Dendroidal Sets
A short story

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The following is an attempt at a non-technical explanation of the motivation for developing dendroidal sets.

Algebraic structures

Modern Algebra has traditionally busied itself with the study of algebraic structures, such as groups, monoids, semi-groups, rings, etc. An algebraic structure is usually described as a set together with one or more operations, satisfying certain axioms.
While it's apparent that any approach must begin with such a definition, it's seldom obvious from the start what the right axioms should be. A correct axiomatization can give rise to significant progress, as shown by the case of the axiomatization of groups, one of the first algebraic structures to be properly axiomatized, with many fruitful results in Physics, Computer Science, and Chemistry.

Weak algebraic structures

Algebraic structures abound in many areas of Mathematics, and quite often the underlying set upon which the algebraic structure is defined has a richer structure than that of a mere set. It can be a topological space, a differentially graded vector space, or one of many other possibilities. In many such cases, such as for topological spaces, there exists a notion of sameness that's weaker than equality, called homotopy. One can then distinguish between two types of algebraic structures: In the first type, called a strict algebraic structure, the same equational relations hold as in classical set-based structures (an example would be topological groups). The second, and more illusive, type is the weak (or up-to-homotopy) algebraic structure. In the weak version the equational relations are replaced by homotopies. However, the homotopies themselves have to satisfy certain homotopical equations of their own, and this usually results in an infinite tower of homotopies, homotopies between homotopies, homotopies between homotopies between homotopies, and so on. The resulting structure is clearly very complicated.

This distinction between strict and weak algebraic structures is more than just a curiosity. In many situations in Mathematics, Theoretical Physics, and Computer Science, one encounters algebraic structures that fail to obey equational relations strictly but do obey them up-to-homotopy. Since the first step in an effective study of algebraic structures is obtaining a workable definition of the structure at hand, finding a way of describing weak algebraic structures and a process that deforms a strict one to its weak version have become important research goals. The ubiquity of algebraic structures that fail to obey strict laws also makes these problems important for the development of many areas of Mathematics, Computer Science, and Theoretical Physics. A book that surveys such structures in Algebra, Topology, and Physics is [MSS].

Operads as tools for studying weak algebraic structures

Just as in the strict case, the first step in studying a weak algebraic structure is to define it. While in the strict case this usually amounts to a set, some operations, and a relatively short list of axioms, for the weak version there is little hope of the list of axioms being short due to the extra complexity of the relations the homotopies themselves need to satisfy. Stasheff [Sta] in 1963 first considered such a problem in the topological setting. His solution consists of constructing spaces (later named the Stasheff associahedra) that parametrize all the homotopies and their intricate interrelations in a manageable way.

In 1972 May [May], in his work on the geometry of n-fold loop spaces, introduced operads. An operad is a mathematical gadget that allows for the codification of very complicated algebraic structures in a manageable way. Indeed a basic example of an operad consists of (a slight variation of) the Stasheff associahedra. Around the same time Boardman and Vogt [BoVo] introduced a structure, essentially identical to an operad, that also codifies algebraic structures. They gave a general construction that takes an operad codifying a particular algebraic structure and produces a new operad codifying the weak version of that structure in the context of topological spaces. More recently, Berger and Moerdijk [BeMo] showed how to modify this construction to apply in the general context of Abstract Homotopy Theory, not just for topological spaces.

Thus operads became an essential tool in the study of weak algebraic structures. The theory of operads went through a renaissance period in the 1990's [LSV] and applications of operads are found in more and more areas of mathematics.

Problems with the operadic approach

As stated above, operads are used to codify algebraic structures. Such an operad can be thought of as a blueprint for the construction of an algebraic structure. For example, there is an operad that codifies the presence of a single associative binary operation. An operad is only the blueprint of a structure - to actually construct an algebraic structure according to the blueprint, one needs to interpret the operad in some universe of interest. For example, interpreting the operad mentioned above in the universe of sets amounts precisely to a set upon which an associative binary operation is given. Interpreting the same operad in the universe of topological spaces amounts to a topological space with a continuous associative binary operation on it. There are, in general, many different ways to interpret an operad in a given universe, resulting in different algebraic structures.

It's a pleasant coincidence that quite often the universe in question is itself an operad. Thus operads occur in two very different roles: An operad can be a codification of an algebraic structure, or it can be a universe in which to interpret such a codification. Let us call an operad seen as a codification a 'blueprint operad' and an operad in which one wishes to interpret a codification a 'universe operad'. Thus, if one chooses a blueprint operad for a certain algebraic structure, then interpreting it in a universe operad results in an algebraic structure in that universe. Varying the universe operad and keeping the blueprint operad fixed will yield different versions of the same algebraic structure.

Operads provide a powerful setting for the study of both strict and weak algebraic structures, but they also carry some weaknesses. To understand one such weakness consider the following: Given a fixed blueprint operad and a universe operad, if one considers all the interpretations of the blueprint operad in the universe operad, then this entire collection of algebraic structures, together with their *strict* mappings, carries itself the structure of a universe operad. This property of 'closure under interpretations' is very useful. For example, it allows immediately for the definition of iterated algebraic structures: Again, fix a blueprint operad and a universe operad. Consider the new universe operad of all interpretations of the blueprint operad in the given universe operad. Now keep the same blueprint operad fixed, but interpret it in the new universe operad. The result is an algebraic structure given by the blueprint operad and doubled upon itself. This process can go on indefinitely to produce n-fold iterations of strict algebraic structures.

However, for strict algebraic structures iteration is usually not very interesting, since such iterations quickly degenerate to a steady structure. For example, Eckman-Hilton duality basically says that for the operad codifying a single associative binary operation (with unit), already the first iteration results in a single commutative, associative binary operation (with unit), as do all the higher iterations.

The situation is quite different for weak algebraic structures. The weakness alluded to above is that the theory of operads does not allow for iteration of weak algebraic structures, whereas it's precisely such weak iterated structures that are interesting and abound in areas of interest. The reason that iteration is impossible is that, where with strict algebraic structures the collection of all interpretations of the blueprint operad with their strict mappings results in a new universe operad, the same does not hold true for weak structures and their weak mappings. It is in fact extremely rare that the collection of all interpretations of a blueprint operad codifying a weak algebraic structure with their weak mappings forms a new universe operad.

Dendroidal sets

A solution to the above problem was developed in [MoWe1, MoWe2]. The solution proposes a new mathematical structure, that of a dendroidal set, which extends the notion of an operad. Just as for operads, here too one can distinguish between a blueprint dendroidal set and a universe dendroidal set, and one can consider interpretations of a blueprint in a universe. However, unlike for operads, the collection of all interpretations of a blueprint dendroidal set in a universe dendroidal set is always a new universe dendroidal set, regardless of whether the blueprint dendroidal set codifies a strict or a weak structure. This immediately implies that iteration is always possible. Moreover, dendroidal sets faithfully extend operads in the sense that, in most situations, nothing is lost by passing from operads to dendroidal sets. Thus dendroidal sets provide a better environment for the study of weak algebraic structures than that offered by operads, and at little to no cost.

References:

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