- Jacobimatrix en Jacobiaan

$F: \mathbb{R}^n \to \mathbb{R}^m$

Vb: $F: \mathbb{R}^3 \to \mathbb{R}^2$

$(x,y,z) \longmapsto \left(\frac{x}{z+1}, \frac{y}{z+1}\right) = (\xi, \eta)$

$= (F_1, F_2)$

maw: $F_1(x,y,z) = \xi$ en $F_2(x,y,z) = \eta$

m.a.w. $\xi = \frac{x}{z+1}$, $\eta = \frac{y}{z+1}$

Vraag: als je aan x,y,z wiebelt, wat gebeurt er dan met ξ, η?

Herinner: de totale afgeleide

Als $\xi = F_1(x,y,z)$

$\longrightarrow \quad d\xi = \frac{\partial F_1}{\partial x} dx + \frac{\partial F_1}{\partial y} dy + \frac{\partial F_1}{\partial z} dz$

Als $\eta = F_2(x,y,z)$

$\longrightarrow \quad d\eta = \frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} dz$

Concreet vb

$d\xi = \frac{1}{z+1} dx - \frac{x}{(z+1)^2} dz$

$d\eta = \frac{1}{z+1} dy - \frac{y}{(z+1)^2} dz$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} F_1(x,y,z) \\ F_2(x,y,z) \end{pmatrix} = F(x,y,z)$$

Vectornotatie

$$\begin{pmatrix} d\xi \\ d\eta \end{pmatrix} = \begin{pmatrix} \frac{\partial F_1}{\partial x} dx + \frac{\partial F_1}{\partial y} dy + \frac{\partial F_1}{\partial z} dz \\ \frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} dz \end{pmatrix}$$

Matrix-vector!

$$\underbrace{\begin{pmatrix} d\xi \\ d\eta \end{pmatrix}}_{\text{wiebelen van } \xi, \eta} = \underbrace{\begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{pmatrix}}_{\text{Jacobimatrix}} \underbrace{\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}}_{\text{wiebelen van } x, y, z}$$

Jacobimatrix

wiebelen van ξ, η

wiebelen van x, y, z

geeft de verandering in x, y, z = (dx, dy, dz) INPUT

door een verandering in ξ, η = $(d\xi, d\eta)$ OUTPUT

(althans het lineaire deel ervan)

Voorbeeld verder uitwerken:

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} x/_{z+1} \\ y/_{z+1} \end{pmatrix}$$

$$J = \begin{pmatrix} \frac{1}{z+1} & 0 & -\frac{x}{(z+1)^2} \\ 0 & \frac{1}{z+1} & -\frac{y}{(z+1)^2} \end{pmatrix}$$

dus
$$\begin{pmatrix} d\xi \\ d\eta \end{pmatrix} = \begin{pmatrix} \frac{1}{z+1} & 0 & -\frac{x}{(z+1)^2} \\ 0 & \frac{1}{z+1} & -\frac{y}{(z+1)^2} \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

Bijv: neem $(x,y,z) = (2,4,1)$:

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = F(2,4,1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \xi = \frac{2}{1+1}, \quad \eta = \frac{4}{1+1}$$

Vraag: schat ξ, η in $(x,y,z) = (2{,}01 \;\; 4{,}02 \;\; 0{,}99)$

Antw: we hebben hier $J(2,4,1) = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & -1 \end{pmatrix}$

$$\begin{pmatrix} d\xi \\ d\eta \end{pmatrix} = J(2,4,1)\cdot\begin{pmatrix} 0{,}01 \\ 0{,}02 \\ -0{,}01 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{4} & -1 \end{pmatrix}\begin{pmatrix} 0{,}01 \\ 0{,}02 \\ -0{,}01 \end{pmatrix}$$

$$= \begin{pmatrix} 0{,}01 \\ 0{,}02 \end{pmatrix}$$

Conclusie $\begin{pmatrix} \xi \\ \eta \end{pmatrix} \approx \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0{,}01 \\ 0{,}02 \end{pmatrix} = \begin{pmatrix} 1{,}01 \\ 2{,}02 \end{pmatrix}$

Algemeen J bij functies $\mathbb{R}^n \to \mathbb{R}^m$

$$F(x_1, x_2, \ldots x_n) = (F_1, F_2, \ldots F_m)$$

$$J = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \cdots & \frac{\partial F_m}{\partial x_n} \end{pmatrix}$$

Nog een voorbeeld:

$$\Psi : \mathbb{R}^2 \to \mathbb{R}^2$$

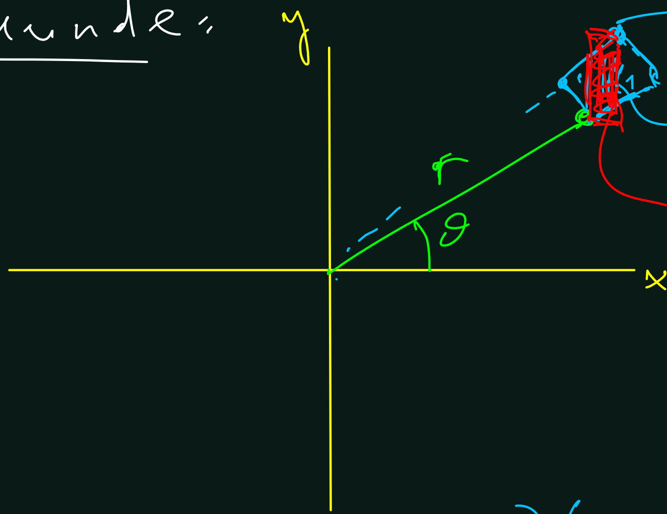
$$(r,\vartheta) \mapsto (x,y) \quad \text{met} \quad \begin{array}{l} x = r\cos\vartheta \\ y = r\sin\vartheta \end{array}$$

(omrekenen pool → rechthoek)

Jacobimatrix
$$J = \begin{pmatrix} \dfrac{\partial\Psi_1}{\partial r} & \dfrac{\partial\Psi_1}{\partial\vartheta} \\ \dfrac{\partial\Psi_2}{\partial r} & \dfrac{\partial\Psi_2}{\partial\vartheta} \end{pmatrix} = \begin{pmatrix} \cos\vartheta & -r\sin\vartheta \\ \sin\vartheta & r\cos\vartheta \end{pmatrix}$$

dus
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \cos\vartheta & -r\sin\vartheta \\ \sin\vartheta & r\cos\vartheta \end{pmatrix} \begin{pmatrix} dr \\ d\vartheta \end{pmatrix}$$

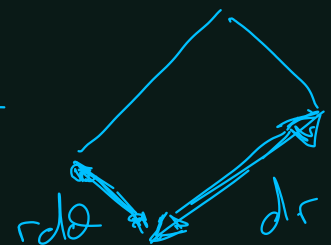
Meetkunde:



$(r+dr, \vartheta + d\vartheta)$

Opp $(r)\, d\vartheta\, dr$

↑ Jacobiaan



opp $\neq dx\, dy$

$$\det J = |J| = \frac{\partial(x,y)}{\partial(r,\vartheta)}$$

← Jacobiaan

← nieuwe notatie

$$= (\cos\vartheta)(r\cos\vartheta) - (-r\sin\vartheta)(\sin\vartheta)$$

$$= r\cos^2\vartheta + r\sin^2\vartheta = r$$

<u>Onthoud</u>: 1) de Jacobiaan is det van Jacobimatrix

2) Jacobiaan geeft "netto strek- en krimp"

dwz (in dit geval)

de opp van een stukje tussen

(r, ϑ) en $(r + dr, \vartheta + d\vartheta)$

Nodig volgende week en later.

Jacobiaan geeft verhouding tussen  opp en 

UPGAVEN (OUD)

12.5:16 Als $f(x,y)$ harmonisch, dan $f\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$ ook.

Check: harmonisch wil zeggen: $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

met $g(x,y) = f\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right) = f(u,v)$

Er moet gelden: $\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0.$

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \qquad \frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}$$

$$\frac{\partial g}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 g}{\partial x^2} = \partial f$$

$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{2y^2}{(x^2+y^2)^2} - \frac{1}{x^2+y^2}$$

Oh help
wat een
slavenwerk.

12.7.16 $f(r,\theta) = (x,y)$ met

$$x = r\cos\theta$$
$$y = r\sin\theta$$

gevraagd: ∇f

$$\left.\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}\\ \frac{\partial f}{\partial y} &= \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y}\end{aligned}\right\}\quad \operatorname{grad} f = \begin{pmatrix}\frac{\partial r}{\partial x} & \frac{\partial \theta}{\partial x}\\ \frac{\partial r}{\partial y} & \frac{\partial \theta}{\partial y}\end{pmatrix}\begin{pmatrix}\frac{\partial f}{\partial r}\\ \frac{\partial f}{\partial \theta}\end{pmatrix}$$

$$\cancel{= \left(\quad\right)^{-1}\begin{pmatrix}\frac{\partial f}{\partial r}\\ \frac{\partial f}{\partial \theta}\end{pmatrix}}$$

Je kunt $\frac{\partial r}{\partial x}$ etc makkelijk als volgt berekenen:

Uit $\begin{cases} x = r\cos\vartheta \\ y = r\sin\vartheta \end{cases}$ weten we $\begin{cases} r^2 = x^2+y^2 \\ \tan\vartheta = y/x \end{cases}$

diff naar x resp y:

$$\begin{cases} 2r\frac{\partial r}{\partial x} = 2x \\ (1+\tan^2\vartheta)\frac{\partial \vartheta}{\partial x} = -\frac{y}{x^2} \end{cases} \quad \text{resp.} \quad \begin{cases} 2r\frac{\partial r}{\partial y} = 2y \\ (1+\tan^2\vartheta)\frac{\partial \vartheta}{\partial y} = \frac{1}{x} \end{cases}$$

en merk op dat $1+\tan^2\vartheta = \frac{x^2+y^2}{x^2} = \frac{r^2}{x^2}$

dan krijg je

$$\begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial \vartheta}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial \vartheta}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{x}{r} & -\frac{y}{r^2} \\ \frac{y}{r} & \frac{x}{r^2} \end{pmatrix}$$

dus grad $f = \begin{pmatrix} x/r & -y/r^2 \\ y/r & x/r^2 \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial r} \\ \frac{\partial f}{\partial \vartheta} \end{pmatrix}$

achteraf: deze tussenstap was niet zo zinvol en zou weg kunnen

$$= \begin{pmatrix} \frac{x}{r}\frac{\partial f}{\partial r} - \frac{y}{r^2}\frac{\partial f}{\partial \theta} \\ \frac{y}{r}\frac{\partial f}{\partial r} + \frac{x}{r^2}\frac{\partial f}{\partial \theta} \end{pmatrix} = \frac{1}{r}\begin{pmatrix} x \\ y \end{pmatrix}\frac{\partial f}{\partial r} + \frac{1}{r^2}\begin{pmatrix} -y \\ x \end{pmatrix}\frac{\partial f}{\partial \theta}$$

Met $\hat{r} = \frac{1}{r}\begin{pmatrix} x \\ y \end{pmatrix}$ en $\hat{\theta} = \frac{1}{r}\begin{pmatrix} -y \\ x \end{pmatrix}$ kun je dit schrijven als:

$$\boxed{\operatorname{grad} f = \hat{r}\frac{\partial f}{\partial r} + \frac{1}{r}\hat{\theta}\frac{\partial f}{\partial \theta}}$$

en dat was de bedoeling