**Exercise 12.2** *If the roots of the auxiliary equation are* $k_1 > 0$ *and* $-k_2 < 0$ *then the solution is*

$$x(t) = A\mathrm{e}^{k_1 t} + B\mathrm{e}^{-k_2 t}.$$

*For most choices of initial conditions*

$$x(0) = x_0 \qquad \dot{x}(0) = y_0$$

*we will have* $x(t) \to \pm\infty$ *as* $t \to \infty$*. However, there are some special initial conditions for which* $x(t) \to 0$ *as* $t \to \infty$*. Find the relationship between* $x_0$ *and* $y_0$ *that ensures this.*

The solution is

$$x(t) = A\mathrm{e}^{k_1 t} + B\mathrm{e}^{-k_2 t};$$

this will only tend to zero as $t \to \infty$ if $A = 0$, and then the solution is $x(t) = B\mathrm{e}^{-k_2 t}$ for some $B$. In this case, since $\dot{x}(t) = -k_2 B\mathrm{e}^{-k_2 t}$, we should have

$$x(0) = B \qquad \text{and} \qquad \dot{x}(0) = -k_2 B.$$

So the solution only tends to zero if $y_0 = -k_2 x_0$.

**Exercise 12.3** *Solutions of linear equations with constant coefficients cannot blow up in finite time: it follows that their solutions exist for all* $t \in \mathbb{R}$*. To see this, we will consider*

$$\ddot{x} + p\dot{x} + qx = 0 \qquad \text{with} \qquad x(0) = x_0 \qquad \text{and} \qquad \dot{x}(0) = y_0$$

*for* $t \geq 0$ *(a similar argument applies for* $t \leq 0$*). By setting* $y = \dot{x}$*, we can rewrite this as a coupled pair of first order equations*

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -py - qx. \end{aligned}$$

*Show that*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2) = (1 - q)xy - py^2,$$

*and hence that*

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2) \leq (1 + |q| + 2|p|)(x^2 + y^2).$$

*Using the result of Exercise 9.7 deduce that for* $t \geq 0$

$$x(t)^2 + y(t)^2 \leq (x(0)^2 + y(0)^2)\mathrm{e}^{(1+|q|+2|p|)t},$$

*showing that finite-time blowup is impossible. Hint:* $xy \le \frac{1}{2}(x^2 + y^2)$. *(The same argument works, essentially unchanged, for*

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0$$

*provided that* $|p(t)| \le p$ *and* $|q(t)| \le q$ *for all* $t \in \mathbb{R}$.*)*

We have

$$\begin{aligned}
\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2) &= \tfrac{1}{2}(2x\dot{x} + 2y\dot{y}) \\
&= x\dot{x} + y\dot{y} \\
&= xy + y(-py - qx) \\
&= (1 - q)xy - py^2.
\end{aligned}$$

Now it follows (using the hint) that

$$\begin{aligned}
\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2) &\le (1 + |q|)\frac{x^2 + y^2}{2} + |p|y^2 \\
&\le \left(\frac{1 + |q|}{2} + |p|\right)(x^2 + y^2),
\end{aligned}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2) \le (1 + |q| + 2|p|)(x^2 + y^2).$$

It follows using the result of Exercise 9.7 that

$$x(t)^2 + y(t)^2 \le (x(0)^2 + y(0)^2)\mathrm{e}^{(1+|q|+2|p|)t},$$

and so $x(t)$ and $\dot{x}(t)$ remain bounded for all $t \ge 0$.