Comment on the paper by Kirkinis on the renormalization method

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Abstract

In this note we draw attention to the literature on the comparison of various perturbation methods. We present examples where the anticipation of timescales is difficult and multiple timing may be deficient.

1 Introduction

The paper [3] by E. Kirkinis discusses the renormalization method with the aim of increasing its visibility in the applied mathematics curriculum. The discussion of various perturbation methods can be useful providing a comparison of methods, but it is then very important to analyze the role of timescales and to give precise error estimates. In this respect, the paper [3] has to be supplemented.

Considering the scientific literature, one observes that the use of asymptotic series to approximate solutions of differential equations takes all kind of different forms: averaging, multiple timing, renormalization, WKBJ etc. The choice of a particular method seems to be a matter of taste and local customs. This adds to the widespread opinion that perturbation theory or even applied mathematics is 'a bag of tricks' and not an accomplished and respectable part of mathematics. This opinion, however, needs revision. There exists now a wealth of knowledge on the use and meaning of perturbation methods. In this respect it has been very important to have comparative and unifying studies as [7], [4], [5], [8] and [2], to name a few. A basic aspect of the discussion is of course that there is some freedom in using perturbation methods as asymptotic expansions are not unique.

The relation between averaging, multiple timing and the renormalization method was discussed in [1], [2] and [4]. In [7], the equivalence of the averaging method and multiple timing was established for standard equations like

$$\dot{x} = \varepsilon f(t, x)$$

on intervals of time of order $1/\varepsilon$. This was a major step forward. See also the extensive discussions in [5] and [8].

Asymptotic equivalence of methods would imply that, considering a solution of a differential equation $x(t)$, expressions $\bar{x}_1(t)$ and $\bar{x}_2(t)$ obtained by different methods, would both represent an approximation of $x(t)$ with error $\delta(\varepsilon) = o(1)$ as $\varepsilon \to 0$ on the same interval of time (for instance of size $1/\varepsilon$).
Often, a clear statement of equivalence of methods is lacking; this is the case in [3] but even in [1] and [2]. As we will show, without these specifications, statements that multiple timing and averaging or renormalization are equivalent are too vague and in general not correct.

2 The origin of timescales t and εt

Many small ε parameter problems are studied using timescales like $t, \varepsilon t$, $\varepsilon^2 t$ and in general $\varepsilon^n t$ with $n \in \mathbb{N}$. Where do the timescales come from? Consider a nonlinear equation in the form

$$\dot{x} = f(x, t, \varepsilon) = f_0(x, t) + \varepsilon f_1(x, t) + \varepsilon^2 \ldots,$$

As this is supposed to be a perturbation problem we should be able to solve the 'unperturbed' problem

$$\dot{x}_0 = f_0(x_0, t),$$

to obtain $x_0(t) = \phi(t, C)$ where C is an n-dimensional constant of integration. Apply variation of constants (Lagrange)

$$x = \phi(t, y).$$

For y we obtain the equation:

$$\dot{y} = \varepsilon g(y, t) + \varepsilon^2 \ldots,$$

a so-called slowly-varying system. Perturbation methods produces from the equation for y a transformed (normal form) equation:

$$\dot{\bar{y}} = \varepsilon \bar{g}(\bar{y}) + \varepsilon^2 \ldots,$$

a slowly varying equation for $\bar{y}$. We have transformed $x \to y \to \bar{y}$ but until this point, no approximation has been applied. Omitting the $O(\varepsilon^2)$ terms to start the approximation process, and dividing the equation for $\bar{y}$ by ε, we note that the 'natural' first order timescale for $\bar{y}$ is $\tau = \varepsilon t$.
Because of the transformation $x \to y$, t is the zero order timescale for the original perturbation problem in x and so we have the timescales $t, \varepsilon t$. If in the normal form $\bar{g}$ vanishes, $\varepsilon^2 t$ might be the next 'natural' timescale.
The only assumptions until now are smoothness of the vector functions on a suitable domain and the possibility of inversion of the variation of constants relations.

3 Algebraic timescales

A basic problem of multiple timing is that one anticipates the timescales that rule the solutions. Such a guess can be correct for simple or even more complex, but well-understood problems. However, for most research problems the anticipation of timescales is an unnecessary and dangerous restriction. We discuss briefly two classes of problems where multiple timing may be deficient and for instance averaging gives the correct result.

- In bifurcation problems one encounters, after linearization, structural stability problems of matrices, this is characteristic for such problems. A $n \times n$ matrix is called *structurally stable* if it is nonsingular and all eigenvalues have nonzero real part. If we have a zero eigenvalue or purely imaginary eigenvalues, we can expect bifurcations. Apart from this, the presence of multiple eigenvalues affects the form of the expansions and the timescales. In such cases unexpected algebraic timescales can not be avoided. A simple example is the Mathieu-equation, discussed in detail in [1] and [10]:

$$\ddot{x} + (1 + \varepsilon a + \varepsilon^2 b + \varepsilon \cos 2t)x = 0, \tag{1}$$

  a and b are free parameters independent of ε. To first order in ε, the instability tongue is found for $a^2 = 1/4$. Choosing the boundary of the tongue for $a = 1/2$, we find to second order $b = 1/32$. The timescales characterising the flow near the Floquet tongue are to second order

$$t,\ \varepsilon t,\ \varepsilon^{\frac{3}{2}} t,\ \varepsilon^2 t.$$

- Problems with resonance manifolds may arise in systems of the form

$$\begin{aligned} \dot{x} &= \varepsilon X(x, \phi) + \varepsilon^2 \ldots, \\ \dot{\phi} &= \Omega(x) + \varepsilon \ldots \end{aligned}$$

  with x a Euclidean n-vector, ϕ an angle-vector; the order functions multiplying the righthand sides are different, but the choice of size here is just an example. Such problems arise in Hamiltonian systems and in dissipative systems; see for instance [8] or [10] and references there. Also in these problems higher order algebraic timescales and asymptotically small domains are natural. A typical example from [10], example 12.11, describes a slightly eccentric flywheel. The vertical displacement x of a small mass on the flywheel and its rotation angle ϕ are given by

$$\begin{aligned} \ddot{x} + x &= \varepsilon(-x^3 - \dot{x} + \dot{\phi}^2 \cos \phi) + O(\varepsilon^2), \\ \ddot{\phi} &= \varepsilon(\frac{1}{4}(2 - \dot{\phi}) + (1 - x) \sin \phi) + O(\varepsilon^2). \end{aligned}$$

  It turns out that there exists a resonance zone near $\dot{\phi} = 1$; the zone is of size $O(\varepsilon^{\frac{1}{2}})$, the timescale of the dynamics is $\sqrt{\varepsilon} t$.

  In Hamiltonian systems one encounters resonance zones of various sizes, algebraic in ε and with interaction on very long timescales (longer than $1/\varepsilon^2$), see [9].

Conclusion

Normal form methods, averaging and renormalization have no need to anticipate the timescales that are relevant for the approximations. The timescales present themselves naturally in the course of the analysis. Multiple timing, on the other hand, makes restricting choices of timescales but this method is safe to use if we confine the analysis to time intervals of order $1/\varepsilon$ and in general if we understand apriori the nature of the solutions. Extension of validity of approximations beyond order $1/\varepsilon$ is usually not expedient for the multiple timescale method; see the discussions in [8] and [5].

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