROOF. The first step of the proof is to reduce the general problem to showing the injectivity of the map $\iota : \mathscr{E}(A, \mathfrak{m}) \to H^n(X, \hat{\mathscr{K}}_{2,X}(A, \mathfrak{m}))$ for certain special objects $(A, \mathfrak{m})$.

Consider a surjective homomorphism $(A, \mathfrak{m}) \to (A', \mathfrak{m}')$ in $\mathscr{C}$. It gives rise to a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & T_0 & \longrightarrow & T_1 & \longrightarrow & T_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & T_3 \longrightarrow & \mathscr{E}(A, \mathfrak{m}) & \xrightarrow{\iota} & H^n(X, \hat{\mathscr{K}}_{2,X}(A, \mathfrak{m})) & \xrightarrow{\jmath} & \mathscr{G}(A, \mathfrak{m}) & \longrightarrow & 0 \\
 & \downarrow & \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & T_4 \longrightarrow & \mathscr{E}(A', \mathfrak{m}') & \xrightarrow{\iota} & H^n(X, \hat{\mathscr{K}}_{2,X}(A', \mathfrak{m}')) & \xrightarrow{\jmath} & \mathscr{G}(A', \mathfrak{m}') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The groups $T_0, \ldots, T_4$ are defined by the requirement that the rows and columns be exact. The snake lemma shows that the map $T_3 \to T_4$ is surjective if and only if the map $T_2 \to \mathscr{G}(A, \mathfrak{m})$ is injective, i.e. if and only if the surjection $T_2 \twoheadrightarrow \ker\ (\mathscr{G}(A, \mathfrak{m}) \to \mathscr{G}(A', \mathfrak{m}'))$ has an inverse. Let $M = \ker\ (A \twoheadrightarrow A')$. The definition of $\mathscr{G}$ in (2.1) implies immediately that

$$\ker\ (\mathscr{G}(A, \mathfrak{m}) \to \mathscr{G}(A', \mathfrak{m}')) = H^n(X, \Omega^1_{X/k}) \otimes_k M.$$

Let

$$\Omega^1_{A,M} = \ker\ (\Omega^1_{A/\mathbb{Q}} \to \Omega^1_{A'/\mathbb{Q}}) = \ker\ (\Omega^1_{A,\mathfrak{m}} \to \Omega^1_{A',\mathfrak{m}'}).$$

and let $I(A, M)$ be the subgroup of $I(A, \mathfrak{m})$ which is generated by those elements $\omega \otimes db + d\omega \otimes b$ with $b \in M$ and $\omega \in H^n(X, \mathscr{O}_X)$ (cf. (1.2)). Using the fact that the natural maps $P(A, \mathfrak{m}) \to P(A', \mathfrak{m}')$ and $I(A, \mathfrak{m}) \to I(A', \mathfrak{m}')$ are surjective, the ubiquitous snake lemma, theorem 3 and the definition of $\mathscr{E}$ one can easily see that there are surjective homomorphisms

$$\left[\frac{H^n(X, \mathscr{O}_X) \otimes_k \Omega^1_{A,M} \oplus H^n(X, \Omega^1_{X/\mathbb{Q}}) \otimes_k M}{I(A, M)}\right] \twoheadrightarrow T_1$$

and

$$H^n(X, \mathcal{O}_X)\otimes_k \Omega^1_{A,M} \twoheadrightarrow T_0.$$

Taking the quotient of the first expression by the second one gets a surjection $H^n(X, \Omega^1_{X/k})\otimes_k M \twoheadrightarrow T_2$.

This map is obviously right-inverse to the map $T_2 \twoheadrightarrow H^n(X, \Omega^1_{X/k})\otimes_k M$. So they are isomorphisms. The conclusion is that the map $T_3 \to T_4$ is surjective and that therefore T_4 will be zero if T_3 is so. Thus the problem has been reduced to proving the injectivity of $\mathcal{E}(A, m) \to H^n(X, \hat{\mathcal{K}}_{2,X}(A, m))$ only for

$$A = k[t_1, \ldots, t_q]/(t_1, \ldots, t_q)^r$$
$$m = (t_1, \ldots, t_q).$$

It is obvious from theorem 3 that there is a surjection

$$H^n(X, \hat{\mathcal{K}}_{2,X}(A, m)) \twoheadrightarrow H^n(X, \mathcal{O}_X)\otimes_k [\Omega^1_{A/k}/dm].$$

The composite of this map with $\iota : \mathcal{E}(A, m) \to H^n(X, \hat{\mathcal{K}}_{2,X}(A, m))$ is the obvious projection, the kernel of which is generated by the classes of the elements $\omega\otimes dc\mathbf{t}^\alpha$, with $\omega \in H^n(X, \mathcal{O}_X)$, $c \in k$, $\mathbf{t}^\alpha = t_1^{\alpha_1} \ldots t_q^{\alpha_q}$. This kernel contains ker ι.

CLAIM. If

$$\Sigma_\alpha \ \Sigma_i \ \omega_{i,\alpha}\otimes dc_{i,\alpha}\mathbf{t}^\alpha \bmod P(A, m) \text{ is in ker } \iota$$

then

$$\Sigma_i \ \omega_{i,\alpha}\otimes dc_{i,\alpha}\mathbf{t}^\alpha \bmod P(A, m) \text{ is in ker } \iota \text{ for every } \alpha.$$

PROOF OF THIS CLAIM. By means of the substitutions $t_1 \mapsto \ell t_1$ for $\ell = 1, 2, \ldots, r$ we obtain from the one given element r elements:

$$\sum_{s=0}^{r} \ell^s \sum_{\alpha, \alpha_1 = s} \ \sum_i \ \omega_{i,\alpha}\otimes dc_{i,\alpha}\mathbf{t}^\alpha \bmod P(A, m).$$

By functoriality each of these r elements belongs to ker ι. Let

$$D = \det \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 2 & 2^2 & \ldots & 2^r \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ 1 & r & r^2 & \ldots & r^r \end{pmatrix}$$

Then D is a non-zero integer. For each s one can write

$$D \sum_{\alpha, \alpha_1 = s} \ \sum_i \ \omega_{i,\alpha}\otimes dc_{i,\alpha}\mathbf{t}^\alpha \bmod P(A, m)$$

as a $\mathbb{Z}$-linear combination of the above r elements. So each of these terms is in ker ι. Using the substitution $t_1 \mapsto D^{-1}t_1$ and functoriality one can now conclude that for each s

$$\sum_{\alpha,\alpha_1=s} \sum_i \omega_{i,\alpha}\otimes dc_{i,\alpha}\mathbf{t}^\alpha \bmod P(A,\mathfrak{m})$$

belongs to ker ι.

Repeat this trick with each of these terms and with t_2 instead of t_1. And so on. This proves the claim.

Thus the problem of showing ker $\iota=0$ has been reduced to the question: If $\sum_i \omega_i\otimes dc_i\mathbf{t}^\alpha \bmod P(A,\mathfrak{m})$ belongs to ker ι, is then necessarily $\sum_i c_i d\omega_i=0$ in $H^n(X,\Omega^1_{X/\mathbb{Q}})$?

Consider the substitution

$$A=k[t_1,\dots,t_q]/(t_1,\dots,t_q)^r\to B=k[t]/(t^r)$$

sending t_i to t for $i=1,\dots,q$. It transforms $\sum_i \omega_i\otimes dc_i\mathbf{t}^\alpha \bmod P(A,\mathfrak{m})$ into $\sum_i \omega_i\otimes dc_i t^s \bmod P(B,tB)$ with $s=\alpha_1+\dots+\alpha_q$. If the former element is in ker ι, then the latter element is in ker ι too.

Take the description of $H^n(X,\hat{\mathscr{K}}_{2,X}(B,tB))$ which is given in theorem 3. There is a homomorphism

$$H^n(X,\mathcal{O}_X)\otimes_k\Omega^1_{B,tB}\oplus H^n(X,\Omega^1_{X/\mathbb{Q}})\otimes_k(tB)\to H^n(X,\Omega^1_{X/\mathbb{Q}})\otimes_k(tB)$$

which on $H^n(X,\Omega^1_{X/\mathbb{Q}})\otimes_k(tB)$ is the identity map and which on $H^n(X,\mathcal{O}_X)\otimes_k\Omega^1_{B,tB}$ is defined by

$$\omega\otimes(\sum_i a_i t^i)d(\sum_j b_j t^j)\mapsto -\sum_{i,j}(b_j d(a_i\omega)-i(i+j)^{-1}d(a_i b_j\omega))\otimes t^{i+j}.$$

This map obviously annihilates the group $I(B,tB)$. So it induces a homomorphism $H^n(X,\hat{\mathscr{K}}_{2,X}(B,tB))\to H^n(X,\Omega^1_{X/\mathbb{Q}})\otimes_k(tB)$.

Let us compose it with the map $\iota:\mathscr{E}(B,tB)\to H^n(X,\hat{\mathscr{K}}_{2,X}(B,tB))$. Evaluating the composite map at the element $\sum_i \omega_i\otimes d(c_i t^s) \bmod P(B,tB)$ one finds $(-\sum_i c_i d\omega_i)\otimes t^s$. Since the former element is in ker ι, this implies $\sum_i c_i d\omega_i=0$. This concludes the proof of theorem 4. □

Thus the problem of understanding ker $[\jmath : H^n(X,\hat{\mathscr{K}}_{2,X})\to\mathscr{G}]$ is essentially reduced to the question: Which relations of the form $\sum_i c_i d\omega_i=0$ do exist in $H^n(X,\Omega^1_{X/\mathbb{Q}})$?

The answer is most simple if the canonical map $H^n(X,\Omega^1_{X/\mathbb{Q}})\to H^n(X,\Omega^1_{X/k})$ is an isomorphism; for then $d:H^n(X,\mathcal{O}_X)\to H^n(X,\Omega^1_{X/\mathbb{Q}})$ is the zero map, since $d:H^n(X,\mathcal{O}_X)\to H^n(X,\Omega^1_{X/k})$ is always zero.

In that case the group $P(A,\mathfrak{m})$ is generated by the elements $\omega\otimes dt$ with $\omega\in H^n(X,\mathcal{O}_X)$, $t\in\mathfrak{m}$ i.e.

$$P(A,\mathfrak{m})=H^n(X,\mathcal{O}_X)\otimes_k k\,d\mathfrak{m}.$$

As a consequence we find (cf. (3.1))

$$\mathscr{E}(A,\mathfrak{m})=H^n(X,\mathcal{O}_X)\otimes_k(\Omega^1_{A/k}/dA)$$

for all $(A, m) \in \mathscr{C}$. Hence to prove theorem 2 we check that the condition mentioned in this theorem, implies $H^n(X, \Omega^1_{X/\mathbb{Q}}) \simeq H^n(X, \Omega^1_{X/k})$. Consider the exact sequence of sheaves on X:

$$(3.3) \qquad 0 \to \mathscr{O}_X \otimes_k \Omega^1_{k/\mathbb{Q}} \to \Omega^1_{X/\mathbb{Q}} \to \Omega^1_{X/k} \to 0.$$

and the following part of the associated long exact sequence of cohomology groups

$$H^{n-1}(X, \Omega^1_{X/k}) \to H^n(X, \mathscr{O}_X) \otimes_k \Omega^1_{k/\mathbb{Q}} \to H^n(X, \Omega^1_{X/\mathbb{Q}}) \to H^n(X, \Omega^1_{X/k}) \to 0.$$

It shows that $H^n(X, \Omega^1_{X/\mathbb{Q}})$ is isomorphic to $H^n(X, \Omega^1_{X/k})$ if and only if the coboundary map $H^{n-1}(X, \Omega^1_{X/k}) \to H^n(X, \mathscr{O}_X) \otimes_k \Omega^1_{k/\mathbb{Q}}$ is surjective.

The fact that this coboundary is equal to the map induced by the Gauss-Manin connection on $H^n_{DR}(X/k)$, as well as to cup-product with the Kodaira-Spencer mapping, is proved in ([5] (1.3.2), (1.4.1.7)). (See also below for a summary of the construction and main properties of the Gauss-Manin connection.)

Thus we have proved most of the following theorem.

THEOREM 2BIS. Let X be a smooth projective n-dimensional variety over a field k, which has finite transcendence degree over $\mathbb{Q}$. Then the following statements are equivalent.

(i) The canonical map $H^n(X, \Omega^1_{X/\mathbb{Q}}) \to H^n(X, \Omega^1_{X/k})$ is a isomorphism.

(ii) The map $H^{n-1}(X, \Omega^1_{X/k}) \to H^n(X, \mathscr{O}_X) \otimes_k \Omega^1_{k/\mathbb{Q}}$ which is induced by the Gauss-Manin connection, is surjective.

(iii) The functor $\ker [j : H^n(X, \hat{\mathscr{K}}_{2,X}) \to \mathscr{G}]$ is naturally isomorphic to the one which assigns to an object (A, m) of $\mathscr{C}$ the group

$$H^n(X, \mathscr{O}_X) \otimes_k (\Omega^1_{A/k}/dA).$$

PROOF. The implications (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) have been shown above. We are left with (iii) $\Rightarrow$ (i). So, assume (iii) holds. Take $A = k[\varepsilon]/(\varepsilon^2)$. Then $\Omega^1_{A/k}/dA = 0$. Hence

$$j : H^n(X, \hat{\mathscr{K}}_{2,X}(A, m)) \to \mathscr{G}(A, m)$$

is an isomorphism for this A. On the other hand we have for this A

$$\mathscr{G}(A, m) \simeq H^n(X, \Omega^1_{X/k}) \text{ by } (2.1)$$

$$H^n(X, \hat{\mathscr{K}}_{2,X}(A, m)) \simeq H^n(X, \Omega^1_{X/\mathbb{Q}})$$

by Van der Kallen's theorem [7]. So (iii) does imply (i). □

We conclude this paper with an analysis of the functor $\mathscr{E}$ in case the conditions of the above theorem are not satisfied. Eventually we shall assume that X has genus 1.

First we briefly recall the Katz-Oda construction of the Gauss-Manin connection [6].

The De Rham complex $\Omega^{\cdot}_{X/\mathbb{Q}}$ carries a natural filtration by subcomplexes, which are defined as follows

$$K^{i,j} = \text{image } (\Omega^{j-i}_{X/\mathbb{Q}} \otimes_{k} \Omega^{i}_{k/\mathbb{Q}} \to \Omega^{j}_{X/\mathbb{Q}}).$$

Using (3.3) one can show that the associated graded complex is

$$gr^{i,\cdot} = K^{i,\cdot}/K^{i+1,\cdot} = \Omega^{\cdot-i}_{X/k} \otimes_{k} \Omega^{i}_{k/\mathbb{Q}}.$$

In particular $gr^{0,\cdot} = \Omega^{\cdot}_{X/k}$ and $gr^{1,\cdot} = \Omega^{\cdot-1}_{X/k} \otimes_{k} \Omega^{1}_{k/\mathbb{Q}}$. Take the long exact sequence of hypercohomology groups which is associated with the short exact sequence of complexes of sheaves on X

$$0 \to gr^{1,\cdot} \to K^{0,\cdot}/K^{2,\cdot} \to gr^{0,\cdot} \to 0.$$

The Gauss-Manin connection on $H^{n}_{DR}(X/k)$ $(= \mathbb{H}^{n}(X, \Omega^{\cdot}_{X/k}))$ relative to $\mathbb{Q}$ is by definition the coboundary map

$$\text{(3.4)} \qquad \nabla : H^{n}_{DR}(X/k) \to H^{n}_{DR}(X/k) \otimes_{k} \Omega^{1}_{k/\mathbb{Q}}$$

which appears in this sequence between $\mathbb{H}^{n}$ and $\mathbb{H}^{n+1}$ (cf. [5] p. 14). Recall that the Hodge filtration on $H^{n}_{DR}(X/k)$ is defined by

$$F^{i} = \text{image } (\mathbb{H}^{n}(X, \Omega^{\geq i}_{X/k}) \to \mathbb{H}^{n}(X, \Omega^{\cdot}_{X/k})).$$

Griffths' transversality theorem states $\nabla F^{i} \subset F^{i-1} \otimes_{k} \Omega^{1}_{k/\mathbb{Q}}$ (cf. [5] p. 14). So ∇ induces a map

$$\nabla : F^{0}/F^{2} \to F^{0}/F^{1} \otimes_{k} \Omega^{1}_{k/\mathbb{Q}}.$$

Standard facts in Hodge theory, in particular the degeneracy of the Hodge-De Rham spectral sequence, imply

$$F^{0}/F^{1} = H^{n}(X, \mathcal{O}_{X}), \; F^{0}/F^{2} = \mathbb{H}^{n}(X, \mathcal{O} \xrightarrow{d} \Omega^{1}_{X/k})$$

(the right-hand group is the n-th hypercohomology of the two-term complex $\mathcal{O} \xrightarrow{d} \Omega^{1}_{X/k}$, concentrated in degrees 0 and 1). We summarize these facts in the following commutative square

$$\text{(3.5)} \qquad \begin{array}{ccc} H^{n}_{DR}(X/k) & \xrightarrow{\nabla} & H^{n}_{DR}(X/k) \otimes_{k} \Omega^{1}_{k/\mathbb{Q}} \\ \downarrow & & \downarrow \\ \mathbb{H}^{n}(X, \mathcal{O} \xrightarrow{d} \Omega^{1}_{X/k}) & \xrightarrow{\nabla} & H^{n}(X, \mathcal{O}_{X}) \otimes_{k} \Omega^{1}_{k/\mathbb{Q}} \end{array}$$

Now consider the following commutative diagram in which the lines are short exact sequences of complexes concentrated in degrees 0 and 1.

$$\begin{array}{ccccccccc} 0 \longrightarrow & (0 \longrightarrow \mathcal{O}_{X} \otimes_{k} \Omega^{1}_{k/\mathbb{Q}}) & \longrightarrow & (\mathcal{O}_{X} \longrightarrow \Omega^{1}_{X/\mathbb{Q}}) & \longrightarrow & (\mathcal{O}_{X} \longrightarrow \Omega^{1}_{X/k}) & \longrightarrow 0 \\ & \downarrow & & \| & & \downarrow & \\ 0 \longrightarrow & (0 \longrightarrow \Omega^{1}_{X/\mathbb{Q}}) & \longrightarrow & (\mathcal{O}_{X} \longrightarrow \Omega^{1}_{X/\mathbb{Q}}) & \longrightarrow & (\mathcal{O}_{X} \longrightarrow 0) & \longrightarrow 0 \end{array}$$

The degree 1 part in the top line is precisely (3.3). Taking hypercohomology groups we get the following commutative square which involves the coboundaries between $\mathbb{H}^n$ and $\mathbb{H}^{n+1}$:

$$\begin{array}{ccc} \mathbb{H}^n(X, \mathcal{O}_X \longrightarrow \Omega^1_{X/k}) & \xrightarrow{\nabla} & H^n(X, \mathcal{O}_X) \otimes_k \Omega^1_{k/\mathbb{Q}} \\ \downarrow & & \downarrow \\ H^n(X, \mathcal{O}_X) & \xrightarrow{d} & H^n(X, \Omega^1_{X/\mathbb{Q}}) \end{array} \tag{3.6}$$

It is easy to see that the bottom line of (3.5) and the top line of (3.6) are indeed the same. This gives the relation between the map $d : H^n(X, \mathcal{O}_X) \to H^n(X, \Omega^1_{X/\mathbb{Q}})$ and the Gauss-Manin connection on $H^n_{DR}(X/k)$. We extend (3.6) a bit:

$$\begin{array}{ccccccc} H^{n-1}(X, \Omega^1_{X/k}) & \longrightarrow & \mathbb{H}^n(X, \mathcal{O}_X \longrightarrow \Omega^1_{X/k}) & \longrightarrow & H^n(X, \mathcal{O}_X) & \longrightarrow & H^n(X, \Omega^1_{X/k}) \\ \| & & \downarrow \nabla & & \downarrow d & & \| \\ H^{n-1}(X, \Omega^1_{X/k}) & \longrightarrow & H^n(X, \mathcal{O}_X) \otimes_k \Omega^1_{k/\mathbb{Q}} & \longrightarrow & H^n(X, \Omega^1_{X/\mathbb{Q}}) & \longrightarrow & H^n(X, \Omega^1_{X/k}) \end{array} \tag{3.7}$$

The top line is the long exact sequence associated with the obvious filtration of the complex $(\mathcal{O}_X \xrightarrow{d} \Omega^1_{X/k})$. The bottom line is the long exact sequence associated with (3.3). The left-hand square is commutative according to ([5] (1.3.2), (1.4.1.7)). The middle square is just (3.6); so it is commutative. The right-hand square is trivially commutative. The map $d : H^n(X, \mathcal{O}_X) \to \to H^n(X, \Omega^1_{X/k})$ is zero. So the map $d : H^n(X, \mathcal{O}_X) \to H^n(X, \Omega^1_{X/\mathbb{Q}})$ factors through the subgroup

$$\operatorname{coker} (H^{n-1}(X, \Omega^1_{X/k}) \to H^n(X, \mathcal{O}_X) \otimes_k \Omega^1_{k/\mathbb{Q}}) \text{ of } H^n(X, \Omega^1_{X/\mathbb{Q}}).$$

Giving the Gauss-Manin connection in the form (3.4) is equivalent to giving a k-linear homomorphism

$$\nabla : \operatorname{Der}(k/\mathbb{Q}) \to \operatorname{Hom}_{\mathbb{Q}}(H^n_{DR}(X/k), H^n_{DR}(X/k)). \tag{3.8}$$

Here "$\operatorname{Hom}_{\mathbb{Q}}$" means "space of $\mathbb{Q}$-linear maps". Note that the map in (3.8) is also denoted as ∇. It assigns to a derivation D the composite

$$H^n_{DR}(X/k) \xrightarrow{\nabla} H^n_{DR}(X/k) \otimes_k \Omega^1_{k/\mathbb{Q}} \xrightarrow{1 \otimes D} H^n_{DR}(X/k)$$
$$\underbrace{\qquad\qquad\qquad\qquad\qquad}_{\nabla(D)}$$

Griffths' transversality theorem becomes in this formulation $\nabla(D)F^i \subset F^{i-1}$ for all $D \in \operatorname{Der}(k/\mathbb{Q})$ and $i = 1, \ldots, n$. Thus $\nabla(D)$ induces a k-linear homomorphism $\nabla_i(D) : F^i/F^{i+1} \to F^{i-1}/F^i$. We get in particular k-linear maps

$$\left\{ \begin{array}{l} \nabla_1 : \operatorname{Der}(k/\mathbb{Q}) \to \operatorname{Hom}_k (H^{n-1}(X, \Omega^1_{X/k}), H^n(X, \mathcal{O}_X)) \\ \nabla_n : \operatorname{Der}(k/\mathbb{Q}) \to \operatorname{Hom}_k (H^0(X, \Omega^n_{X/k}), H^1(X, \Omega^{n-1}_{X/k})). \end{array} \right. \tag{3.9}$$

This time the target space consists of k-linear homomorphisms. Serre duality gives isomorphisms

$$H^0(X,\Omega^n_{X/k})^\vee \simeq H^n(X,\mathcal{O}_X) \text{ and } H^1(X,\Omega^{n-1}_{X/k})^\vee \simeq H^{n-1}(X,\Omega^1_{X/k}).$$

And ∇_1 and ∇_n are related by

(3.10) $\quad \nabla_n(D)^\vee = -\nabla_1(D)$ for all $D \in \mathrm{Der}\,(k/\mathbb{Q})$

(cf. [6] p. 204 formula (11)). In particular $\ker \nabla_n = \ker \nabla_1$. From now on we assume

(3.11) $\quad$ *X has genus* 1 *i.e.* $\dim_k H^n(X,\mathcal{O}_X) = \dim_k H^0(X,\Omega^n_{X/k}) = 1$.

This condition allows us to view ∇_1 as the 'dual' of the map $H^{n-1}(X,\Omega^1_{X/k}) \to \to H^n(X,\mathcal{O}_X)\otimes_k \Omega^1_{k/\mathbb{Q}}$. This yields an isomorphism

$$\mathrm{coker}\,(H^{n-1}(X,\Omega^1_{X/k}) \to H^n(X,\mathcal{O}_X)\otimes_k\Omega^1_{k/\mathbb{Q}}) \simeq H^n(X,\mathcal{O}_X)\otimes_k(\ker \nabla_1)^\vee.$$

As we have seen the left-hand side is a subgroup of $H^n(X,\Omega^1_{X/\mathbb{Q}})$ which contains the image of the map $d: H^n(X,\mathcal{O}_X) \to H^n(X,\Omega^1_{X/\mathbb{Q}})$. We now see that d factors as the composite of a map

(3.12) $\quad \delta: H^n(X,\mathcal{O}_X) \to H^n(X,\mathcal{O}_X)\otimes_k(\ker \nabla_1)^\vee$

and an injection. This injection is a k-linear map. The definition of the group $P(A,m)$ at the beginning of this section can now be reformulated as follows: $P(A,m)$ is the subgroup of $H^n(X,\mathcal{O}_X)\otimes_k\Omega^1_{A,m}$ which is generated by the elements $\sum_i \omega_i \otimes dc_i t$ with $\omega_i \in H^n(X,\mathcal{O}_X)$, $c_i \in k$, $t \in m$ and $\sum_i c_i \delta\omega_i = 0$ in $H^n(X,\mathcal{O}_X)\otimes_k(\ker \nabla_1)^\vee$.

In the introduction we discussed extensively the situation which arises when ∇_1 is injective. So let us assume from now on that ∇_1 is not injective. Fix a basis $D_1,\dots,D_r$ of $\ker \nabla_1$ and let $D_1^\vee,\dots,D_r^\vee$ be the dual basis in $(\ker \nabla_1)^\vee$. Let

$$k' = \{x \in k \mid Dx = 0 \text{ for all } D \in \ker \nabla_1\}.$$

Then k' is a subfield of k and $\ker \nabla_1 \subset \mathrm{Der}\,(k/k')$. For reasons which will become clear below, we have to assume

(3.13) $\quad$ *assumption:* $\ker \nabla_1 = \mathrm{Der}\,(k/k')$.

Dualizing (3.13) we get $(\ker \nabla_1)^\vee = \Omega^1_{k/k'}$. We will consider $D_1^\vee,\dots,D_r^\vee$ also as basis for $\Omega^1_{k/k'}$.

Let $P'(A,m)$ be the subgroup of $H^n(X,\mathcal{O}_X)\otimes_k\Omega^1_{A,m}$ which is generated by the elements $\omega\otimes tdc$ with $\omega \in H^n(X,\mathcal{O}_X)$, $t \in m$ and $c \in k'$. This group is contained in $P(A,m)$ because $\omega\otimes tdc = \omega\otimes dct - c\omega\otimes dt$ and

$$c\delta\omega - \delta c\omega = -\omega\otimes \sum_{q=1}^{r} D_q(c)D_q^\vee = 0 \text{ for } c \in k'.$$

It is obvious that

$$\frac{H^n(X,\mathcal{O}_X)\otimes_k\Omega^1_{A,m}}{P'(A,m)} = H^n(X,\mathcal{O}_X)\otimes_k\Omega^1_{A,m/k'}$$

where

$$\Omega^1_{A,\mathfrak{m}/k'} = \ker\ (\Omega^1_{A/k'} \to \Omega^1_{k/k'}).$$

Let $\bar{P}(A,\mathfrak{m}) = P(A,\mathfrak{m})/P'(A,\mathfrak{m})$. Then we find, in view of (3.1)

$$(3.14) \qquad \mathscr{E}(A,\mathfrak{m}) = \frac{H^n(X,\mathcal{O}_X)\otimes_k \Omega^1_{A,\mathfrak{m}/k'}}{\bar{P}(A,\mathfrak{m})}.$$

Now fix a non-zero element $\omega \in H^n(X,\mathcal{O}_X)$. Then $\omega\otimes D_1^\vee,\dots,\omega\otimes D_r^\vee$ is a basis for $H^n(X,\mathcal{O}_X)\otimes_k(\ker\ \nabla_1)^\vee$.

Define $s_1,\dots,s_r$ by

$$(3.15) \qquad \delta\omega = \sum_{q=1}^{r} s_q \omega\otimes D_q^\vee.$$

Then

$$\delta(h\omega) = \sum_{q=1}^{r} (hs_q + D_q(h))\omega\otimes D_q^\vee \text{ for every } h\in k.$$

A relation $\sum_i c_i\delta(h_i\omega)=0$ in $H^n(X,\mathcal{O}_X)\otimes_k(\ker\ \nabla_1)^\vee$ is therefore equivalent to

$$(*) \qquad \sum_i (c_ih_is_q + c_iD_q(h_i)) = 0 \text{ for } q=1,\dots,r.$$

By definition, $\bar{P}(A,\mathfrak{m})$ is the subgroup of $H^n(X,\mathcal{O}_X)\otimes_k\Omega^1_{A,\mathfrak{m}/k'}$ which is generated by the elements $\sum_i h_i\omega\otimes dc_it$ with $h_i, c_i \in k$, $t\in\mathfrak{m}$, satisfying the relation (*). For such a generator we calculate

$$\begin{aligned}\sum_i h_i\omega\otimes dc_it &= \sum_i \omega\otimes dh_ic_it - \sum_i \omega\otimes c_itdh_i\\ &= \sum_i \omega\otimes dh_ic_it - \sum_{i,q} \omega\otimes c_itD_q(h_i)D_q^\vee\\ &= \sum_i \omega\otimes[dh_ic_it + h_ic_it\sum_{q=1}^{r} s_qD_q^\vee].\end{aligned}$$

The assumption (3.13) has been used on the second line of this computation to allow the replacement of the form c_itdh_i by $\sum_q c_itD_q(h_i)D_q^\vee$; both forms are in $\Omega^1_{A,\mathfrak{m}/k'}$, or rather in the image of $\mathfrak{m}\otimes_k\Omega^1_{k/k'}$. The above calculation shows that every generator of $\bar{P}(A,\mathfrak{m})$ can be written as a sum of elements of the form

$$\# \qquad \omega\otimes[dt + t\sum_{q=1}^{r} s_qD_q^\vee]$$

with $t\in\mathfrak{m}$ variable and $\omega, s_q, D_q^\vee$ fixed as before.

Now look at # mod $\bar{P}(A,\mathfrak{m})$, which is an element of $\mathscr{E}(A,\mathfrak{m})$. The image of this element in $H^n(X,\hat{\mathscr{K}}_{2,X})$ is clearly the same as the image of $\omega\otimes dt + d\omega\otimes t$, as one can see by using (3.15) and the definition of the map δ in (3.12). Since $\omega\otimes dt + d\omega\otimes t$ is an element of $I(A,\mathfrak{m})$, this means that this image is zero (cf. theorem 3). This implies by theorem 4 and (3.14) that # is in $\bar{P}(A,\mathfrak{m})$. Thus we have shown that $\bar{P}(A,\mathfrak{m})$ is actually generated by the elements of the form #.

Before stating our final conclusion as theorem 5 we give another interpretation of the elements s_q which were defined in (3.15). From (3.8) and (3.9) one sees:

(3.16) $D \in \ker \nabla_1 = \ker \nabla_n \Leftrightarrow \nabla(D)F^1 \subset F^1 \Leftrightarrow \nabla(D)F^n \subset F^n.$

The first of these equivalences implies that for $D \in \ker \nabla_1$ the following diagram is commutative

$$\begin{array}{ccccc} H^n_{DR}(X/k) & \xrightarrow{\nabla} & H^n_{DR}(X/k)\otimes_k \Omega^1_{k/\mathbb{Q}} & \xrightarrow{1\otimes D} & H^n_{DR}(X/k) \\ & & \nabla(D) & & \\ \downarrow & & & & \downarrow \\ H^n(X, \mathcal{O}_X) & \xrightarrow{\delta} & H^n(X, \mathcal{O}_X)\otimes_k \Omega^1_{k/k'} & \xrightarrow{1\otimes D} & H^n(X, \mathcal{O}_X) \end{array}$$

In particular, if $\tilde{\omega} \in H^n_{DR}(X/k)$ lifts ω, then

$$\nabla(D_q)\tilde{\omega} \bmod F^1 = s_q\omega \text{ for } q = 1, \ldots, r.$$

Let $\omega^\vee \in H^0(X, \Omega^n_{X/k}) = F^n$ be dual to $\tilde{\omega}$ and to ω under the standard pairing on $H^n_{DR}(X/k)$ and Serre duality respectively. Recall that $F^1 = F^{n\perp}$ for the standard pairing, and that therefore $\langle \omega^\vee, \nabla(D_q)\tilde{\omega}\rangle$ depends only on the class of $\nabla(D_q)\tilde{\omega} \bmod F^1$. Thus we get

$$\langle \omega^\vee, \nabla(D_q)\tilde{\omega}\rangle = \langle \omega^\vee, s_q\omega\rangle = s_q.$$

On the other hand

$$\langle \omega^\vee, \nabla(D_q)\tilde{\omega}\rangle = -\langle \nabla(D_q)\omega^\vee, \tilde{\omega}\rangle.$$

The second equivalence in (3.16) implies that $\nabla(D_q)\omega^\vee$ is a multiple of $\omega^\vee$. In view of the preceding computation we have in fact

$$\nabla(D_q)\omega^\vee = -s_q\omega^\vee.$$

This gives a nicer interpretation of s_q than the one in (3.15). Summarizing the above analysis we find

THEOREM 5. Let k be a field of finite transcendence degree over $\mathbb{Q}$. Let X be a smooth projective variety over k of dimension n and genus 1. Let

$$\nabla_n : \mathrm{Der}\,(k/\mathbb{Q}) \to \mathrm{Hom}_k(H^0(X, \Omega^n_{X/k}), H^1(X, \Omega^{n-1}_{X/k}))$$

be the map induced by the Gauss-Manin connection

$$\nabla : \mathrm{Der}\,(k/\mathbb{Q}) \to \mathrm{Hom}_{\mathbb{Q}}(H^n_{DR}(X/k), H^n_{DR}(X/k)).$$

Let $k' = \{x \in k \mid Dx = 0 \text{ for all } D \in \ker \nabla_n\}$. And assume

$$\ker \nabla_n = \mathrm{Der}\,(k/k').$$

Let $D_1, \ldots, D_r$ be a basis of $\ker \nabla_n$ and let $D_1^\vee, \ldots, D_r^\vee$ be the dual basis in $\Omega^1_{k/k'}$. Fix a non-zero n-form $\omega^\vee \in H^0(X, \Omega^n_{X/k})$.

Then the functor ker $(j : H^n(X, \hat{\mathcal{K}}_{2,X}) \to \mathcal{G})$ is naturally isomorphic to the one which assigns to an object (A, m) of $\mathscr{C}$ the group

$$\Omega^1_{A,m/k'}/\{dt + t\sum_{q=1}^{r} s_q D_q^\vee | t \in m\}$$

where

$$\Omega^1_{A,m/k'} = \ker\ (\Omega^1_{A/k'} \to \Omega^1_{k/k'})$$

and where $s_1, \ldots, s_r \in k$ are defined by $\nabla(D_q)\omega^\vee = -s_q\omega^\vee$. □

COROLLARY. Let k and k' be fields of finite transcendence degree over $\mathbb{Q}$, with $k' \subset k$. Let X' be a smooth projective variety over k' of dimension n and genus 1. Assume that the map

$$\nabla_n : \mathrm{Der}\ (k'/\mathbb{Q}) \to \mathrm{Hom}_k(H^0(X', \Omega^n_{X'/k'}), H^1(X', \Omega^{n-1}_{X'/k'}))$$

is injective. Let

$$X = X' \times_{\mathrm{Spec}\ k'} \mathrm{Spec}\ k.$$

Then the functor ker $(j : H^n(X, \hat{\mathcal{K}}_{2,X}) \to \mathcal{G})$ (for X) isomorphic to the one which assigns to $(A, m) \in \mathscr{C}$ the group

$$\Omega^1_{A,m/k'}/dm.$$

PROOF. The functoriality properties of the construction of the Gauss-Manin connection yield a commutative diagram

$$\begin{array}{ccc}
\mathrm{Der}\ (k/\mathbb{Q}) & \xrightarrow{\nabla_n} & \mathrm{Hom}_k(H^0(X, \Omega^n_{X/k}), H^1(X, \Omega^{n-1}_{X/k})) \\
\downarrow & & \| \wr \\
k \otimes_{k'} \mathrm{Der}\ (k'/\mathbb{Q}) & \xrightarrow{1 \otimes \nabla_n} & k \otimes_{k'} \mathrm{Hom}_{k'}(H^0(X', \Omega^n_{X'/k'}), H^1(X', \Omega^{n-1}_{X'/k'}))
\end{array}$$

It shows that Der $(k/k') = \ker \nabla_n$ (this is ∇_n for X). So the hypotheses of the theorem are satisfied. The functoriality properties of the construction of the Gauss-Manin connection also show that the following square is commutative

$$\begin{array}{ccc}
H^n_{DR}(X'/k') & \xrightarrow{\nabla} & H^n_{DR}(X'/k') \otimes_{k'} \Omega^1_{k'/\mathbb{Q}} \\
\downarrow & & \downarrow \\
H^n_{DR}(X/k) & \xrightarrow{\nabla} & H^n_{DR}(X/k) \otimes_k \Omega^1_{k/\mathbb{Q}}
\end{array}$$

From this one can easily conclude that for $\omega^\vee \in H^0(X', \Omega^n_{X'/k'}) \subset H^n_{DR}(X/k)$ and $D \in \ker \nabla_n$ (∇_n for X) one has $\nabla(D)\omega^\vee = 0$. So the s_q which occur in theorem 5 are all zero. □

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