12

Homogeneous linear 2nd order equations with constant coefficients

Exercise 12.1 *Find the general solution of the following differential equations, and then the solution satisfying the specified initial conditions.*

(i) $\ddot{x} - 3\dot{x} + 2x = 0$ *with* $x(0) = 2$ *and* $\dot{x}(0) = 6$*;*
(ii) $y'' - 4y' + 4y = 0$ *with* $y(0) = 0$ *and* $y'(0) = 3$*;*
(iii) $z'' - 4z' + 13z = 0$ *with* $z(0) = 7$ *and* $z'(0) = 42$*;*
(iv) $\ddot{y} + \dot{y} - 6y = 0$ *with* $y(0) = -1$ *and* $\dot{y}(0) = 8$*;*
(v) $\ddot{y} - 4\dot{y} = 0$ *with* $y(0) = 13$ *and* $\dot{y}(0) = 0$*;*
(vi) $\ddot{\theta} + 4\theta = 0$ *with* $\theta(0) = 0$ *and* $\dot{\theta}(0) = 10$*;*
(vii) $\ddot{y} + 2\dot{y} + 10y = 0$ *with* $y(0) = 3$ *and* $\dot{y}(0) = 0$*;*
(viii) $2\ddot{z} + 7\dot{z} - 4z = 0$ *with* $z(0) = 0$ *and* $\dot{z}(0) = 9$*;*
(ix) $\ddot{y} + 2\dot{y} + y = 0$ *with* $y(0) = 0$ *and* $\dot{y}(0) = -1$*;*
(x) $\ddot{x} + 6\dot{x} + 10x = 0$ *with* $x(0) = 3$ *and* $\dot{x}(0) = 1$*;*
(xi) $4\ddot{x} - 20\dot{x} + 21x = 0$ *with* $x(0) = -4$ *and* $\dot{x}(0) = -12$*;*
(xii) $\ddot{y} + \dot{y} - 2y = 0$ *with* $y(0) = 4$ *and* $\dot{y}(0) = -4$*;*
(xiii) $\ddot{y} - 4y = 0$ *with* $y(0) = 10$ *and* $\dot{y}(0) = 0$*;*
(xiv) $y'' + 4y' + 4y = 0$ *with* $y(0) = 27$ *and* $y'(0) = -54$*; and*
(xv) $\ddot{y} + \omega^2 y = 0$ *with* $y(0) = 0$ *and* $\dot{y}(0) = 1$.

(i)

$$\ddot{x} - 3\dot{x} + 2x = 0 \qquad \text{with} \qquad x(0) = 2 \quad \text{and} \quad \dot{x}(0) = 6.$$

Try $x = \mathrm{e}^{kt}$ and obtain the auxiliary equation for k,

$$k^2 - 3k + 2 = 0$$

with solutions $k = 2$ and $k = 4$. The general solution is therefore

$$x(t) = A\mathrm{e}^{2t} + B\mathrm{e}^{4t}.$$

Since $\dot{x}(t) = 2A\mathrm{e}^{2t} + 4B\mathrm{e}^{4t}$ the particular solution with

$$x(0) = A + B = 2 \qquad \text{and} \qquad \dot{x}(0) = 2A + 4B = 6$$

has $A = B = 1$ and is

$$x(t) = \mathrm{e}^{2t} + \mathrm{e}^{4t}.$$

(ii)

$$y'' - 4y' + 4y = 0 \qquad \text{with} \qquad y(0) = 0 \quad \text{and} \quad y'(0) = 3.$$

Try $y(x) = \mathrm{e}^{kx}$ to obtain the auxiliary equation

$$k^2 - 4k + 4 = 0$$

which has $k = 4$ as a repeated root. The general solution is therefore

$$y(x) = A\mathrm{e}^{4x} + Bx\mathrm{e}^{4x}.$$

Since $y'(x) = 4A\mathrm{e}^{4x} + B[\mathrm{e}^{4x} + 4x\mathrm{e}^{4x}]$ the solution with

$$y(0) = A + B = 0 \qquad \text{and} \qquad y'(0) = 4A + B = 3$$

has $A = 1$ and $B = -1$: it is

$$y(x) = (1 - x)\mathrm{e}^{4x}.$$

(iii)

$$z'' - 4z' + 13z = 0 \qquad \text{with} \qquad z(0) = 7 \quad \text{and} \quad z'(0) = 42.$$

Try $z(x) = \mathrm{e}^{kx}$ to obtain

$$k^2 - 4k + 13 = 0,$$

with solutions $k = 4 \pm 6\mathrm{i}$. The general solution is therefore

$$z(x) = \mathrm{e}^{4x}(A\cos 6x + B\sin 6x).$$

We have

$$\begin{aligned} z'(x) &= \mathrm{e}^{4x}(4A\cos 6x + 4B\sin 6x - 6A\sin 6x + 6B\cos 6x) \\ &= \mathrm{e}^{4x}[(4A + 6B)\cos 6x + (4B - 6A)\sin 6x], \end{aligned}$$

and so the solution with

$$z(0) = A + B = 7 \qquad \text{and} \qquad z'(0) = 4A + 6B = 42$$

has $A = 0$ and $B = 7$, and is

$$z(x) = 7\mathrm{e}^{4x}\sin 6x.$$

(iv)

$$\ddot{y} + \dot{y} - 6y = 0 \qquad \text{with} \qquad y(0) = -1 \quad \text{and} \quad \dot{y}(0) = 8.$$

Trying $y(t) = \mathrm{e}^{kt}$ gives the auxiliary equation

$$k^2 + k - 6 = 0$$

with solutions $k = 2$ or $k = -3$. So the general solution is

$$y(t) = A\mathrm{e}^{2t} + B\mathrm{e}^{-3t}.$$

Since $\dot{y}(t) = 2A\mathrm{e}^{2t} - 3B\mathrm{e}^{-3t}$ the solution with

$$y(0) = A + B = -1 \qquad \text{and} \qquad \dot{y}(0) = 2A - 3B = 8$$

has $A = 1$ and $B = -2$; it is

$$y(t) = \mathrm{e}^{2t} - 2\mathrm{e}^{-3t}.$$

(v)

$$\ddot{y} - 4y = 0 \qquad \text{with} \qquad y(0) = 13 \quad \text{and} \quad \dot{y}(0) = 0.$$

We try $y(t) = \mathrm{e}^{kt}$ then

$$k^2 - 4k = 0$$

with solutions $k = 0$ or $k = 4$. The general solution is

$$y(t) = A + B\mathrm{e}^{4t}.$$

We have $\dot{y}(t) = 4B\mathrm{e}^{4t}$. The initial conditions

$$y(0) = A + B = 13 \qquad \text{and} \qquad \dot{y}(0) = 4B = 0$$

imply that $A = 13$ and $B = 0$, so the solution is

$$y(t) = 13.$$

(vi)

$$\ddot{\theta} + 4\theta = 0 \qquad \text{with} \qquad \theta(0) = 0 \quad \text{and} \quad \dot{\theta}(0) = 10.$$

With $\theta(t) = \mathrm{e}^{kt}$ we have

$$k^2 + 4 = 0,$$

and so $k = \pm 2\mathrm{i}$. The general solution is

$$\theta(t) = A\cos 2t + B\sin 2t.$$

The derivative is given by $\dot{\theta} = -2A\sin 2t + 2B\cos 2t$. In order to satisfy

$$\theta(0) = A = 0 \qquad \text{and} \qquad \dot{\theta}(0) = 2B = 10$$

we must have $A = 0$ and $B = 5$, and the solution is

$$\theta(t) = 5\sin 2t.$$

(vii)

$$\ddot{y} + 2\dot{y} + 10y = 0 \qquad \text{with} \qquad y(0) = 3 \quad \text{and} \quad \dot{y}(0) = 0.$$

Try $y(t) = \mathrm{e}^{kt}$ and then

$$k^2 + 2k + 10 = 0$$

giving the complex conjugate roots $k = -1 \pm 3\mathrm{i}$ and the general solution

$$y(t) = \mathrm{e}^{-t}(A\cos 3t + B\sin 3t).$$

We have

$$\begin{aligned}\dot{y}(t) &= \mathrm{e}^{-t}(-A\cos 3t - B\sin 3t - 3A\sin 3t + 3B\cos 3t)\\ &= \mathrm{e}^{-t}[(3B - A)\cos 3t - (3A + B)\sin 3t].\end{aligned}$$

The solution with

$$y(0) = A = 3 \qquad \text{and} \qquad \dot{y}(0) = 3B - A = 0$$

has $A = 3$ and $B = 1$, and is

$$y(t) = \mathrm{e}^{-t}(3\cos 3t + \sin 3t).$$

(viii)

$$2\ddot{z} + 7\dot{z} - 4z = 0 \qquad \text{with} \qquad z(0) = 0 \quad \text{and} \quad \dot{z}(0) = 9.$$

When we substitute $z(t) = \mathrm{e}^{kt}$ we obtain the equation

$$2k^2 + 7k - 4 = 0$$

for k. This equation has roots $k = 1/2$ and $k = -4$, so the general solution is

$$z(t) = A\mathrm{e}^{t/2} + B\mathrm{e}^{-4t}.$$

We have $\dot{z}(t) = (A/2)\mathrm{e}^{t/2} - 4B\mathrm{e}^{-4t}$, and so when

$$z(0) = A + B = 0 \qquad \text{and} \qquad \dot{z}(0) = (A/2) - 4B = 9$$

$A=2$ and $B=-2$; the solution is

$$z(t)=2(\mathrm{e}^{t/2}-\mathrm{e}^{-4t}).$$

(ix)

$$\ddot{y}+2\dot{y}+y=0 \qquad \text{with} \qquad y(0)=0 \quad \text{and} \quad \dot{y}(0)=-1.$$

With $y(t)=\mathrm{e}^{kt}$ we obtain

$$k^2+2k+1=0$$

and so $k=-1$ is a repeated root, giving the general solution

$$y(t)=A\mathrm{e}^{-t}+Bt\mathrm{e}^{-t}.$$

The derivative is $\dot{y}(t)=-A\mathrm{e}^{-t}+B(\mathrm{e}^{-t}-t\mathrm{e}^{-t})$, and so the solution with

$$y(0)=A=0 \qquad \text{and} \qquad \dot{y}(0)=-A+B=-1$$

has $A=0$ and $B=-1$: it is

$$y(t)=-t\mathrm{e}^{-t}.$$

(x)

$$\ddot{x}+6\dot{x}+10x=0 \qquad \text{with} \qquad x(0)=3 \quad \text{and} \quad \dot{x}(0)=1.$$

We try $x(t)=\mathrm{e}^{kt}$ and then

$$k^2+6k+10=0;$$

this implies that $k=-3\pm\mathrm{i}$, and gives the general solution

$$x(t)=\mathrm{e}^{-3t}(A\cos t+B\sin t).$$

We have

$$\begin{aligned}\dot{x}(t) &= \mathrm{e}^{-3t}(-3A\cos t-3B\sin t-A\sin t+B\cos t)\\ &= \mathrm{e}^{-3t}[(B-3A)\cos t-(A+3B)\sin t].\end{aligned}$$

The solution with

$$x(0)=A=3 \qquad \text{and} \qquad \dot{x}(0)=B-3A=1$$

has $A=3$ and $B=10$:

$$x(t)=\mathrm{e}^{-3t}(3\cos t+10\sin t).$$

(xi)

$$4\ddot{x} - 20\dot{x} + 21x = 0 \qquad \text{with} \qquad x(0) = -4 \quad \text{and} \quad \dot{x}(0) = -12.$$

Substituting $x(t) = \mathrm{e}^{kt}$ produces the auxiliary equation

$$4k^2 - 20k + 21 = 0$$

with roots $k = 3/2$ and $k = 7/2$. The general solution is

$$x(t) = A\mathrm{e}^{3t/2} + B\mathrm{e}^{7t/2}.$$

We have $\dot{x}(t) = \frac{1}{2}[3A\mathrm{e}^{3t/2} + 7B\mathrm{e}^{7t/2}]$, and so for

$$x(0) = A + B = -4 \qquad \text{and} \qquad \dot{x}(0) = \frac{3A + 7B}{2} = -12$$

we need $A = -1$ and $B = -3$ and the solution is

$$x(t) = -\mathrm{e}^{3t/2} - 3\mathrm{e}^{7t/2}.$$

(xii)

$$\ddot{y} + \dot{y} - 2y = 0 \qquad \text{with} \qquad y(0) = 4 \quad \text{and} \quad \dot{y}(0) = -4.$$

Trying $y(t) = \mathrm{e}^{kt}$ gives the equation for k

$$k^2 - k - 2 = 0$$

with roots $k = 2$ and $k = -1$. The general solution is

$$y(t) = A\mathrm{e}^{2t} + B\mathrm{e}^{-t}.$$

With $\dot{y}(t) = 2A\mathrm{e}^{2t} - B\mathrm{e}^{-t}$ we need

$$y(0) = A + B = 4 \qquad \text{and} \qquad \dot{y}(0) = 2A - B = -4$$

i.e. $A = 0$ and $B = 4$, and so

$$y(t) = 4\mathrm{e}^{-t}.$$

(xiii)

$$\ddot{y} - 4y = 0 \qquad \text{with} \qquad y(0) = 10 \quad \text{and} \quad \dot{y}(0) = 0.$$

Try $y(t) = \mathrm{e}^{kt}$ and then

$$k^2 - 4 = 0$$

which gives $k = -2$ or $k = 2$ and the general solution

$$y(t) = A\mathrm{e}^{2t} + B\mathrm{e}^{-2t}.$$

The derivative is given by $\dot{y}(t) = 2A\mathrm{e}^{2t} - 2B\mathrm{e}^{-2t}$, and so for

$$y(0) = A + B = 10 \qquad \text{and} \qquad \dot{y}(0) = 2A - 2B = 0$$

we need $A = B = 5$ and the solution

$$y(t) = 5(\mathrm{e}^{2t} + \mathrm{e}^{-2t}).$$

(xiv)

$$y'' + 4y' + 4y = 0 \qquad \text{with} \qquad y(0) = 27 \quad \text{and} \quad y'(0) = -54.$$

Try $y(x) = \mathrm{e}^{kx}$ and obtain

$$k^2 + 4k + 4 = 0$$

with $k = -2$ a repeated root. The general solution is

$$y(x) = A\mathrm{e}^{-2x} + Bx\mathrm{e}^{-2x}.$$

We have $y'(x) = -2A\mathrm{e}^{-2x} + B(\mathrm{e}^{-2x} - 2x\mathrm{e}^{-2x})$, and so to match the initial conditions we need

$$y(0) = A = 27 \qquad \text{and} \qquad y'(0) = -2A + B = -54,$$

i.e. $A = 27$ and $B = 0$. The solution is

$$y(x) = 27\mathrm{e}^{-2x}.$$

(xv)

$$\ddot{y} + \omega^2 y = 0 \qquad \text{with} \qquad y(0) = 0 \quad \text{and} \quad \dot{y}(0) = 1.$$

Try $y(t) = \mathrm{e}^{kt}$ and the k must satisfy

$$k^2 + \omega^2 = 0,$$

so $k = \pm\mathrm{i}\omega$, and we obtain the general solution

$$y(t) = A\cos\omega t + B\sin\omega t.$$

Since $\dot{y}(t) = -A\omega\sin\omega t + B\omega\cos\omega t$ we need

$$y(0) = A = 0 \qquad \text{and} \qquad \dot{y}(0) = -A\omega + B\omega = 1$$

which gives $A = 0$ and $B = 1/\omega$; the solution we require is

$$y(t) = \frac{\sin\omega t}{\omega}.$$

Exercise 12.2 *If the roots of the auxiliary equation are* $k_1 > 0$ *and* $-k_2 < 0$ *then the solution is*

$$x(t) = A\mathrm{e}^{k_1 t} + B\mathrm{e}^{-k_2 t}.$$

For most choices of initial conditions

$$x(0) = x_0 \qquad \dot{x}(0) = y_0$$

we will have $x(t) \to \pm\infty$ *as* $t \to \infty$*. However, there are some special initial conditions for which* $x(t) \to 0$ *as* $t \to \infty$*. Find the relationship between* x_0 *and* y_0 *that ensures this.*

The solution is

$$x(t) = A\mathrm{e}^{k_1 t} + B\mathrm{e}^{-k_2 t};$$

this will only tend to zero as $t \to \infty$ if $A = 0$, and then the solution is $x(t) = B\mathrm{e}^{-k_2 t}$ for some B. In this case, since $\dot{x}(t) = -k_2 B\mathrm{e}^{-k_2 t}$, we should have

$$x(0) = B \qquad \text{and} \qquad \dot{x}(0) = -k_2 B.$$

So the solution only tends to zero if $y_0 = -k_2 x_0$.

Exercise 12.3 *Solutions of linear equations with constant coefficients cannot blow up in finite time: it follows that their solutions exist for all* $t \in \mathbb{R}$*. To see this, we will consider*

$$\ddot{x} + p\dot{x} + qx = 0 \qquad \text{with} \qquad x(0) = x_0 \qquad \text{and} \qquad \dot{x}(0) = y_0$$

for $t \geq 0$ *(a similar argument applies for* $t \leq 0$*). By setting* $y = \dot{x}$*, we can rewrite this as a coupled pair of first order equations*

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -py - qx. \end{aligned}$$

Show that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2) = (1 - q)xy - py^2,$$

and hence that

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2) \leq (1 + |q| + 2|p|)(x^2 + y^2).$$

Using the result of Exercise 9.7 deduce that for $t \geq 0$

$$x(t)^2 + y(t)^2 \leq (x(0)^2 + y(0)^2)\mathrm{e}^{(1+|q|+2|p|)t},$$

showing that finite-time blowup is impossible. Hint: $xy \leq \frac{1}{2}(x^2 + y^2)$. *(The same argument works, essentially unchanged, for*

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0$$

provided that $|p(t)| \leq p$ *and* $|q(t)| \leq q$ *for all* $t \in \mathbb{R}$*.)*

We have

$$\begin{aligned} \tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2) &= \tfrac{1}{2}(2x\dot{x} + 2y\dot{y}) \\ &= x\dot{x} + y\dot{y} \\ &= xy + y(-py - qx) \\ &= (1-q)xy - py^2. \end{aligned}$$

Now it follows (using the hint) that

$$\begin{aligned} \tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2) &\leq (1+|q|)\frac{x^2+y^2}{2} + |p|y^2 \\ &\leq \left(\frac{1+|q|}{2} + |p|\right)(x^2+y^2), \end{aligned}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2) \leq (1 + |q| + 2|p|)(x^2 + y^2).$$

It follows using the result of Exercise 9.7 that

$$x(t)^2 + y(t)^2 \leq (x(0)^2 + y(0)^2)\mathrm{e}^{(1+|q|+2|p|)t},$$

and so $x(t)$ and $\dot{x}(t)$ remain bounded for all $t \geq 0$.

13

Oscillations

Exercise 13.1 *A spring of natural length l and spring constant k is suspended vertically from a fixed point, and a weight of mass m attached. If the system is at rest ($\ddot{x} = \dot{x} = 0$) how far has the spring extended? If the mass is pulled down slightly from this rest position and then released, show that it then oscillates about its equilibrium position with period $2\pi/\omega$, where $\omega^2 = k/m$.*

Denoting by x the extension from the spring's natural length, then the forces on the spring are that due to the weight of the mass, mg, and the restoring force $-kx$ from the spring, so

$$m\ddot{x} = -kx + mg.$$

When the spring is at rest we have $\ddot{x} = 0$ and so $x = mg/k$. If we write $x = y + (mg/k)$ then y satisfies

$$m\ddot{y} = -ky.$$

The solution of this can either be found by trying $y(t) = \mathrm{e}^{\alpha t}$ which gives $m\alpha^2 = -k$ and so $\alpha = \pm\mathrm{i}\sqrt{k/m}$; or preferably you should just recognise the equation as giving rise to simple harmonic motion. Whichever way, the solution is

$$y(t) = A\cos\sqrt{k/m}\,t + B\sin\sqrt{k/m}\,t,$$

and the spring oscillates about its equilibrium position with period $2\pi/\omega$ with $\omega^2 = k/m$ as claimed.

Exercise 13.2 *The acceleration due to gravity in fact depends on the distance R from the centre of the earth: $g = GM/R^2$, where M is the mass of the earth and G Newton's gravitational constant. Show that the period of oscillation of a pendulum will increase as it is taken higher.*

The equation of motion for a pendulum is (cf. (13.5))

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\omega^2\theta \qquad \text{with} \qquad \omega^2 = \frac{g}{L}.$$

The period of oscillation is $2\pi/\omega$, i.e.

$$2\pi\sqrt{\frac{L}{g}}.$$

Since $g = GM/R^2$ decreases as R increases, it follows that the period will increase as the pendulum moves further from the centre of the earth.

Exercise 13.3 *The earth bulges at the equator: at a latitude θ, the distance to the centre of the earth (measured in kilometres) is approximately*

$$R(\theta) = \sqrt{R_\mathrm{e}^2\cos^2\theta + R_\mathrm{p}^2\sin\theta},$$

where $R_\mathrm{e} = 6378$ and $R_\mathrm{p} = 6357$.

I decide to move from Leamington Spa, at a latitude of $52°$, to Seville, which is lies at a latitude of $37°$. My grandfather clock, which keeps perfect time, has a pendulum of length 75 cm. How long would the pendulum need to be to keep perfect time in Seville?

The approximate distance of Leamington from the centre of the earth is 6365 km, while for Seville the Figure is roughly 6370 km. To keep the period of oscillations constant I would need to keep L/g constant. Since g is proportional to R^{-2}, this is the same as keeping LR^2 constant, i.e.

$$L_\mathrm{Leam}R_\mathrm{Leam}^2 = L_\mathrm{Sev}R_\mathrm{Sev}^2,$$

which gives

$$L_\mathrm{Sev} = 75 \times \frac{6365^2}{6370^2}\,\mathrm{cm} = 74.88\,\mathrm{cm}$$

a minimal adjustment.

Exercise 13.4 *The buoyancy force on an object is equal to the weight of water that it displaces: if an object has mass M and displaces a volume V of water then the forces on it are $Mg - Vg$, in units where the density of water is one; see Figure 13.1.*

A bird of mass m is sitting on a cylindrical buoy of density ρ, radius R, and height h, which is floating at rest. How much of the buoy lies below the surface?

The bird flies away. Show that the buoy now bobs up and down, with

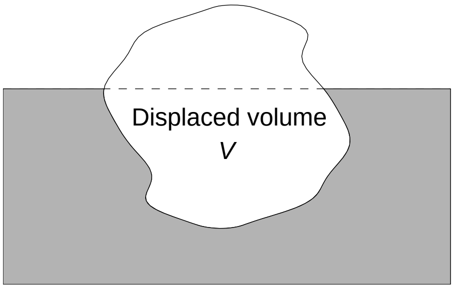

Fig. 13.1. The buoyancy force on an object is equal to the weight of water that it displaces.

the amount below the surface oscillating about ρh *with period* $2\pi\sqrt{\rho h/g}$ *and amplitude* $m/\pi R^2$*.*

Suppose that the buoy is immersed to a depth y. While the bird is there the forces on the buoy are a buoyancy force $\pi R^2 yg$ and gravitational forces due to the mass of the buoy ($M = \rho\pi R^2 h$) and the bird (m). So y obeys the equation

$$(m + M)\ddot{y} = (M + m)g - \pi R^2 yg.$$

At rest y is constant, and so $\ddot{y} = 0$, and so

$$y = \frac{M + m}{\pi R^2}.$$

When the bird leaves the equation of motion becomes

$$M\ddot{y} = Mg - \pi R^2 yg.$$

The new equilibrium depth of immersion is $y = M/\pi R^2 = \rho h$; the initial condition when the bird flies away is

$$y(0) = \frac{M + m}{\pi R^2} = \rho h + \frac{m}{\pi R^2}$$

and $\dot{y}(0) = 0$. The displacement from the equilibrium position, $z = y - \rho h$, satisfies

$$M\ddot{z} = -(\pi R^2 g)z.$$

The general solution of this equation is

$$z(t) = A\cos\omega t + B\sin\omega t,$$

with $\omega = \sqrt{\pi R^2 g/M} = \sqrt{g/\rho h}$. The particular solution satisfying the initial conditions

$$z(0) = m/\pi R^2 \qquad \text{and} \qquad \dot{z}(0) = 0$$

is

$$z(t) = \frac{m}{\pi R^2}\cos \omega t.$$

Therefore the buoy oscillations about $y = \rho h$ with amplitude $m/\pi R^2$ and frequency $2\pi\sqrt{\rho h/g}$, as claimed.

Exercise 13.5 *An open tin can, half full of water, is floating in a canal. The can is 11cm tall, has a diameter of 7.5 cm, and has a mass of 50 . Show that at rest the can is submerged a distance of approximately 6.63 cm below the surface of the canal. If the can is pushed down further it will then perform oscillations about its equilibrium position. Show that the can bobs up and down every 0.21 seconds (a little under five times per second). The acceleration due to gravity is approximately 9.8 m/s^2=980 cm/s^2; the density of water is 1 g/cm^3. You can check your answers in a sink with a baked bean can.*

Denote by y the amount in excess of half the height of the can that is submerged. The forces on the can are then $50g$ downwards and a buoyancy force of $\pi(7.5/2)^2 yg$. So, since $(7.5/2)^2 = 14.0625$,

$$50\ddot{y} = 50g - 14.0625\pi g y,$$

and the equilibrium position is

$$y_{\mathrm{e}} = \frac{50}{14.0625\pi} \approx 1.13,$$

which means that the can is submerged a distance of roughly 6.63 cm below the surface.

If $z = y - y_{\mathrm{e}}$ then z satisfies

$$50\ddot{z} = -14.0625\pi g z,$$

and so the can oscillates with period

$$2\pi/\sqrt{14.0625\pi g/50} \approx 0.21s.$$

[You need to use g in the correct units, 980 cm/s^2.]

Exercise 13.6 *A right circular cone, of height h, radius ρ, and with base radius R, is placed point downward in a lake. Assuming that the apex remains point vertically downwards, show that if the cone is submerged to a depth x then*

$$\ddot{x} = g - \left(\frac{x}{h}\right)^3 \frac{g}{\rho}.$$

(You need not solve this equation.) At equilibrium how far is the cone submerged?

When the cone is submerged to a height x the volume submerged is

$$\frac{1}{3}\pi[(x/h)R]^2 x.$$

Then

$$\frac{1}{3}\pi R^2 h\rho\ddot{x} = \frac{1}{3}\pi R^2 h\rho g - \frac{1}{3}\pi[(x/h)R]^2 xg,$$

which simplifies to give

$$\ddot{x} = g - \left(\frac{x}{h}\right)^3 \frac{g}{\rho}.$$

In equilibrium we must have

$$g = \left(\frac{x}{h}\right)^3 \frac{g}{\rho},$$

and so $x = h\rho^{1/3}$.

Exercise 13.7 *A dashpot is a device designed to add damping to a system, consisting essentially of a plunger in a cylinder of liquid or gas, see Figure 13.2.*

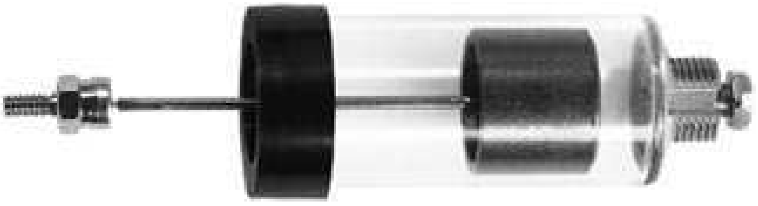

Fig. 13.2. A dashpot. Illustration © 2001 Airpot Corporation. Airpot is a registered trademark of Airpot Corporation.

It produces a resisting force proportional to the velocity, precisely the kind of 'damping' that we used in our model

$$m\ddot{x} + \mu\dot{x} + kx = 0, \tag{S13.1}$$

with μ indicating the 'strength' of the dashpot. Dashpots are used in a variety of applications, for example, cushioning the opening mechanism on a tape recorder, or in car shock absorbers.

A mass-spring-dashpot system consists of a mass attached to a spring and a dashpot, as shown in Figure 13.3. A weight of mass 10 kg is attached to a spring with spring constant 5, and to a dashpot of strength μ. How strong should the dashpot be to ensure that the system is over-damped? What would the period of oscillations be if $\mu = 14$?

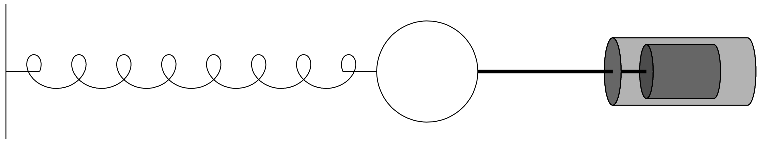

Fig. 13.3. A mass-spring-dashpot system.

Trying $x = \mathrm{e}^{\alpha t}$ in the model

$$m\ddot{x} + \mu\dot{x} + kx = 0$$

yields the auxiliary equation for α,

$$m\alpha^2 + \mu\alpha + k = 0,$$

with solutions

$$\alpha = \frac{-\mu \pm \sqrt{\mu^2 - 4mk}}{2m}.$$

For the system to be overdamped we require

$$\mu^2 - 4mk > 0,$$

so we need $\mu > 2\sqrt{50} \approx 14.14$. If $\mu = 14$ then the system is under-damped: we would have

$$\alpha = \frac{-14 \pm \sqrt{196 - 200}}{20} = \frac{-14 \pm 2\mathrm{i}}{20}$$

giving oscillations that decay like $\mathrm{e}^{-0.7t}$ and have period $2\pi/\sqrt{0.1} \approx 19.9$ seconds.

Exercise 13.8 *When first opened, the Millennium Bridge in London (see Figure 13.4) wobbled from side to side as people crossed; you can see this on video at* `www.arup.com/MillenniumBridge`*. Footfalls created small side-to-side movements of the bridge, which were then enhanced by the tendency of*