The fourth book investigates how many times the sections of cones can meet one another and the circumference of a circle; in addition it contains other things, none of which have been discussed by previous writers, namely, in how many points a section of a cone or a circumference of a circle can meet [the opposite branches of hyperbolas].

The remaining books are thrown in by way of addition: one of them discusses fully *minima* and *maxima*, another deals with equal and similar sections of cones, another with theorems about the determinations of limits, and the last with determinate conic problems. When they are all published it will be possible for anyone who reads them to form his own judgement. Farewell.[a]

(iii.) *Definitions*

Ibid., Definitions, Apoll. Perg. ed. Heiberg i. 6. 2-8. 20

If a straight line be drawn from a point to the circumference of a circle, which is not in the same plane with the point, and be produced in either direction, and if, while the point remains stationary, the straight line be made to move round the circumference of the circle until it returns to the point whence it set out, I call the surface described by the straight line a *conical surface*; it is composed of two surfaces lying vertically opposite to each other, of which each

surviving books have been put into mathematical notation by T. L. Heath, *Apollonius of Perga* (Cambridge, 1896) and translated into French by Paul Ver Eecke, *Les Coniques d'Apollonius de Perga* (Bruges, 1923).

In ancient times Eutocius edited the first four books with a commentary which still survives and is published in Heiberg's edition. Serenus and Hypatia also wrote commentaries, and Pappus a number of lemmas.

extends to infinity when the straight line which describes them is produced to infinity; I call the fixed point the *vertex*, and the straight line drawn through this point and the centre of the circle I call the *axis*.

The figure bounded by the circle and the conical surface between the vertex and the circumference of the circle I term a *cone*, and by the *vertex of the cone* I mean the point which is the vertex of the surface, and by the *axis* I mean the straight line drawn from the vertex to the centre of the circle, and by the *base* I mean the circle.

Of cones, I term those *right* which have their axes at right angles to their bases, and *scalene* those which have their axes not at right angles to their bases.

In any plane curve I mean by a *diameter* a straight line drawn from the curve which bisects all straight lines drawn in the curve parallel to a given straight line, and by the *vertex of the curve* I mean the extremity of the straight line on the curve, and I describe each of the parallels as being drawn *ordinate-wise* to the diameter.

Similarly, in a pair of plane curves I mean by a *transverse diameter* a straight line which cuts the two curves and bisects all the straight lines drawn in either curve parallel to a given straight line, and by the *vertices of the curves* I mean the extremities of the diameter on the curves; and by an *erect diameter* I mean a straight line which lies between the two curves and bisects the portions cut off between the curves of all straight lines drawn parallel to a given

straight line; and I describe each of the parallels as drawn *ordinate-wise to the diameter*.

By *conjugate diameters* in a curve or pair of curves I mean straight lines of which each, being a diameter, bisects parallels to the other.

By an *axis* of a curve or pair of curves I mean a straight line which, being a diameter of the curve or pair of curves, bisects the parallels at right angles.

By *conjugate axes* in a curve or pair of curves I mean straight lines which, being conjugate diameters, bisect at right angles the parallels to each other.

(iv.) *Construction of the Sections*

Ibid., Props. 7-9, Apoll. Perg. ed. Heiberg i. 22. 26-36. 5

Prop. 7[a]

If a cone be cut by a plane through the axis, and if it be also cut by another plane cutting the plane containing the base of the cone in a straight line perpendicular to the base of the axial triangle,[b] *or to the base produced, a section will be made on the surface of the cone by the cutting plane, and straight lines drawn in it parallel to the straight line perpendicular to the base of the axial triangle will meet the common section of the cutting plane and the axial*

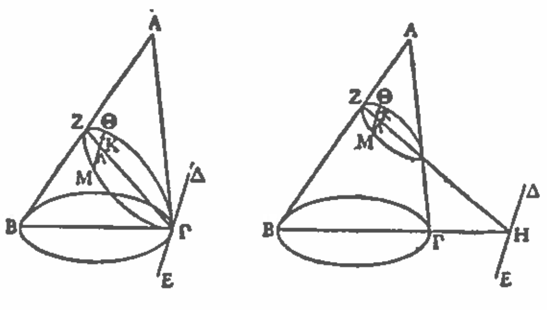
later in the work (i. 52-58) that the principal axes are introduced as diameters at right angles to their ordinates. The proposition is an excellent example of the generality of Apollonius's methods.

Apollonius followed rigorously the Euclidean form of proof. In consequence his general enunciations are extremely long and often can be made tolerable in an English rendering only by splitting them up; but, though Apollonius seems to have taken a malicious pleasure in their length, they are formed on a perfect logical pattern without a superfluous word.

[b] Lit. "the triangle through the axis."

triangle and, if produced to the other part of the section, will be bisected by it; if the cone be right, the straight line in the base will be perpendicular to the common section of the cutting plane and the axial triangle; but if it be scalene, it will not in general be perpendicular, but only when the plane through the axis is perpendicular to the base of the cone.

Let there be a cone whose vertex is the point A and whose base is the circle ΒΓ, and let it be cut by a



plane through the axis, and let the section so made be the triangle ΑΒΓ. Now let it be cut by another plane cutting the plane containing the circle ΒΓ in a straight line ΔΕ which is either perpendicular to ΒΓ or to ΒΓ produced, and let the section made on the surface of the cone be ΔΖΕ[a]; then the common section of the cutting plane and of the triangle ΑΒΓ

[a] This applies only to the first two of the figures given in the mss.

is ZH. Let any point Θ be taken on ΔZE, and through Θ let ΘK be drawn parallel to ΔE. I say that ΘK intersects ZH and, if produced to the other part of the section ΔZE, it will be bisected by the straight line ZH.

For since the cone, whose vertex is the point A and base the circle BΓ, is cut by a plane through the axis and the section so made is the triangle ABΓ, and there has been taken any point Θ on the surface, not being on a side of the triangle ABΓ, and ΔH is perpendicular to BΓ, therefore the straight line drawn through Θ parallel to ΔH, that is ΘK, will meet the triangle ABΓ and, if produced to the other part of the surface, will be bisected by the triangle [Prop. 6]. Therefore, since the straight line drawn through Θ parallel to ΔE meets the triangle ABΓ and is in the plane containing the section ΔZE, it will fall upon the common section of the cutting plane and the triangle ABΓ. But the common section of those planes is ZH; therefore the straight line drawn through Θ parallel to ΔE will meet ZH; and if it be produced to the other part of the section ΔZE it will be bisected by the straight line ZH.

Now the cone is right, or the axial triangle ABΓ is perpendicular to the circle BΓ, or neither.

First, let the cone be right; then the triangle ABΓ will be perpendicular to the circle BΓ [Def. 3; Eucl. xi. 18]. Then since the plane ABΓ is perpendicular to the plane BΓ, and ΔE is drawn in one of the planes perpendicular to their common section BΓ, therefore

ΔE is perpendicular to the triangle ABΓ [Eucl. xi. Def. 4]; and therefore it is perpendicular to all the straight lines in the triangle ABΓ which meet it [Eucl. xi. Def. 3]. Therefore it is perpendicular to ZH.

Now let the cone be not right. Then, if the axial triangle is perpendicular to the circle BΓ, we may similarly show that ΔE is perpendicular to ZH. Now let the axial triangle ABΓ be not perpendicular to the circle BΓ. I say that neither is ΔE perpendicular to ZH. For if it is possible, let it be; now it is also perpendicular to BΓ; therefore ΔE is perpendicular to both BΓ, ZH. And therefore it is perpendicular to the plane through BΓ, ZH [Eucl. xi. 4]. But the plane through BΓ, HZ is ABΓ; and therefore ΔE is perpendicular to the triangle ABΓ. Therefore all the planes through it are perpendicular to the triangle ABΓ [Eucl. xi. 18]. But one of the planes through ΔE is the circle BΓ; therefore the circle BΓ is perpendicular to the triangle ABΓ. Therefore the triangle ABΓ is perpendicular to the circle BΓ; which is contrary to hypothesis. Therefore ΔE is not perpendicular to ZH.

Corollary

From this it is clear that ZH is a diameter of the section ΔZE [Def. 4], inasmuch as it bisects the straight lines drawn parallel to the given straight line ΔE, and also that parallels can be bisected by the diameter ZH without being perpendicular to it.

ιγʹ

Ἐὰν κῶνος ἐπιπέδῳ τμηθῇ διὰ τοῦ ἄξονος, τμηθῇ δὲ καὶ ἑτέρῳ ἐπιπέδῳ συμπίπτοντι μὲν ἑκατέρᾳ πλευρᾷ τοῦ διὰ τοῦ ἄξονος τριγώνου, μήτε δὲ παρὰ τὴν βάσιν τοῦ κώνου ἠγμένῳ μήτε ὑπεναντίως, τὸ δὲ ἐπίπεδον, ἐν ᾧ ἐστιν ἡ βάσις τοῦ κώνου, καὶ τὸ τέμνον ἐπίπεδον συμπίπτῃ κατ' εὐθεῖαν πρὸς ὀρθὰς οὖσαν ἤτοι τῇ βάσει τοῦ διὰ τοῦ ἄξονος τριγώνου ἢ τῇ ἐπ' εὐθείας αὐτῇ, ἥτις ἂν ἀπὸ τῆς τομῆς τοῦ κώνου παράλληλος ἀχθῇ τῇ κοινῇ τομῇ τῶν ἐπιπέδων ἕως τῆς διαμέτρου τῆς τομῆς, δυνήσεταί τι χωρίον παρακείμενον παρά τινα εὐθεῖαν, πρὸς ἣν λόγον ἔχει ἡ διάμετρος τῆς τομῆς, ὃν τὸ τετράγωνον τὸ ἀπὸ τῆς ἠγμένης ἀπὸ τῆς κορυφῆς τοῦ κώνου παρὰ τὴν διάμετρον τῆς τομῆς ἕως τῆς βάσεως τοῦ τριγώνου πρὸς τὸ περιεχόμενον ὑπὸ τῶν ἀπολαμβανομένων ὑπ' αὐτῆς πρὸς ταῖς τοῦ τριγώνου εὐθείαις, πλάτος ἔχον τὴν ἀπολαμβανομένην ὑπ' αὐτῆς ἀπὸ τῆς διαμέτρου πρὸς τῇ κορυφῇ τῆς τομῆς, ἐλλεῖπον εἴδει ὁμοίῳ τε καὶ ὁμοίως κειμένῳ τῷ περιεχομένῳ ὑπό τε τῆς διαμέτρου καὶ τῆς παρ' ἣν δύναται· καλείσθω δὲ ἡ τοιαύτη τομὴ ἔλλειψις.

Ἔστω κῶνος, οὗ κορυφὴ μὲν τὸ Α σημεῖον,

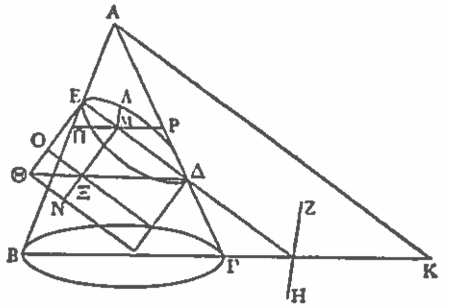
Prop. 13

Let a cone be cut by a plane through the axis, and let it be cut by another plane meeting each side of the axial triangle, being neither parallel to the base nor subcontrary, and let the plane containing the base of the cone meet the cutting plane in a straight line perpendicular either to the base of the axial triangle or to the base produced; then if a straight line be drawn from any point of the section of the cone parallel to the common section of the planes as far as the diameter of the section, its square will be equal to an area applied to a certain straight line; this line is such that the diameter of the section will bear to it the same ratio as the square on the line drawn from the vertex of the cone parallel to the diameter of the section as far as the base of the triangle bears to the rectangle contained by the intercepts made by it on the sides of the triangle; the breadth of the applied figure will be the intercept made by it on the diameter in the direction of the vertex of the section; and the applied figure will be deficient by a figure similar and similarly situated to the rectangle bounded by the diameter and the parameter;[a] *and let such a section be called an ellipse.*

Let there be a cone, whose vertex is the point A

[a] The *erect* and *transverse* side, that is to say, of the figure (εἶδος) applied to the diameter. In the case of the parabola, the transverse side is infinite.

βάσις δὲ ὁ ΒΓ κύκλος, καὶ τετμήσθω ἐπιπέδῳ διὰ τοῦ ἄξονος, καὶ ποιείτω τομὴν τὸ ΑΒΓ τρίγωνον, τετμήσθω δὲ καὶ ἑτέρῳ ἐπιπέδῳ συμπίπτοντι μὲν ἑκατέρᾳ πλευρᾷ τοῦ διὰ τοῦ ἄξονος τριγώνου, μήτε δὲ παραλλήλῳ τῇ βάσει τοῦ κώνου μήτε ὑπεναντίως ἠγμένῳ, καὶ ποιείτω τομὴν ἐν τῇ ἐπιφανείᾳ τοῦ κώνου τὴν ΔΕ γραμμήν· κοινὴ



δὲ τομὴ τοῦ τέμνοντος ἐπιπέδου καὶ τοῦ, ἐν ᾧ ἐστιν ἡ βάσις τοῦ κώνου, ἔστω ἡ ΖΗ πρὸς ὀρθὰς οὖσα τῇ ΒΓ, ἡ δὲ διάμετρος τῆς τομῆς ἔστω ἡ ΕΔ, καὶ ἀπὸ τοῦ Ε τῇ ΕΔ πρὸς ὀρθὰς ἤχθω ἡ ΕΘ, καὶ διὰ τοῦ Α τῇ ΕΔ παράλληλος ἤχθω ἡ ΑΚ, καὶ πεποιήσθω ὡς τὸ ἀπὸ ΑΚ πρὸς τὸ ὑπὸ ΒΚΓ, οὕτως ἡ ΔΕ πρὸς τὴν ΕΘ, καὶ εἰλήφθω τι σημεῖον ἐπὶ τῆς τομῆς τὸ Λ, καὶ διὰ τοῦ Λ τῇ ΖΗ παράλληλος ἤχθω ἡ ΛΜ. λέγω, ὅτι ἡ ΛΜ δύναταί τι χωρίον, ὃ παράκειται παρὰ τὴν ΕΘ, πλάτος ἔχον τὴν ΕΜ, ἐλλεῖπον εἴδει ὁμοίῳ τῷ ὑπὸ τῶν ΔΕΘ.

Ἐπεζεύχθω γὰρ ἡ ΔΘ, καὶ διὰ μὲν τοῦ Μ τῇ

and whose base is the circle ΒΓ, and let it be cut by a plane through the axis, and let the section so made be the triangle ΑΒΓ, and let it be cut by another plane meeting either side of the axial triangle, being drawn neither parallel to the base nor subcontrary, and let the section made on the surface of the cone be the curve ΔΕ; let the common section of the cutting plane and of that containing the base of the cone be ΖΗ, perpendicular to ΒΓ, and let the diameter of the section be ΕΔ, and from Ε let ΕΘ be drawn perpendicular to ΕΔ, and through Α let ΑΚ be drawn parallel to ΕΔ, and let $AK^2 : BK . K\Gamma = \Delta E : E\Theta$, and let any point Λ be taken on the section, and through Λ let ΛΜ be drawn parallel to ΖΗ. I say that the square on ΛΜ is equal to an area applied to the straight line ΕΘ, having ΕΜ for its breadth, and being deficient by a figure similar to the rectangle contained by ΔΕ, ΕΘ.

For let ΔΘ be joined, and through Μ let ΜΞΝ be

ΘΕ παράλληλος ἤχθω ἡ ΜΞΝ, διὰ δὲ τῶν Θ, Ξ τῇ ΕΜ παράλληλοι ἤχθωσαν αἱ ΘΝ, ΞΟ, καὶ διὰ τοῦ Μ τῇ ΒΓ παράλληλος ἤχθω ἡ ΠΜΡ. ἐπεὶ οὖν ἡ ΠΡ τῇ ΒΓ παράλληλός ἐστιν, ἔστι δὲ καὶ ἡ ΛΜ τῇ ΖΗ παράλληλος, τὸ ἄρα διὰ τῶν ΛΜ, ΠΡ ἐπίπεδον παράλληλόν ἐστι τῷ διὰ τῶν ΖΗ, ΒΓ ἐπιπέδῳ, τουτέστι τῇ βάσει τοῦ κώνου. ἐὰν ἄρα ἐκβληθῇ διὰ τῶν ΛΜ, ΠΡ ἐπίπεδον, ἡ τομὴ κύκλος ἔσται, οὗ διάμετρος ἡ ΠΡ. καί ἐστι κάθετος ἐπ' αὐτὴν ἡ ΛΜ· τὸ ἄρα ὑπὸ τῶν ΠΜΡ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΛΜ. καὶ ἐπεί ἐστιν, ὡς τὸ ἀπὸ τῆς ΑΚ πρὸς τὸ ὑπὸ τῶν ΒΚΓ, οὕτως ἡ ΕΔ πρὸς τὴν ΕΘ, ὁ δὲ τοῦ ἀπὸ τῆς ΑΚ πρὸς τὸ ὑπὸ τῶν ΒΚΓ λόγος σύγκειται ἐκ τοῦ, ὃν ἔχει ἡ ΑΚ πρὸς ΚΒ, καὶ ἡ ΑΚ πρὸς ΚΓ, ἀλλ' ὡς μὲν ἡ ΑΚ πρὸς ΚΒ, οὕτως ἡ ΕΗ πρὸς ΗΒ, τουτέστιν ἡ ΕΜ πρὸς ΜΠ, ὡς δὲ ἡ ΑΚ πρὸς ΚΓ, οὕτως ἡ ΔΗ πρὸς ΗΓ, τουτέστιν ἡ ΔΜ πρὸς ΜΡ, ὁ ἄρα τῆς ΔΕ πρὸς τὴν ΕΘ λόγος σύγκειται ἔκ τε τοῦ τῆς ΕΜ πρὸς ΜΠ καὶ τοῦ τῆς ΔΜ πρὸς ΜΡ. ὁ δὲ συγκείμενος λόγος ἔκ τε τοῦ, ὃν ἔχει ἡ ΕΜ πρὸς ΜΠ, καὶ ἡ ΔΜ πρὸς ΜΡ, ὁ τοῦ ὑπὸ τῶν ΕΜΔ ἐστι πρὸς τὸ ὑπὸ τῶν ΠΜΡ. ἔστιν ἄρα ὡς τὸ ὑπὸ τῶν ΕΜΔ πρὸς τὸ ὑπὸ τῶν ΠΜΡ, οὕτως ἡ ΔΕ πρὸς τὴν ΕΘ, τουτέστιν ἡ ΔΜ πρὸς τὴν ΜΞ. ὡς δὲ ἡ ΔΜ πρὸς ΜΞ, τῆς ΜΕ κοινοῦ ὕψους λαμβανομένης, οὕτως τὸ ὑπὸ ΔΜΕ πρὸς τὸ ὑπὸ ΞΜΕ. καὶ ὡς ἄρα τὸ ὑπὸ ΔΜΕ πρὸς τὸ ὑπὸ ΠΜΡ, οὕτως τὸ ὑπὸ ΔΜΕ πρὸς τὸ ὑπὸ ΞΜΕ. ἴσον ἄρα ἐστὶ τὸ ὑπὸ ΠΜΡ τῷ ὑπὸ ΞΜΕ. τὸ δὲ ὑπὸ ΠΜΡ ἴσον ἐδείχθη τῷ ἀπὸ τῆς ΛΜ· καὶ τὸ ὑπὸ ΞΜΕ ἄρα ἐστὶν ἴσον τῷ ἀπὸ τῆς ΛΜ. ἡ ΛΜ

drawn parallel to ΘΕ, and through Θ, Ξ, let ΘΝ, ΞΟ be drawn parallel to ΕΜ, and through Μ let ΠΜΡ be drawn parallel to ΒΓ. Then since ΠΡ is parallel to ΒΓ, and ΛΜ is parallel to ΖΗ, therefore the plane through ΛΜ, ΠΡ is parallel to the plane through ΖΗ, ΒΓ [Eucl. xi. 15], that is to the base of the cone. If, therefore, the plane through ΛΜ, ΠΡ be produced, the section will be a circle with diameter ΠΡ [Prop. 4]. And ΛΜ is perpendicular to it; therefore

$$\Pi M . MP = \Lambda M^2.$$

And since $AK^2 : BK . K\Gamma = E\Delta : E\Theta$,

and $AK^2 : BK . K\Gamma = (AK : KB)(AK : K\Gamma)$,

while $AK : KB = EH : HB$

$= EM : M\Pi$, [Eucl. vi. 4

and $AK : K\Gamma = \Delta H : H\Gamma$

$= \Delta M : MP$, [*ibid.*

therefore $\Delta E : E\Theta = (EM : M\Pi)(\Delta M : MP)$.

But $(EM : M\Pi)(\Delta M : MP) = EM . M\Delta : \Pi M . MP$.

Therefore

$EM . M\Delta : \Pi M . MP = \Delta E : E\Theta$

$= \Delta M : M\Xi$. [*ibid.*

But $\Delta M : M\Xi = \Delta M . ME : \Xi M . ME$,

by taking a common height ΜΕ.

Therefore $\Delta M . ME : \Pi M . MP = \Delta M . ME : \Xi M . ME$.

Therefore $\Pi M . MP = \Xi M . ME$. [Eucl. v. 9

But $\Pi M . MP = \Lambda M^2$,

as was proved;

and therefore $\Xi M . ME = \Lambda M^2$.

ἄρα δύναται τὸ ΜΟ, ὃ παράκειται παρὰ τὴν ΘΕ, πλάτος ἔχον τὴν ΕΜ, ἐλλεῖπον εἴδει τῷ ΟΝ ὁμοίῳ ὄντι τῷ ὑπὸ ΔΕΘ. καλείσθω δὲ ἡ μὲν τοιαύτη τομὴ ἔλλειψις, ἡ δὲ ΕΘ παρ' ἣν δύνανται αἱ καταγόμεναι ἐπὶ τὴν ΔΕ τεταγμένως, ἡ δὲ αὐτὴ καὶ ὀρθία, πλαγία δὲ ἡ ΕΔ.

[a] Let p be the parameter of a conic section and d the corresponding diameter, and let the diameter of the section and the tangent at its extremity be taken as axes of co-ordinates (in general oblique). Then Props. 11-13 are equivalent to the Cartesian equations

Therefore the square on ΑΜ is equal to ΜΟ, which is applied to ΘΕ, having ΕΜ for its breadth, and being deficient by the figure ΟΝ similar to the rectangle ΔΕ . ΕΘ. Let such a section be called an *eclipse*, let ΕΘ be called *the parameter to the ordinates* to ΔΕ, and let this line be called the *erect side* (*latus rectum*), and ΕΔ the *transverse side*.[a]

$y^2 = px$ (the parabola),

and

$y^2 = px \pm \frac{p}{d}x^2$ (the hyperbola and ellipse respectively).

It is the essence of Apollonius's treatment to express the fundamental properties of the conics as equations between *areas*, whereas Archimedes had given the fundamental properties of the central conics as *proportions*

$$y^2 : (a^2 \pm x^2) = a^2 : b^2.$$

This form is, however, equivalent to the Cartesian equations referred to axes through the centre.

(vi.) *Transition to New Diameter*

Ibid., Prop. 50, Apoll. Perg. ed. Heiberg i. 148. 17–151. 8

ν'

Ἐὰν ὑπερβολῆς ἢ ἐλλείψεως ἢ κύκλου περιφερείας εὐθεῖα ἐπιψαύουσα συμπίπτῃ τῇ διαμέτρῳ, καὶ διὰ τῆς ἁφῆς καὶ τοῦ κέντρου εὐθεῖα ἐκβληθῇ, ἀπὸ δὲ τῆς κορυφῆς ἀναχθεῖσα εὐθεῖα παρὰ τεταγμένως κατηγμένην συμπίπτῃ τῇ διὰ τῆς ἁφῆς καὶ

[a] Apollonius is the first person known to have recognized the opposite branches of a hyperbola as portions of the same

(vi.) *Transition to New Diameter*

Ibid., Prop. 50, Apoll. Perg. ed. Heiberg i. 148. 17–154. 8

Prop. 50

In a hyperbola, ellipse or circumference of a circle let a straight line be drawn to touch [*the curve*] *and meet the diameter, and let the straight line through the point of contact and the centre be produced, and from the vertex let a straight line be drawn parallel to a straight line drawn ordinate-wise so as to meet the straight line drawn*

curve. It is his practice, however, where possible to discuss the single-branch hyperbola (or the hyperbola *simpliciter* as he would call it) together with the ellipse and circle, and to deal with the opposite branches separately. But occasionally, as in i. 30, the double-branch hyperbola and the ellipse are included in one enunciation.

τοῦ κέντρου ἠγμένῃ εὐθείᾳ, καὶ ποιηθῇ, ὡς τὸ τμῆμα τῆς ἐφαπτομένης τὸ μεταξὺ τῆς ἁφῆς καὶ τῆς ἀνηγμένης πρὸς τὸ τμῆμα τῆς ἠγμένης διὰ τῆς ἁφῆς καὶ τοῦ κέντρου τὸ μεταξὺ τῆς ἁφῆς καὶ τῆς ἀνηγμένης, εὐθεῖά τις πρὸς τὴν διπλασίαν τῆς ἐφαπτομένης, ἥτις ἂν ἀπὸ τῆς τομῆς ἀχθῇ ἐπὶ τὴν διὰ τῆς ἁφῆς καὶ τοῦ κέντρου ἠγμένην εὐθεῖαν παράλληλος τῇ ἐφαπτομένῃ, δυνήσεταί τι χωρίον ὀρθογώνιον παρακείμενον παρὰ τὴν ποριςθεῖσαν, πλάτος ἔχον τὴν ἀπολαμβανομένην ὑπ' αὐτῆς πρὸς τῇ ἁφῇ, ἐπὶ μὲν τῆς ὑπερβολῆς ὑπερβάλλον εἴδει ὁμοίῳ τῷ περιεχομένῳ ὑπὸ τῆς διπλασίας τῆς μεταξὺ τοῦ κέντρου καὶ τῆς ἁφῆς καὶ τῆς ποριςθείσης εὐθείας, ἐπὶ δὲ τῆς ἐλλείψεως καὶ τοῦ κύκλου ἐλλεῖπον.

Ἔστω ὑπερβολὴ ἢ ἔλλειψις ἢ κύκλου περιφέρεια, ἧς διάμετρος ἡ ΑΒ, κέντρον δὲ τὸ Γ, ἐφαπτομένη δὲ ἡ ΔΕ, καὶ ἐπιζευχθεῖσα ἡ ΓΕ ἐκβεβλήσθω ἐφ' ἑκάτερα, καὶ κείσθω τῇ ΕΓ ἴση ἡ ΓΚ, καὶ διὰ τοῦ Β τεταγμένως ἀνήχθω ἡ ΒΖΗ, διὰ δὲ τοῦ Ε τῇ ΕΓ πρὸς ὀρθὰς ἤχθω ἡ ΕΘ, καὶ γινέσθω, ὡς ἡ ΖΕ πρὸς ΕΗ, οὕτως ἡ ΕΘ πρὸς τὴν διπλασίαν τῆς ΕΔ, καὶ ἐπιζευχθεῖσα ἡ ΘΚ ἐκβεβλήσθω, καὶ εἰλήφθω τι ἐπὶ τῆς τομῆς σημεῖον τὸ Λ, καὶ δι' αὐτοῦ τῇ ΕΔ παράλληλος ἤχθω ἡ ΛΜΞ, τῇ δὲ

[a] To save space, the figure is here given for the hyperbola only; in the mss. there are figures for the ellipse and circle as well.

The general enunciation is not easy to follow, but the particular enunciation will make it easier to understand. The

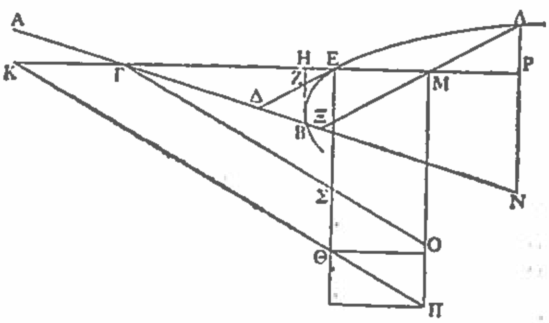
through the point of contact and the centre, and let the segment of the tangent between the point of contact and the line drawn ordinate-wise bear to the segment of the line drawn through the point of contact and the centre between the point of contact and the line drawn ordinate-wise the same ratio as a certain straight line bears to double the tangent; then if any straight line be drawn from the section parallel to the tangent so as to meet the straight line drawn through the point of contact and the centre, its square will be equal to a certain rectilineal area applied to the postulated straight line, having for its breadth the intercept between it and the point of contact, in the case of the hyperbola exceeding by a figure similar to the rectangle bounded by double the straight line between the centre and the point of contact and the postulated straight line, in the case of the ellipse and circle falling short.[a]

In a hyperbola, ellipse or circumference of a circle, with diameter ΑΒ and centre Γ, let ΔΕ be a tangent, and let ΓΕ be joined and produced in either direction, and let ΓΚ be placed equal to ΕΓ, and through Β let ΒΖΗ be drawn ordinate-wise, and through Ε let ΕΘ be drawn perpendicular to ΕΓ, and let $ΖΕ : ΕΗ = ΕΘ : 2ΕΔ$, and let ΘΚ be joined and produced, and let any point Λ be taken on the section, and through it let ΛΜΞ be drawn parallel to ΕΔ and

purpose of this important proposition is to show that, if any other diameter be taken, the ordinate-property of the conic with reference to this diameter has the same form as the ordinate-property with reference to the original diameter. The theorem amounts to a transformation of co-ordinates from the original diameter and the tangent at its extremity to any diameter and the tangent at its extremity. In succeeding propositions, showing how to construct conics from certain data, Apollonius introduces the axes for the first time as special cases of diameters.

ΒΗ ἡ ΛΡΝ, τῇ δὲ ΕΘ ἡ ΜΠ. λέγω, ὅτι τὸ ἀπὸ ΛΜ ἴσον ἐστὶ τῷ ὑπὸ ΕΜΠ.

Ἤχθω γὰρ διὰ τοῦ Γ τῇ ΚΠ παράλληλος ἡ ΓΣΟ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΕΓ τῇ ΓΚ, ὡς δὲ ἡ



ΕΓ πρὸς ΚΓ, ἡ ΕΣ πρὸς ΣΘ, ἴση ἄρα καὶ ἡ ΕΣ τῇ ΣΘ. καὶ ἐπεί ἐστιν, ὡς ἡ ΖΕ πρὸς ΕΗ, ἡ ΘΕ πρὸς τὴν διπλασίαν τῆς ΕΔ, καί ἐστι τῆς ΕΘ ἡμίσεια ἡ ΕΣ, ἔστιν ἄρα, ὡς ἡ ΖΕ πρὸς ΕΗ, ἡ ΣΕ πρὸς ΕΔ. ὡς δὲ ἡ ΖΕ πρὸς ΕΗ, ἡ ΛΜ πρὸς ΜΡ· ὡς ἄρα ἡ ΛΜ πρὸς ΜΡ, ἡ ΣΕ πρὸς ΕΔ. καὶ ἐπεὶ τὸ ΡΝΓ τρίγωνον τοῦ ΗΒΓ τριγώνου, τουτέστι τοῦ ΓΔΕ, ἐπὶ μὲν τῆς ὑπερβολῆς μεῖζον ἐδείχθη, ἐπὶ δὲ τῆς ἐλλείψεως καὶ τοῦ κύκλου ἔλασσον τῷ ΛΝΞ, κοινῶν ἀφαιρεθέντων ἐπὶ μὲν τῆς ὑπερβολῆς τοῦ τε ΕΓΔ τριγώνου καὶ τοῦ ΝΡΜΞ τετραπλεύρου, ἐπὶ δὲ τῆς ἐλλείψεως καὶ τοῦ κύκλου τοῦ ΜΞΓ τριγώνου, τὸ ΛΜΡ τρίγωνον τῷ ΜΕΔΞ τετραπλεύρῳ ἐστὶν ἴσον. καί ἐστι

ΛΡΝ parallel to ΒΗ, and let ΜΠ be drawn parallel to ΕΘ. I say that $ΛΜ^2 = ΕΜ . ΜΠ$.

For through Γ let ΓΣΟ be drawn parallel to ΚΠ. Then since

$$ΕΓ = ΓΚ$$

and $ΕΓ : ΓΚ = ΕΣ : ΣΘ$, [Eucl. vi. 2

therefore $ΕΣ = ΣΘ$.

And since $ΖΕ : ΕΗ = ΘΕ : 2ΕΔ$,

and $ΕΣ = \frac{1}{2}ΕΘ$,

therefore $ΖΕ : ΕΗ = ΣΕ : ΕΔ$.

But $ΖΕ : ΕΗ = ΛΜ : ΜΡ$; [Eucl. vi. 4

therefore $ΛΜ : ΜΡ = ΣΕ : ΕΔ$.

And since it has been proved [Prop. 43] that in the hyperbola

triangle ΡΝΓ = triangle ΗΒΓ + triangle ΛΝΞ,

i.e., triangle ΡΝΓ = triangle ΓΔΕ + triangle ΛΝΞ,[a]

while in the ellipse and the circle

triangle ΡΝΓ = triangle ΗΒΓ − triangle ΛΝΞ,

i.e., triangle ΡΝΓ + triangle ΛΝΞ = triangle ΓΔΕ,[b]

therefore by taking away the common elements—in the hyperbola the triangle ΕΓΔ and the quadrilateral ΝΡΜΞ, in the ellipse and the circle the triangle ΜΞΓ,

triangle ΛΜΡ = quadrilateral ΜΕΔΞ.

[a] For this step *v.* Eutocius's comment on Prop. 43.
[b] See Eutocius.

παράλληλος ἡ ΜΞ τῇ ΔΕ, ἡ δὲ ὑπὸ ΛΜΡ τῇ ὑπὸ ΕΜΞ ἐστιν ἴση· ἴσον ἄρα ἐστὶ τὸ ὑπὸ ΛΜΡ τῷ ὑπὸ τῆς ΕΜ καὶ συναμφοτέρου τῆς ΕΔ, ΜΞ. καὶ ἐπεί ἐστιν, ὡς ἡ ΜΓ πρὸς ΓΕ, ἥ τε ΜΞ πρὸς ΕΔ καὶ ἡ ΜΟ πρὸς ΕΣ, ὡς ἄρα ἡ ΜΟ πρὸς ΕΣ, ἡ ΜΞ πρὸς ΔΕ. καὶ συνθέντι, ὡς συναμφότερος ἡ ΜΟ, ΣΕ πρὸς ΕΣ, οὕτως συναμφότερος ἡ ΜΞ, ΕΔ πρὸς ΕΔ· ἐναλλάξ, ὡς συναμφότερος ἡ ΜΟ, ΣΕ πρὸς συναμφότερον τὴν ΞΜ, ΕΔ ἡ ΣΕ πρὸς ΕΔ. ἀλλ' ὡς μὲν συναμφότερος ἡ ΜΟ, ΕΣ πρὸς συναμφότερον τὴν ΜΞ, ΔΕ, τὸ ὑπὸ συναμφοτέρου τῆς ΜΟ, ΕΣ καὶ τῆς ΕΜ πρὸς τὸ ὑπὸ συναμφοτέρου τῆς ΜΞ, ΕΔ καὶ τῆς ΕΜ, ὡς δὲ ἡ ΣΕ πρὸς ΕΔ, ἡ ΖΕ πρὸς ΕΗ, τουτέστιν ἡ ΛΜ πρὸς ΜΡ, τουτέστι τὸ ἀπὸ ΛΜ πρὸς τὸ ὑπὸ ΛΜΡ· ὡς ἄρα τὸ ὑπὸ συναμφοτέρου τῆς ΜΟ, ΕΣ καὶ τῆς ΜΕ πρὸς τὸ ὑπὸ συναμφοτέρου τῆς ΜΞ, ΕΔ καὶ τῆς ΕΜ, τὸ ἀπὸ ΛΜ πρὸς τὸ ὑπὸ ΛΜΡ. καὶ ἐναλλάξ, ὡς τὸ ὑπὸ συναμφοτέρου τῆς ΜΟ, ΕΣ καὶ τῆς ΜΕ πρὸς τὸ ἀπὸ ΜΛ, οὕτως τὸ ὑπὸ συναμφοτέρου τῆς ΜΞ, ΕΔ καὶ τῆς ΜΕ πρὸς τὸ ὑπὸ ΛΜΡ. ἴσον δὲ τὸ ὑπὸ ΛΜΡ τῷ ὑπὸ τῆς ΜΕ καὶ συναμφοτέρου τῆς ΜΞ, ΕΔ· ἴσον ἄρα καὶ τὸ ἀπὸ ΛΜ τῷ ὑπὸ ΕΜ καὶ συναμφοτέρου τῆς ΜΟ, ΕΣ. καί ἐστιν ἡ μὲν ΣΕ τῇ ΣΘ ἴση, ἡ δὲ ΣΘ τῇ ΟΠ· ἴσον ἄρα τὸ ἀπὸ ΛΜ τῷ ὑπὸ ΕΜΠ.

But ΜΞ is parallel to ΔΕ and angle ΛΜΡ = angle ΕΜΞ [Eucl. i. 15];

therefore $\Lambda M . M P = E M . (E\Delta + M\Xi)$.

And since $M\Gamma : \Gamma E = M\Xi : E\Delta$,

and $M\Gamma : \Gamma E = MO : E\Sigma$, [Eucl. vi. 4

therefore $MO : E\Sigma = M\Xi : \Delta E$.

Componendo, $MO + \Sigma E : E\Sigma = M\Xi + E\Delta : E\Delta$;

and *permutando*

$$MO + \Sigma E : \Xi M + E\Delta = \Sigma E : E\Delta.$$

But $MO + \Sigma E : \Xi M + E\Delta = (MO + E\Sigma) . EM : (M\Xi + E\Delta) . EM$,

and $\Sigma E : E\Delta = ZE : EH$

$= \Lambda M : MP$ [Eucl. vi. 4

$= \Lambda M^2 : \Lambda M . MP$;

therefore

$$(MO + E\Sigma) . ME : (M\Xi + E\Delta) . EM = \Lambda M^2 : \Lambda M . MP.$$

And *permutando*

$$(MO + E\Sigma) . ME : M\Lambda^2 = (M\Xi + E\Delta) . ME : \Lambda M . MP.$$

But $\Lambda M . MP = ME . (M\Xi + E\Delta)$;

therefore $\Lambda M^2 = EM . (MO + E\Sigma)$.

And $\Sigma E = \Sigma\Theta$, while $\Sigma\Theta = O\Pi$ [Eucl. i. 34];

therefore $\Lambda M^2 = EM . M\Pi$.

Apollonius of Perga (ca. 200 BCE), Conics

Source: Greek Mathematical Works, trl. Ivor Thomas, vol. II, Aristarchos to Pappus of Alexandria.

See also, for ex:

T.L. Heath, The Conics of Apollonius

available via www.wilbourhall.org