Nash Equilibria in the Response Strategy of Correlated Games

1. Abstract

→ In nature and society problems arise when different interests are difficult to reconcile. These can be modeled by game theory. Most applications assume uncorrelated games, but a more detailed modeling is necessary to consider the correlations that influence the decisions of the players. The current theory for correlated games enforces the players to obey the instructions from a third party or "correlation device" to reach equilibrium, but this cannot be achieved for all initial correlations.

→ We extend here the existing framework of correlated games and find that there are other interesting and previously unknown Nash equilibria that make use of correlations to obtain the best payoff and go beyond the correlated equilibrium and mixed-strategy solutions. We look particularly at the Snowdrift Game, which we find to be describable by Ising Models in thermal equilibrium.

2. Snowdrift Game

- We study the Snowdrift Game, which has parameters $0 < s < 1$ and $1 < t < 2$. The players must choose a strategy, i.e., a probability with which they will play C and D. The players are symmetric, so the probabilities should be similar for both players.
- In the uncorrelated case, there are two pure Nash equilibria, where one player plays C and the other plays D. However, these equilibria are not reachable because the players would need to have different strategies. Another equilibrium exists, the mixed strategy Nash equilibria, that ensures that, whatever the opponent plays, a player will not have any incentive to changes their strategy. The mixed strategy is given, for both players, by $P_C^* = s/(t+s-1)$.

Player 1 \ Player 2	C	D
C	1,1	s,t
D	t,s	0,0

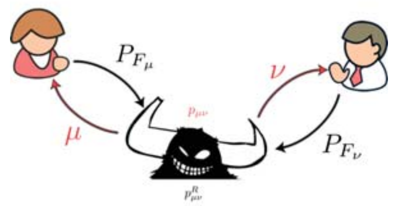
Figure 1: Payoff table for symmetric, two by two coordination games.

3. Correlated Games

- Achieving the best possible strategy requires the introduction of correlations between the players. To that end a **correlation device** is introduced, which assigns a publicly known probability $p_{\mu\nu}$ to a certain state $\mu\nu \in \{C, D\}$ and then informs each player of what they should play to achieve it.
- If the correlation is such that each player has a better payoff, given by $u^i_{\mu\nu}$ for player i in the final state $\mu\nu$, by doing what the correlation device tells them to, namely μ, than by doing anything else, $\bar{\mu}$, while the opponent always follows their instructions, the game is in a "correlated equilibrium:

$$\sum_{\nu} u^i_{\mu\nu} p_{\mu\nu} \geq \sum_{\nu} u^i_{\bar{\mu}\nu} p_{\mu\nu}. \tag{1}$$

- The existing theory predicts that they should always follow the correlation device if it allows them to reach the correlated equilibrium, and otherwise they should fall back to the uncorrelated mixed-strategy solution. This ensures that the probabilities in correlated equilibrium coincide with the final distribution of outcomes, such that they represent the actual statistics of the game. However, we want to study if there is still possible to use the correlations even if the parameters and correlation probabilities do not allow for a correlated equilibrium.



5. Nash Equilibria

- A Nash equilibrium in the response strategy is achieved if there is no incentive for player i to change the probabilities $P^i_{F\mu}$. The intuition is that equilibrium is reached when the payoff of the players cannot be improved anymore by changing their own response probabilities while keeping those of the other players fixed at the equilibrium values.
- Each slope is calculated assuming that the response probabilities of the other players are in equilibrium, such that a self-consistent solution is obtained. We find that the conditions where "always follow", i.e. $P_{F\mu} = 1$, is a stable solution corresponde to the Bayes rational conditions of the correlated equilibrium, but this is only one possible response equilibrium.
- Each renormalized set of probabilities generates a new correlated game for which the response equilibrium exactly matches a new correlated equilibrium.

4. Response Strategy

→ We now allow the players to deviate from the instructions of the correlation device in a controlled manner. The decisions to follow or not to follow the instructions become the new actions that the players can take, while they are still not able to communicate. To implement this, each player i can follow with probability $P^i_{F\mu}$, and thus not follow with probability $P^i_{NF\mu} = 1 - P^i_{F\mu}$ the instruction μ that they receive. To these we call the "response probabilities". The renormalized probability $p^R_{\mu\nu}$ that a certain final state $\mu\nu$ is reached is given by the sum over the initial probability distribution weighted by the probability that the initial states $\mu'\nu'$ gets converted to a specific final state $\mu\nu$ through the players' response. Hence

$$p^R_{\mu\nu} = \sum_{\mu',\nu'} P^1_{\mu\leftarrow\mu'} P^2_{\nu\leftarrow\nu'} p_{\mu'\nu'}, \tag{2}$$

with $P^i_{\mu\leftarrow\mu'}$ the probability that player i is told to play μ' but plays μ. As an example, the probability that the final state is CC is now

$$p^R_{CC} = P^1_{FC}P^2_{FC}p_{CC} + P^1_{FC}P^2_{NFD}p_{CD} + P^1_{NFD}P^2_{FC}p_{DC} + P^1_{NFD}P^2_{NFD}p_{DD}. \tag{3}$$

→ The expected payoff of a player is given by the payoffs averaged over the renormalized probabilities, which depends linearly on the response probabilities of that player as

$$\langle u^i \rangle = \sum_{\mu,\nu} u^i_{\mu\nu} p^R_{\mu\nu} = C_C P^i_{FC} + C_D P^i_{FD} + C_E. \tag{4}$$

Here the coefficients C_C, C_D and C_E depend linearly on the initial correlation probabilities and on the response probabilities of the other player.

→ If the players choose the response probabilities such that they do not want to change them anymore, then the renormalized probability will be in correlated equilibrium. To this effects, we calculate the equilibria when the strategy consists of always following C and sometimes follow D, for example, by solving the following system of equations:

$$\begin{cases} C_C\left(p_{CC}, p_{DD}, 1, P^{1*}_{FD}\right) > 0, & \text{(5a)} \\ C_D\left(p_{CC}, p_{DD}, 1, P^{1*}_{FD}\right) = 0. & \text{(5b)} \end{cases}$$

6. Results

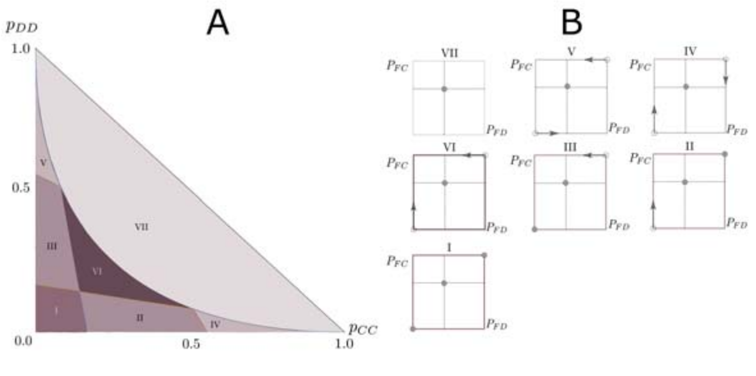


Figure 2: In **A** we see the regions that contain different equilibria, which are schematic in **B**, as a function of the initial correlations for the Snowdrift Game with parameters $s = 0.5$ and $t = 1.2$. Regions I and II contain the correlated equilibrium, but we can see that not only is it not the only equilibrium in those regions, other regions also have equilibria that make use of the correlations. As indicated by the central dot in all figures of B, all regions have the mixed strategy as an equilibrium. A dot in a corner indicates an equilibrium given by a probability of 0 or 1 of always following the correlation device, and an arrow indicates the existence of an equilibrium of which the value of the probability of following one of the instructions will vary between one extreme of the interval and the mixed strategy value.

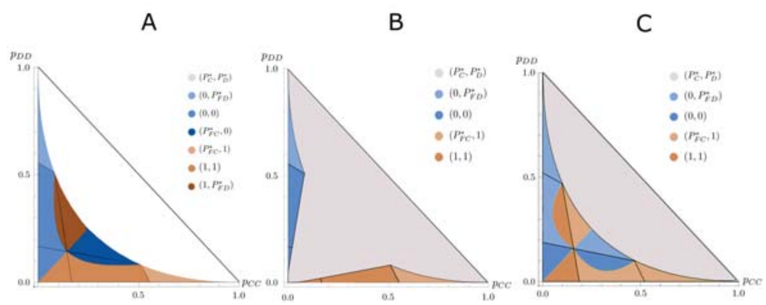


Figure 3: Best payoffs when comparing between the possible equilibria in each region. **A** and **B** correspond to the Snowdrift Game with parameter $s = 0.5$ and $t = 1.2$, but the first does not include the payoff of the mixed strategy. **C** shows the best payoffs for the same game with parameters $s = 0.23$ and $t = 1.5$. We see that for $s < t - 1$ and $s > t - 1$ there is a qualitative difference between which equilibria exist and, within these, which represent the best payoff.

7. Ising Model

- We can map the previous results into an Ising Model, where playing C or D correspond to having a spin up or down, and the correlations can be described as a Boltzman weight with an associate energy, since the renormalized probabilities definitively describe the final statistics. If the initial probabilities are given by

$$p_{\mu\nu} = \frac{e^{-\beta H_{\mu\nu}}}{Z} \tag{6}$$

and each response probability is given by

$$P^i_{\mu\leftarrow\mu'} = \frac{e^{-\beta B^i_{\mu\leftarrow\mu'}}}{Z^i_{\mu'}}, \tag{7}$$

we can rewrite the renormalized probabilities as

$$p^R_{\mu\nu} = \frac{e^{-\beta H^R_{\mu\nu}}}{Z^R}. \tag{8}$$

with associated energy

$$H^R_{\mu\nu} = -\frac{1}{\beta} \ln\left(\sum_{\mu'\nu'} Z^1_{\bar{\mu}'} Z^2_{\bar{\nu}'} e^{-\beta\left(B^1_{\mu\leftarrow\mu'} + B^2_{\nu\leftarrow\nu'} + H_{\mu'\nu'}\right)}\right), \tag{9}$$

. The partition functions associated with each probability are given, respectively, by Z, $Z^i_{\mu'}$ and Z^R.

- This mapping allows us to start tackling the problem of games played on a network, as the behavior of players in a regular grid can be studied with the tools of statistical physics. It remains an interesting open question how well this model describes the non-local effects.

8. Discussion and Conclusions

- The correlated equilibrium is only a particular response equilibrium and other Nash equilibria exist. These new equilibria renormalize to a correlated equilibrium even if the initial game is out of correlated equilibrium, showing that the players even then can still use the correlations to achieve a better payoff.
- The extra information in the correlations is two-fold: either the final distributions of outcomes informs us about an underlying correlation structure, or the players can independently improve on externally imposed initial correlations, motivated by stability and payoff maximization.
- Possible applications involve the study of new equilibria in Evolutionary Game Theory and modeling emergent behavior when games are played on networks. While all the related research relies on numerical methods, our approach may provide some analytical insight to the results.
- Bridging the gap between correlated and uncorrelated games will also prove useful to better model decision-making in economics, since the response probabilities allow us to include interactions between agents that influence their decisions.

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